Some Topics in Continuum Dislocations : *General Framework of Generalized Continuum Model*

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Différents points de vue sur la théorie des dislocations: physique, mécanique, mathématiques.

ENS Paris-Saclay, April 11^{th}, 2025

Planning

1. Basics on Plastic Deformation
2. Generalized Continuum Model (GCM)
3. GCM and Continuum Dislocations
4. Conservation Laws for GCM
5. Example of Constitutive Laws
6. Concluding Remarks

I. Basics on Plastic Deformation

- Severe Plastic Deformation (SPD)
- Relation with Dislocations

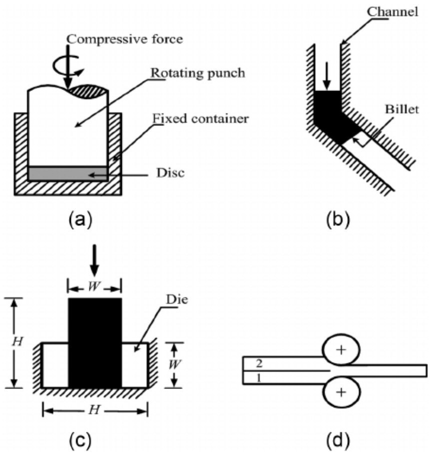


Figure: Material Processing with **Grain Refinement** by Severe Plastic Deformation : Strengthening of material. **(a) HPT : High Pressure Torsion**

SPD : HPT Experimental Results

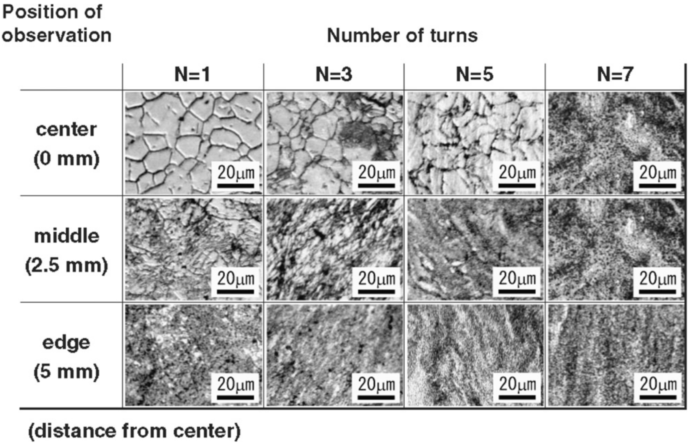


Figure: **Optical Micrographs** : Microstructures after HPT at the center, half-way position and edge of the disk in a magnesium AZ61 alloy after processing N turns at 423 K **(Zhilayev & Langdon 2008)**.

SPD : Dislocation and Grains Mechanisms

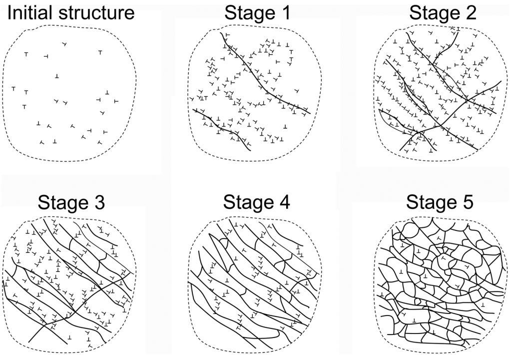


Figure: Grain size refinement under SPD **(Cao et al. 2018)**:
(1) Initial : formation of large size dislocations cell blocks containing **dislocations** and **dislocation cells** structures;
(2) Formation of micro-bands and transformation of some nearly dislocations cells into cells blocks;
(3) Formation of lamellar sub-grains containing **numerous dislocations**;
(4) Formation of well-developed lamellar **sub-grains** and some equiaxed sub-grains;
(5) Homogeneous distribution of **equiaxed ultrafine grains** or nano-grains.

SPD : Mechanisms of Plastic Deformation

1. **Twinning** and **relative motions** of grain with refinement are mostly the mechanisms underlying plastic deformation.

2. **Density of dislocations** augments when the plastic deformation increases.

3. **Asympotic response**. For complicated deformation operating in SPD, microstructure mostly reach a steady state at which further deformation does not change the overall microstructure.

MuD : Approach with Continuum Plasticity

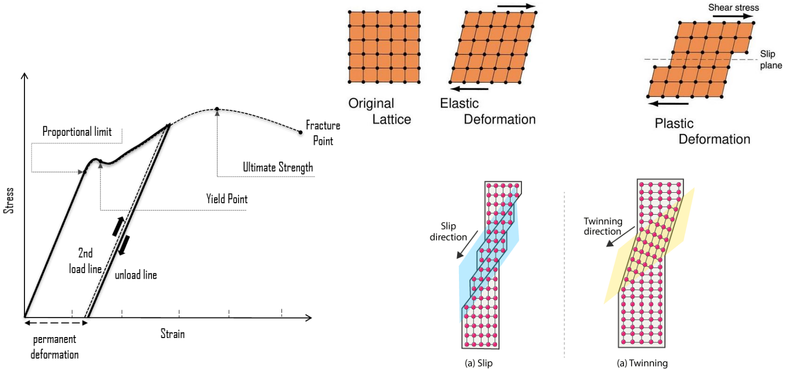


Figure: Stress σ strain ε curve of elementary traction. **Slip** and **Twinning** of Crystallin Material

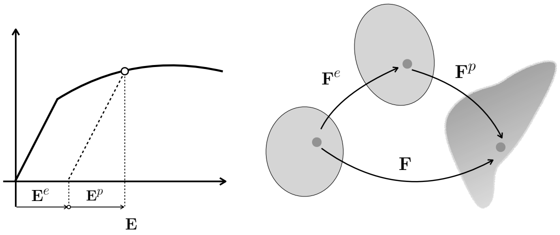


Figure: (a) Additive Decomposition (small deformation) and (b) **Multiplicative Decomposition** (Bilby et al. 1957).

Additive Decomposition: $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p$ (Green & Nagdhi 1965)

- $\mathbf{E}^p$: primal variable
- $\phi(\mathbf{E}, \mathbf{E}^p)$: Helmholtz free energy
- $\mathbf{E} - \mathbf{E}^p$: not necessarily an elastic strain (1966)

Application : Mostly used for **small strains** (despite Nagdhi 1990)

MuD : Swift Review of Theories

1. **Multiplicative Decomposition**: $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ (Lee & Liu 1967)
   - Dislocations & crystallin backgrounds : Kondo (1949), **Bilby (1957)**, Kröner (1959)
   - Incompatible intermediate deformation (non Euclidean configuration): e.g. Le & Stumpf, 1996
   - Existence of numerous partitions of rate of deformations : $\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p$, $\mathbf{L} = \mathbf{L}^e + \mathbf{L}^p, \cdots$, and problem of plastic spin.

   Some problems: Unicity of intermediate configuration, choice of plastic deformation and plastic spin and their objective rates, covariance of model ...

2. **More recent results**:
   - Rigorous relation with crystallin plasticity (Reina et al. 2017)
   - Existence and uniqueness of $\mathbf{F}^p$ (Reina & Conti 2017)
   - Advances on rates of deformations : Lubarda (2004), Bruhns (2009), Volokh (2013), ...

MuD: Towards GND and Dislocations Approach

1. **Geometrically Necessary Dislocation (GND)**: Use of Multiplicative Decomposition leads to
   - Measure of incompatibility $\mathbf{G} := (\mathrm{Det}\mathbf{F}^p)^{-1}\mathbf{F}^p\ \overline{\mathrm{Rot}}\mathbf{F}^p$ (Cermelli & Gurtin 2001, Reina & Conti 2014)
   - $\mathscr{F}$ invariant iff $\mathscr{F}\left(\mathbf{F}^p, \overline{\nabla}\mathbf{F}^p\right) = \mathscr{F}(\mathbf{G})$

   Hypotheses : isovolume plastic strain, no plastic spin,...

2. **Physically**, SPD in metallic systems is mostly accompanied by **grain refinement** (e.g. Zhylaiev & Langdon 2008), and by generation and organization of **high density of crystal defects** (e.g. Cao et al. 2018).

   Observation: **Metric** captures change of shape but the SPD involves the deformation **incompatibilities** and **discontinuous relative motions** of sub-regions (grains).

III. Generalized Continuum Model

- **Goal: Define a model allowing density of dislocations and disclinations within a continuum framework.**
- **Method**: Use of **differential geometry** for capturing incompatibilities of deformations.

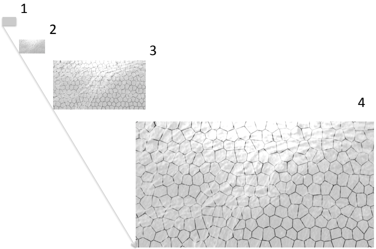


Figure: From (1) : Simple Material Model to (4) Generalized Continuum Model. (Optical Micrographs of an Al alloy).

GCM: Existence of Numerous Length Scales

We remind we are always working at a **chosen length scale** of a continuum.

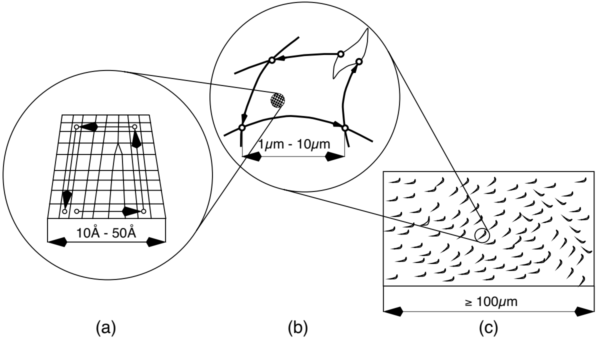


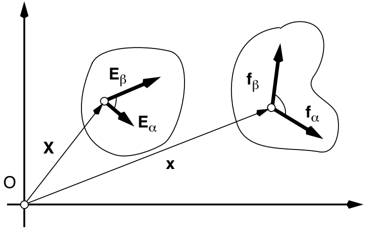
Figure: Crystal plasticity occurs at length scale (a); microscopic and **continuum defects** at scales (b) and (c) $\Longrightarrow$ Rather working with **Defects Density**.

Generalized Continuum is a compact and connected manifold $\mathscr{B}$ endowed with **(Whyburn, 1935)**:

- a metric tensor $\mathbf{g}$ with components $g_{\alpha\beta}(\mathbf{x})$
- an affine connection ∇ with coefficients $\Gamma^{\gamma}_{\alpha\beta}(\mathbf{x})$

GCM : Metric (Shape Change)

Shape of grains and overall continuum is measured by metric



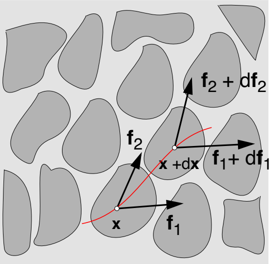
- **Metric** $\mathbf{g}$ measures local deformation : **(length & angle)** :

$$g_{\alpha\beta} = \mathbf{g}\left(\mathbf{f}_\alpha, \mathbf{f}_\beta\right) := \mathbf{f}_\alpha \cdot \mathbf{f}_\beta \quad \Longrightarrow \quad \begin{cases} \|\mathbf{f}_\alpha\| = \sqrt{\mathbf{f}_\alpha \cdot \mathbf{f}_\alpha} \\ \cos\left(\mathbf{f}_\alpha, \mathbf{f}_\beta\right) = \dfrac{\mathbf{f}_\alpha \cdot \mathbf{f}_\beta}{\|\mathbf{f}_\alpha\| . \|\mathbf{f}_\beta\|} \end{cases}$$

- **Triads** $F_\alpha^i\left(\mathbf{X}\right)$ defines local mapping $\mathbf{f}_\alpha = F_\alpha^i \ \mathbf{E}_i$

Connection is a mathematical operator linking two grains.

(Gonseth, 1929)



- **Change of any $\mathbf{f}_\beta$ along $d\mathbf{x}$**: (linear with respect to $d\mathbf{x} := dx^\alpha \mathbf{f}_\alpha$)

$$\mathbf{f}_\beta(\mathbf{x}) \longrightarrow \mathbf{f}_\beta(\mathbf{x} + d\mathbf{x}) := \mathbf{f}_\beta(\mathbf{x}) + \nabla_{d\mathbf{x}} \mathbf{f}_\beta = \mathbf{f}_\beta(\mathbf{x}) + \nabla_{\mathbf{f}_\alpha} \mathbf{f}_\beta \; dx^\alpha$$

- **Connection coefficients** calculated by noticing $\nabla_\alpha \equiv \nabla_{\mathbf{f}_\alpha}$ and projecting onto 1-form $\mathbf{f}^\gamma$: $\Gamma^\gamma_{\alpha\beta} := \mathbf{f}^\gamma \left(\nabla_\alpha \mathbf{f}_\beta\right)$.

V. Generalized Continuum Models for Continuum Dislocations

Motivation : How to measure sharp gradients (defects and grain refinement) in a continuum plasticity model ?

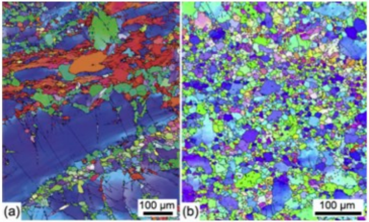


Figure: (a) One Single Crystal Mg after one Equal Canal Angular Pressing (ECAP) (Sedá et al. 2012); (b) a Polycrystalline Mg after four ECAP (Biswas et al. 2010).

Consider a function $f(\mathbf{X})$ on $\mathbb{R}$ with **discontinuity** at points $(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n, n \to \infty)$

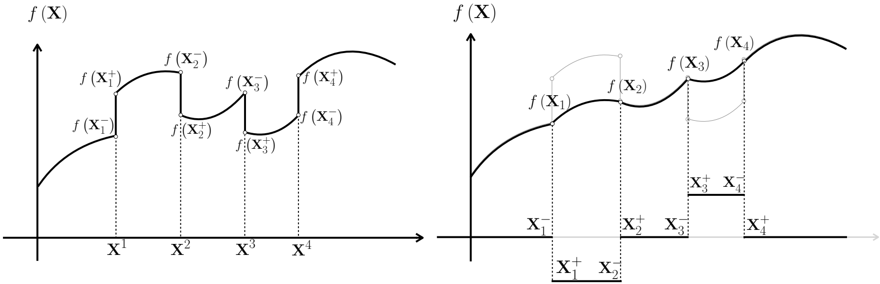
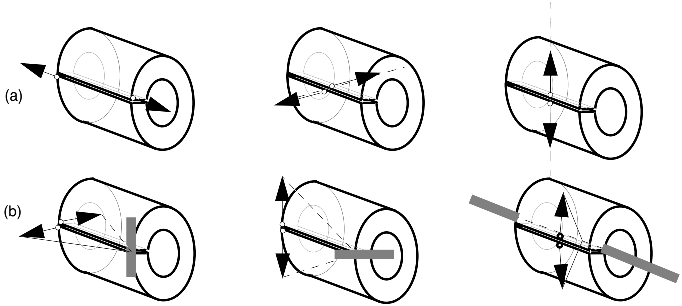


Figure: Discontinuous fields : (Left) **Euclidean approach** : Function (field) is enriched, (Right) **Riemann-Cartan approach** : Domain (manifold) is enriched (Accounting for additional field $[f_1]$, $[f_2]$, $[f_3]$, $[f_4]$ ... ∞ **infinite number** $\longrightarrow$ **density concept**)

Individual dislocations (3D)

(a) Translations discontinuity and (b) Rotations discontinuity[1]



[1]See eg. : *Kondo, 1955; Bilby et al., 1955; Kröner, 1963; Noll, 1967, Wang, 1967; Zorawski, 1967; Povstenko, 1987; Maugin, 1993, Le & Stumpf, 1996, R 1998, Zubov, 1997, Acharya & Bassani, 2000, ...*

Presence of defects as dislocations and disclinations may be approached with a GCM ! (Via defects density)

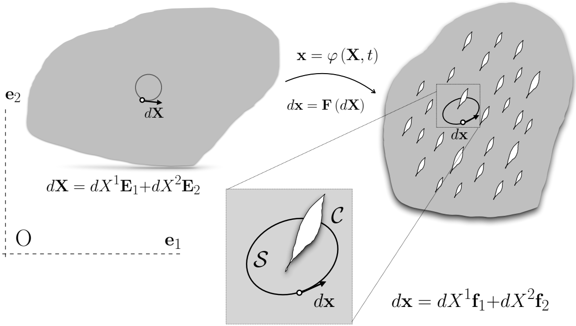


Figure: Cartan Circuit around Defect. Presence of defects induces a **disclosure of Cartan circuit**

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Consider a **scalar field** $\theta(\mathbf{X})$ and **vector field** $\mathbf{w}(\mathbf{X})$ on a **GCM** $\mathscr{B}$ (Example : density, velocity) at a point M

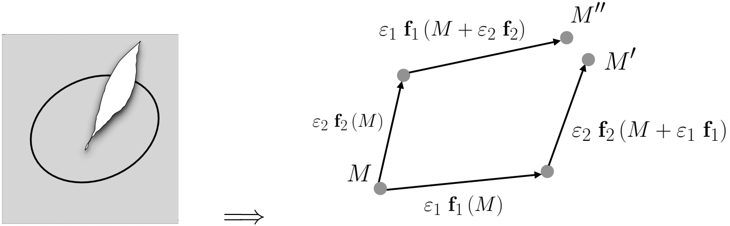


Figure: (Left) : **Field discontinuity** [ex. microcrack within a body]; (Right) Parallelogram of Cartan.

We notice $\theta' := \theta(M')$, and $\theta'' := \theta(M'')$; and $\mathbf{w}' := \mathbf{w}(M')$, and $\mathbf{w}'' := \mathbf{w}(M'')$.

Goal: Define geometric variables modeling discontinuity of fields on a continuum

Theorem

Say an affinely connected manifold $\mathscr{B}$. Let consider two arbitrary vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ at a point M, they define two paths of length ϵ_1 and ϵ_2. Then:

$$\begin{cases} \lim_{(\epsilon_1,\epsilon_2)\to 0} \dfrac{(\theta' - \theta'')}{\epsilon_1 \epsilon_2} = \aleph(\mathbf{f}_1, \mathbf{f}_2)[\theta] \\ \lim_{(\epsilon_1,\epsilon_2)\to 0} \dfrac{(\mathbf{w}' - \mathbf{w}'')}{\epsilon_1 \epsilon_2} = \Re(\mathbf{f}_1, \mathbf{f}_2, \mathbf{w}) - \nabla_{\aleph(\mathbf{f}_1,\mathbf{f}_2)}\mathbf{w} \end{cases}$$

Proof [2] Remind geometric definitions of torsion $\aleph$ and curvature $\Re$:

$$\begin{aligned} \aleph(\mathbf{f}_1, \mathbf{f}_2)[\theta] &:= (\nabla_{\mathbf{f}_1}\mathbf{f}_2 - \nabla_{\mathbf{f}_2}\mathbf{f}_1)[\theta] - [\mathbf{f}_1, \mathbf{f}_2][\theta] \\ \Re(\mathbf{f}_1, \mathbf{f}_2, \mathbf{w}) &:= \nabla_{\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w} - \nabla_{\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w} - \nabla_{[\mathbf{f}_1,\mathbf{f}_2]}\mathbf{w} \end{aligned}$$

[2](Schouten Theorem. Partly R 1998, complete R 2021)

1) Consider a **scalar field** $\theta(M)$, we have respectively the relations:

$$\begin{cases} \theta(M_1) & = & \theta(M + \varepsilon_1 \mathbf{f}_1) = \theta(M) + \nabla_{\varepsilon_1 \mathbf{f}_1(M)} \theta(M) \\ \theta(M') & = & \theta(M_1 + \varepsilon_2 \mathbf{f}_2 (M + \varepsilon_1 \mathbf{f}_1)) = \theta(M_1) + \nabla_{\varepsilon_2 \mathbf{f}_2(M+\varepsilon_1 \mathbf{f}_1)} \theta(M_1) \end{cases}$$

We obtain the value of scalar field at M' in terms of **its value at M**, by noticing $\theta(M') = \theta'$ and $\theta(M) = \theta$, :

$$\begin{aligned} \theta' &= \theta + \varepsilon_1 \nabla_{\mathbf{f}_1} \theta + \varepsilon_2 \nabla_{\mathbf{f}_2} \theta + \varepsilon_2 \varepsilon_1 \nabla_{\nabla_{\mathbf{f}_1} \mathbf{f}_2} \theta + \varepsilon_2 \varepsilon_1 \nabla_{\mathbf{f}_2} \nabla_{\mathbf{f}_1} \theta \\ &+ \varepsilon_2 \varepsilon_1^2 \nabla_{\nabla_{\mathbf{f}_1} \mathbf{f}_2} \nabla_{\mathbf{f}_1} \theta \end{aligned} \qquad (1)$$

2) Similarly, we also obtain, $\theta(M") = \theta"$,:

$$\begin{aligned} \theta'' &= \theta + \varepsilon_2 \nabla_{\mathbf{f}_2} \theta + \varepsilon_1 \nabla_{\mathbf{f}_1} \theta + \varepsilon_1 \varepsilon_2 \nabla_{\nabla_{\mathbf{f}_2} \mathbf{f}_1} \theta + \varepsilon_1 \varepsilon_2 \nabla_{\mathbf{f}_1} \nabla_{\mathbf{f}_2} \theta \\ &+ \varepsilon_1 \varepsilon_2^2 \nabla_{\nabla_{\mathbf{f}_2} \mathbf{f}_1} \nabla_{\mathbf{f}_2} \theta \end{aligned} \qquad (2)$$

GCM: Sketch of the proof (2)

3) Similarly, consider a vector field with $\mathbf{w} := \mathbf{w}(M)$,:

$$\begin{aligned}\mathbf{w}' &= \mathbf{w} + \varepsilon_1 \nabla_{\mathbf{f}_1}\mathbf{w} + \varepsilon_2 \nabla_{\mathbf{f}_2}\mathbf{w} + \varepsilon_2\varepsilon_1 \nabla_{\nabla_{\mathbf{f}_1}\mathbf{f}_2}\mathbf{w} + \varepsilon_2\varepsilon_1 \nabla_{\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w} \\ &+ \varepsilon_2\varepsilon_1^2 \nabla_{\nabla_{\mathbf{f}_1}\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w}\end{aligned} \tag{3}$$

and

$$\begin{aligned}\mathbf{w}'' &= \mathbf{w} + \varepsilon_2 \nabla_{\mathbf{f}_2}\mathbf{w} + \varepsilon_1 \nabla_{\mathbf{f}_1}\mathbf{w} + \varepsilon_1\varepsilon_2 \nabla_{\nabla_{\mathbf{f}_2}\mathbf{f}_1}\mathbf{w} + \varepsilon_1\varepsilon_2 \nabla_{\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w} \\ &+ \varepsilon_1\varepsilon_2^2 \nabla_{\nabla_{\mathbf{f}_2}\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w}\end{aligned} \tag{4}$$

4) Final step:

(1) - (2) induces the first relation on torsion.

(3) - (4) induces the second result on curvature (and torsion). □

Remarque

Calculus is done **exculsively** *at point M (then $\mathscr{T}_M\mathscr{B}$). Without curvature but with torsion, we may have vector field discontinuities.*

1. Physical Plastic Deformation can be related to **discontinuities of scalar and vector fields**. Continuum Plasticity might be approached by **GCM** (Riemann-Cartan manifold) as geometric foundations.

2. Two intrinsic elements of the connection ∇, **torsion** $\aleph$ and **curvature** $\Re$ clearly measure the density of these discontinuities (**dislocations density** and grain refinement, relative motions of grains) !

3. One question is now if they are considered as :
    1. Internal variables
    2. Primal Variables (as for metric / strain)

Goal: Define general shape of Lagrangian for GCM (Material Frame Independent)

Theorem

A GCM defined by the Lagrangian $\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \Gamma^{\gamma}_{\alpha\beta},\ \partial_\lambda \Gamma^{\gamma}_{\alpha\beta})$ *is* **covariant** *if and only if:*

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \aleph^{\gamma}_{\alpha\beta},\ \mathfrak{R}^{\gamma}_{\alpha\beta\lambda})$$

Remarks :

- **Primal / internal variables** are metric $g_{\alpha\beta}$, torsion $\aleph^{\gamma}_{\alpha\beta}$, and curvature $\mathfrak{R}^{\gamma}_{\alpha\beta\lambda}$ (**Continuum physics** : elasticity, fluid mechanics, gravitation, electromagnetism, plastic deformation ...)
- This theorem extends **Cartan** (1922) and **Lovelock-Rund** (1971, 1975) theorems from Riemann to **Riemann-Cartan continuum**.

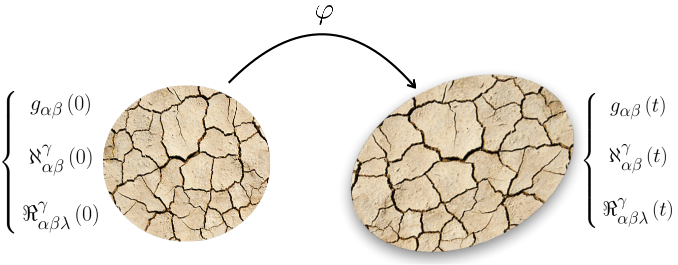
GCM : Main steps of the proof:

1. Consider change of coordinates (C^∞-diffeomorphism)
   $y^\alpha = y^\alpha\left(x^i\right)$, and $J_i^\alpha := \partial_i y^\alpha, \cdots$
   Write : $g_{ij} = J_i^\alpha \; J_j^\beta \; g_{\alpha\beta}$, ...
2. Assume covariance + rules of tensors transformation ($g_{\alpha\beta}$ and $\Gamma^\gamma_{\alpha\beta}$)
   $\mathscr{L}(g_{ij}, \Gamma^k_{ij}, \partial_l \Gamma^k_{ij} + \Gamma^m_{ij}\Gamma^k_{lm}) = \mathscr{L}(g_{\alpha\beta}, \Gamma^\gamma_{\alpha\beta}, \partial_\lambda \Gamma^\gamma_{\alpha\beta} + \Gamma^\mu_{\alpha\beta}\Gamma^\gamma_{\lambda\mu})$
3. Decompose $\Gamma^\gamma_{\alpha\beta}$ and $\partial_\alpha \Gamma^\kappa_{\beta\lambda} + \Gamma^\xi_{\beta\lambda}\Gamma^\kappa_{\alpha\xi}$ into symmetric and skew-symmetric tensors.
4. Derivatives of $\mathscr{L}$ with respect to J_i^α, ...
5. Apply the Quotient Theorem ($\mathbf{h}$ = tensor):
   $A^{\alpha\beta} h_{\alpha\beta} + B^{\alpha\beta\gamma} \nabla_\gamma h_{\alpha\beta}$ scalar $\Rightarrow$ $A^{\alpha\beta}$, $B^{\alpha\beta\gamma}$ tensors. □

GCM: Generalized Deformation

Generalized deformation of a GCM includes:

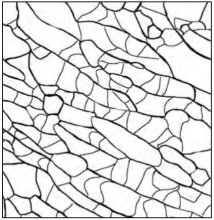
1. classical strain tensor represented by **metric** field $g_{\alpha\beta}$
2. nucleation, drift, and annihilation of dislocations represented by fields of **torsion** $\aleph^{\gamma}_{\alpha\beta}$, and **curvature** $\Re^{\gamma}_{\alpha\beta\lambda}$.



Next step would be the search of conservation laws governing the deformation of $\mathscr{B}$.

VII. Conservation Laws for GCM (= Strain Gradient Plasticity)

Physics of GCM is defined by a Lagrangian $\mathscr{L} = \mathscr{L}(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \Re^{\lambda}_{\alpha\beta\gamma})$



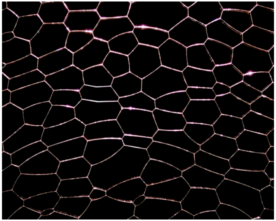


Figure: **Left**: Plastic deformation of (Metallic alloy, set of **microcosms**); **Right**: Spacetime in Loop Quantum Gravitation (set of "**quanta**").

Goal : Derive conservation laws for GCM.

Principle of General Covariance (PGC)

For the general case, we have the action

$$\mathscr{S} := \int_{\mathscr{B}} \mathscr{L} \, \omega_n, \quad \text{with} \quad \omega_n = \text{RC volume} - \text{form}$$

1. **Principle of General Covariance.** [3] (e.g. Souriau 1975, Duval & Künzle 1978, R 2018) The basic method is to express variation of action resulting from the **Lie derivatives**, neglecting $\mathscr{O}(d\lambda)$,:

$$\begin{aligned} \delta_\xi \mathscr{S} &= d\lambda \int_{\mathscr{B}} \left(\sigma^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} + \Sigma_\gamma^{\alpha\beta} \mathcal{L}_\xi \aleph_{\alpha\beta}^\gamma + \Xi_\lambda^{\alpha\beta\mu} \mathcal{L}_\xi \Re_{\alpha\beta\mu}^\lambda \right) \omega_n \\ &= 0 \end{aligned}$$

2. **Conservation laws** are deduced when the trajectory is shifted while the **action is left unchanged**. Infinitesimal **active diffeomorphisms** are defined by **Lie derivative** variations

$$\left(\mathcal{L}_\xi g_{\alpha\beta}, \; \mathcal{L}_\xi \aleph_{\alpha\beta}^\gamma, \; \mathcal{L}_\xi \Re_{\alpha\beta\mu}^\lambda \right)$$

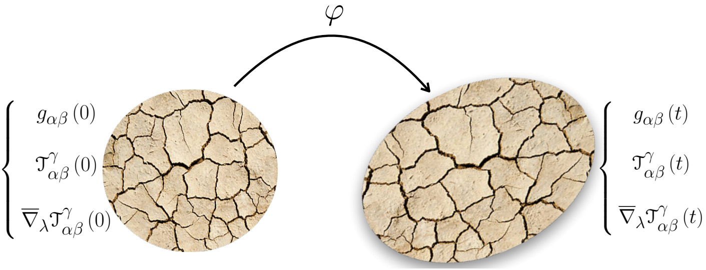
[3] This Principle can be extended to Noether-Klein method and furnishes identities in addition to conservation laws by pointing out the variations ξ, $\nabla\xi$,

We consider **equivalent set of variables** (**contortion** $\mathcal{T}$, **LC gradient of contortion** $\overline{\nabla}\mathcal{T}$) to avoid the use of ∇ which implicitly includes $\aleph$:

$$\begin{cases} \Gamma^{\gamma}_{\alpha\beta} - \overline{\Gamma}^{\gamma}_{\alpha\beta} := \mathcal{T}^{\gamma}_{\alpha\beta} \\ \Re^{\gamma}_{\alpha\beta\lambda} - \overline{\Re}^{\gamma}_{\alpha\beta\lambda} := \overline{\nabla}_{\alpha}\mathcal{T}^{\gamma}_{\lambda\beta} - \overline{\nabla}_{\beta}\mathcal{T}^{\gamma}_{\lambda\alpha} + \left(\mathcal{T}^{\gamma}_{\beta\mu}\mathcal{T}^{\mu}_{\alpha\lambda} - \mathcal{T}^{\gamma}_{\alpha\mu}\mathcal{T}^{\mu}_{\beta\lambda}\right) \end{cases}$$

with the **Levi-Civita** (metric) connection $\overline{\nabla}$:

$$\overline{\Gamma}^{\gamma}_{\alpha\beta} := (1/2)g^{\gamma\lambda}\left(\partial_{\alpha}g_{\lambda\beta} + \partial_{\beta}g_{\alpha\lambda} - \partial_{\lambda}g_{\alpha\beta}\right) \quad \rightarrow \quad \overline{\Re}^{\gamma}_{\alpha\beta\lambda}$$



The main reason of using of these two equivalent dislocations variables is to avoid torsion implicitly present in connection ∇.

Lagrangian of the form $\mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta}, \overline{\nabla}_\lambda \mathcal{T}^{\gamma}_{\alpha\beta})$

1. **GCM action** takes an alternative form:

$$\mathscr{S} := \int_{\mathscr{B}} \mathscr{L}\left(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta}, \overline{\nabla}_\lambda \mathcal{T}^{\gamma}_{\alpha\beta}\right) \omega_n$$

w. volume-form $\omega_n := e^{\vartheta(\mathbf{g},\mathcal{T})}\ \overline{\omega}_n$ for divergence theorem (R 2025).

2. **Constitutive laws**. Dual variables are obtained from action variation: $\delta\mathscr{S} = \int_{\mathscr{B}} \left(\sigma^{\alpha\beta}\ \delta g_{\alpha\beta} + \Sigma^{\alpha\beta}_{\gamma}\ \delta\mathcal{T}^{\gamma}_{\alpha\beta} + \Xi^{\lambda\alpha\beta}_{\gamma}\ \delta\overline{\nabla}_\lambda \mathcal{T}^{\gamma}_{\alpha\beta}\right)\omega_n$

$$\begin{cases} \sigma^{\alpha\beta} := \dfrac{\partial\mathscr{L}}{\partial g_{\alpha\beta}} + \mathscr{L}\left(\dfrac{\partial\vartheta}{\partial g_{\alpha\beta}} - \dfrac{g^{\alpha\beta}}{2}\right) & \text{Stress} \\ \Sigma^{\alpha\beta}_{\gamma} := \dfrac{\partial\mathscr{L}}{\partial\mathcal{T}^{\gamma}_{\alpha\beta}} + \mathscr{L}\dfrac{\partial\vartheta}{\partial\mathcal{T}^{\gamma}_{\alpha\beta}} & \text{Micro} - \text{Stress} \\ \Xi^{\lambda\alpha\beta}_{\gamma} := \dfrac{\partial\mathscr{L}}{\partial\overline{\nabla}_\lambda\mathcal{T}^{\gamma}_{\alpha\beta}} & \text{Polar Stress} \end{cases}$$

1. **Variational Formulation**. For any local translation $\xi(\mathbf{x})$ generating **active diffeomorphism**, the **PGC** takes the form of:

$$\delta \mathscr{S} = \int_{\mathscr{B}} \left[\sigma^{\alpha\beta} \, \mathcal{L}_\xi g_{\alpha\beta} + \Sigma_\gamma^{\alpha\beta} \, \mathcal{L}_\xi \mathcal{T}_{\alpha\beta}^\gamma + \Xi_\gamma^{\lambda\alpha\beta} \, \mathcal{L}_\xi \overline{\nabla}_\lambda \mathcal{T}_{\alpha\beta}^\gamma \right] \omega_n = 0$$

This is considered as an extension of Principle of Virtual Power.

2. The **Principle Virtual Power (PVP** is given by *(Gurtin & Anand 2005)*, assuming $\mathbf{F} := \mathbf{F}^e \mathbf{F}^p \to \mathbf{D}^p := \left\{ \dot{\mathbf{F}}^p \ \mathbf{F}^{p-1} \right\}$:

$$\delta \mathscr{W}_i := \int_{\mathscr{B}} \left(\mathbf{S}^e : \dot{\mathbf{F}}^e + \mathbf{T}^p : \mathbf{D}^p + \mathbb{T}^p : \overline{\nabla} \mathbf{D}^p \right) dV = 0$$

where conditions are : $\mathbf{D}^p = \mathbf{D}^{p\mathrm{T}}, \ \mathrm{Tr} \mathbf{D}^p \equiv 0.$

3. To make **PVP covariant**, $\mathbf{D}^p$ and $\nabla \mathbf{D}^p$ should be merged into the time rate of unique variable $\dot{\mathbf{G}}$ where $\mathbf{G} := (\mathrm{Det} \mathbf{F}^p)^{-1} \mathbf{F}^p \ \overline{\mathrm{Rot}} \mathbf{F}^p$ *(Cermelli & Gurtin 2001)*.

1. **Constitutive laws.** The "stress" σ is defined as :

$$\mathscr{S} := \int_{\mathscr{B}} \mathscr{L} \; \overline{\omega}_n \quad \Longrightarrow \quad \sigma^{\alpha\beta} := \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}} - \frac{\mathscr{L}}{2} \; g^{\alpha\beta}$$

2. **Conservation laws.**

Theorem

Consider a classical continuum model $(\mathscr{B}, \mathbf{g}, \overline{\nabla})$ *with a Lagrangian function* $\mathscr{L}(g_{\alpha\beta})$. *Then :*

$$\overline{\nabla}_\alpha \sigma^\alpha_\gamma = 0$$

Proof: Write the PGC:

$$\Delta \mathscr{S} \quad := \quad \int \sigma^{\alpha\beta} \; \mathcal{L}_\xi g_{\alpha\beta} \; \overline{\omega}_n$$

$$\mathcal{L}_\xi \; g_{\alpha\beta} \quad = \quad \xi^\gamma \; \overline{\nabla}_\gamma g_{\alpha\beta} + g_{\gamma\beta} \; \overline{\nabla}_\alpha \xi^\gamma + g_{\alpha\gamma} \; \overline{\nabla}_\beta \xi^\gamma$$

Integrate by parts and shift the boundary flux terms to obtain:

$$-\int_{\mathscr{B}} 2 \; \xi^\gamma \; \overline{\nabla}_\alpha \sigma^\alpha_\gamma \; \overline{\omega}_n = 0 \qquad \square$$

Say **local transformation** $dx = \mathbf{F}(d\mathbf{X})$ (Pfaffian), with the spatial, initial material, and deformed material **bases**:

$$(\mathbf{e}_i, i = 1, 2, 3), \quad (\mathbf{E}_\alpha, \alpha = 1, 2, 3), \quad (\mathbf{f}_\alpha, \alpha = 1, 2, 3), \quad \mathbf{f}_\alpha := \mathbf{F}(\mathbf{E}_\alpha)$$

Consider the **stress tensor** σ and $J := \mathrm{Det}\mathbf{F}$. Its **projections** hold:

- **"Cauchy"** σ and Kirchhoff τ stresses,

$$\sigma \to \tau := J\sigma \qquad \text{to give} \qquad J\mathbf{e}^i \cdot \sigma\left(\mathbf{e}^j\right) = J\,\sigma^{ij}$$

- **First Piola-Kirchhoff** stress $\mathbf{P} := J\sigma\mathbf{F}^{-\mathrm{T}}$

$$\mathbf{e}^i \cdot \tau\left(\mathbf{f}^\beta\right) = J\mathbf{e}^i \cdot \sigma\left(\mathbf{F}^{-\mathrm{T}}(\mathbf{E}^\beta)\right) = \mathbf{e}^i \cdot J\sigma\mathbf{F}^{-\mathrm{T}}\left(\mathbf{E}^\beta\right) := P^{i\beta}$$

- **Second Piola-Kirchhoff** stress $\mathbf{S} := J\mathbf{F}^{-1}\sigma\mathbf{F}^{-\mathrm{T}}$

$$\mathbf{f}^\alpha \cdot \tau\left(\mathbf{f}^\beta\right) = J\mathbf{F}^{-\mathrm{T}}(\mathbf{E}^\alpha) \cdot \sigma\left(\mathbf{F}^{-\mathrm{T}}(\mathbf{E}^\beta)\right) = \mathbf{E}^\alpha \cdot J\mathbf{F}^{-1}\sigma\mathbf{F}^{-\mathrm{T}}\left(\mathbf{E}^\beta\right) := S^{\alpha\beta}$$

We deal with the same tensor σ projected onto different bases !

1. **Constitutive laws**. We remind the two dual variables :

$$\sigma^{\alpha\beta} := \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}} + \mathscr{L}\left(\frac{\partial \vartheta}{\partial g_{\alpha\beta}} - \frac{g^{\alpha\beta}}{2}\right), \quad \Sigma^{\alpha\beta}_{\gamma} := \frac{\partial \mathscr{L}}{\partial \mathcal{T}^{\gamma}_{\alpha\beta}} + \mathscr{L}\frac{\partial \vartheta}{\partial \mathcal{T}^{\gamma}_{\alpha\beta}}$$

   In RG, dual variables $\sigma^{\alpha\beta}$ and $\Sigma^{\alpha\beta}_{\gamma}$ are respectively called energy-momentum and hypermomentum (Hehl et al. 1976).

2. **Conservations laws**.

Theorem

Consider a **Weitzenböck Continuum Model** $(\mathscr{B}, \mathbf{g}, \nabla)$ *with a Lagrangian function* $\mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta})$. *Then :*

$$\overline{\nabla}_{\beta}\tilde{\sigma}^{\beta}_{\rho} + \mathcal{T}^{\alpha}_{\alpha\beta}\ \tilde{\sigma}^{\beta}_{\rho} = 0$$

with the **generalized stress** $\tilde{\sigma}^{\beta}_{\rho}$ *:*

$$\tilde{\sigma}^{\beta}_{\rho} := \sigma^{\beta}_{\rho} + \frac{1}{2}\left(-\Sigma^{\alpha\gamma}_{\rho}\ \mathcal{T}^{\beta}_{\alpha\gamma} + \Sigma^{\beta\alpha}_{\gamma}\ \mathcal{T}^{\gamma}_{\rho\alpha} + \Sigma^{\alpha\beta}_{\gamma}\ \mathcal{T}^{\gamma}_{\alpha\rho}\right)$$

WCM : Proof (1)

1. Invariance of Poincaré.

1. Express the **PGC as variation** due to **active diffeomorphism**:

$$\delta_\xi \int_{\mathscr{B}} \mathscr{L} \omega_n = \int_{\mathscr{B}} \left(\sigma^{\alpha\beta} \; \mathcal{L}_\xi g_{\alpha\beta} + \Sigma_\gamma^{\alpha\beta} \mathcal{L}_\xi \mathcal{T}_{\alpha\beta}^\gamma\right) \omega_n = 0$$

2. Compute the **Lie derivatives** (tedious !)

$$\begin{cases} \mathcal{L}_\xi g_{\alpha\beta} & = & \overline{\nabla}_\alpha \xi_\beta + \overline{\nabla}_\beta \xi_\alpha + g_{\alpha\gamma} \mathcal{T}_{\rho\beta}^\gamma \xi^\rho + g_{\gamma\beta} \mathcal{T}_{\rho\alpha}^\gamma \xi^\rho \\ \mathcal{L}_\xi \mathcal{T}_{\alpha\beta}^\gamma & = & \xi^\rho \overline{\nabla}_\rho \mathcal{T}_{\alpha\beta}^\gamma - \mathcal{T}_{\alpha\beta}^\rho \overline{\nabla}_\rho \xi^\gamma + \mathcal{T}_{\rho\beta}^\gamma \overline{\nabla}_\alpha \xi^\rho + \mathcal{T}_{\alpha\rho}^\gamma \overline{\nabla}_\beta \xi^\rho \end{cases}$$

3. We write the **integrand** (terms in brackets):

$$\begin{aligned} \delta_\xi \mathscr{L} &= \sigma^{\alpha\beta} \left(\overline{\nabla}_\alpha \xi_\beta + \overline{\nabla}_\beta \xi_\alpha\right) + \sigma^{\alpha\beta} \left(g_{\alpha\gamma} \mathcal{T}_{\rho\beta}^\gamma + g_{\gamma\beta} \mathcal{T}_{\rho\alpha}^\gamma\right) \xi^\rho \\ &+ \left(\Sigma_\gamma^{\alpha\beta} \overline{\nabla}_\rho \mathcal{T}_{\alpha\beta}^\gamma\right) \xi^\rho \\ &+ \Sigma_\gamma^{\alpha\beta} \left(-\mathcal{T}_{\alpha\beta}^\rho \overline{\nabla}_\rho \xi^\gamma + \mathcal{T}_{\rho\beta}^\gamma \overline{\nabla}_\alpha \xi^\rho + \mathcal{T}_{\alpha\rho}^\gamma \overline{\nabla}_\beta \xi^\rho\right) = 0 \end{aligned}$$

It should vanish for **all compatible diffeomorphisms** (as for PVP).

WCM : Proof (2)

2. **Global Invariance.** By considering **uniform** ξ, we deduce an **identity**:

$$\sigma^{\alpha\beta}\left(g_{\alpha\nu}\mathcal{T}^{\nu}_{\rho\beta} + g_{\nu\beta\nu}\mathcal{T}^{\nu}_{\rho\alpha}\right) + \Sigma^{\alpha\beta}_{\gamma}\overline{\nabla}_{\rho}\mathcal{T}^{\gamma}_{\alpha\beta} = 0$$

3. **Local Invariance.** We consider **non uniform** translations $\xi(\mathbf{x})$, factorize by $\overline{\nabla}_{\beta}\xi^{\rho}$, integrate by part, and drop all terms in divergence (for the sake of the simplicity) and obtain:

$$\int_{\mathscr{B}} \tilde{\sigma}^{\beta}_{\rho}\ \overline{\nabla}_{\beta}\xi^{\rho}\ \omega_n = \int_{\mathscr{B}} \left[\nabla_{\beta}\left(\tilde{\sigma}^{\beta}_{\rho}\xi^{\rho}\right) - \left(\overline{\nabla}_{\beta}\tilde{\sigma}^{\beta}_{\rho} + \mathcal{T}^{\alpha}_{\alpha\beta}\ \tilde{\sigma}^{\beta}_{\rho}\right)\xi^{\rho}\right]\omega_n = 0$$

by defining a generalized stress measure:

$$2\tilde{\sigma}^{\beta}_{\rho} := 2\ \sigma^{\beta}_{\rho} - \Sigma^{\alpha\gamma}_{\rho}\ \mathcal{T}^{\beta}_{\alpha\gamma} + \Sigma^{\beta\alpha}_{\gamma}\ \mathcal{T}^{\gamma}_{\rho\alpha} + \Sigma^{\alpha\beta}_{\gamma}\ \mathcal{T}^{\gamma}_{\alpha\rho} \qquad \square$$

Remarque

Presence of the contortion in the conservation laws is due to the use of Levi-Civita connection but Riemann-Cartan volume-form.

WCM : Some remarks (1)

1. **Literature**. Similar conservation equations were obtained in the past obtained for
   - Relative Gravitation *(e.g. Souriau 1975, Hehl et al. 1975, Lompay & Petrov 2013, ...)*,
   - Newton-Cartan Gravitation *(e.g. Duval & Künzle, 1978)*,
   - Noll materially uniform continua *(e.g.* **Noll 1967**, *R 1998)*.

2. The **generalized stress tensor** (not necessarily symmetric) $\tilde{\sigma}_{\rho}^{\beta}$ extends the classical Cauchy stress σ_{ρ}^{β}.

3. The generalized stress includes **two contributions**: "macro" σ_{ρ}^{α} and "micros" $\Sigma_{\gamma}^{\alpha\beta}\mathcal{T}_{\rho\beta}^{\gamma}$ due to change of grain structure and dislocations density associated to plastic deformation.

WCM : Remarks (on the micro-stress / hypermomentum)

The hypermomentum / micro-stress $\Sigma_{\gamma}^{\alpha\beta}$ consists in three terms *(Mindlin 1964, Hehl et al. 1976, 1977, Gordeeva et al. 2010)*:

$$\Sigma_{\gamma}^{\alpha\beta} = \Sigma_{\gamma}^{[\alpha,\beta]} + \frac{\delta_{\gamma}^{\alpha}}{n}\,\overline{\Sigma}^{\beta} + \overline{\Sigma}_{\gamma}^{(\alpha,\beta)}$$

- Spin-angular momentum / **Rotatory micro-stress**

$$\Sigma_{\gamma}^{[\alpha,\beta]} \qquad \text{skew symmetric part}$$

- Proper hypermomentum / **Proper micro-stress**

$$\Sigma_{\gamma}^{(\alpha,\beta)} \qquad \text{symmetric part} \quad \text{and} \quad \Sigma_{\alpha}^{(\alpha,\beta)} = \overline{\Sigma}^{\beta} \quad \text{dilatational}$$

- Traceless proper hypermomentum / **Traceless proper micro-stress**

$$\overline{\Sigma}_{\gamma}^{(\alpha,\beta)} := \Sigma_{\gamma}^{(\alpha,\beta)} - \frac{\delta_{\gamma}^{\alpha}}{n}\,\overline{\Sigma}^{\beta}$$

Theorem

Consider a **Generalized Continuum Model** $(\mathscr{B}, \mathbf{g}, \nabla)$ *with a Lagrangian function* $\mathscr{L} := \mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta}, \overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\beta})$. *Then :*

$$\boxed{\overline{\nabla}_{\beta}\tilde{\sigma}^{\beta}_{\rho} + \mathcal{T}^{\alpha}_{\alpha\beta}\ \tilde{\sigma}^{\beta}_{\rho} = 0}$$

with the **generalized stress** $\tilde{\sigma}^{\beta}_{\rho}$ *:*

$$\begin{aligned}\tilde{\sigma}^{\beta}_{\rho} \quad &:= \quad \sigma^{\beta}_{\rho} + \frac{1}{2}\left(-\Sigma^{\alpha\gamma}_{\rho}\ \mathcal{T}^{\beta}_{\alpha\gamma} + \Sigma^{\beta\alpha}_{\gamma}\ \mathcal{T}^{\gamma}_{\rho\alpha} + \Sigma^{\alpha\beta}_{\gamma}\ \mathcal{T}^{\gamma}_{\alpha\rho}\right)\\ &+ \quad \frac{1}{2}\left(-\Xi^{\lambda\alpha\gamma}_{\rho}\overline{\nabla}_{\lambda}\mathcal{T}^{\beta}_{\alpha\gamma} + \Xi^{\beta\alpha\lambda}_{\gamma}\overline{\nabla}_{\rho}\mathcal{T}^{\gamma}_{\alpha\lambda} + \Xi^{\lambda\beta\alpha}_{\gamma}\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\rho\alpha} + \Xi^{\lambda\alpha\beta}_{\gamma}\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\rho}\right)\end{aligned}$$

Generalized stress includes $\Xi^{\lambda\alpha\beta}_{\gamma}$ that corresponds to **(mass) quadrupole moment** in Relative Gravitation (Dixon 1970, Souriau 1974, Bayley & Israel 1975).

- Express the **PGC** involving arbitrary **active diffeomorphism**:

$$\delta_\xi \int_{\mathscr{B}} \mathscr{L}\, \omega_n = \int_{\mathscr{B}} \left[\sigma^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} + \Sigma_\gamma^{\alpha\beta} \mathcal{L}_\xi \mathcal{T}_{\alpha\beta}^\gamma + \Xi_\gamma^{\lambda\alpha\beta} \mathcal{L}_\xi \overline{\nabla}_\lambda \mathcal{T}_{\alpha\beta}^\gamma \right] \omega_n = 0$$

with the Lie derivatives $\mathcal{L}_\xi g_{\alpha\beta}$ and $\mathcal{L}_\xi \mathcal{T}_{\alpha\beta}^\gamma$.

- Compute the **Lie derivative** of the covariant derivative:

$$\begin{aligned} \mathcal{L}_\xi \overline{\nabla}_\lambda \mathcal{T}_{\alpha\beta}^\gamma &= \xi^\rho \, \overline{\nabla}_\rho \overline{\nabla}_\lambda \mathcal{T}_{\alpha\beta}^\gamma - \overline{\nabla}_\lambda \mathcal{T}_{\alpha\beta}^\rho \overline{\nabla}_\rho \xi^\gamma \\ &+ \overline{\nabla}_\rho \mathcal{T}_{\alpha\beta}^\gamma \overline{\nabla}_\lambda \xi^\rho + \overline{\nabla}_\lambda \mathcal{T}_{\rho\beta}^\gamma \overline{\nabla}_\alpha \xi^\rho + \overline{\nabla}_\lambda \mathcal{T}_{\alpha\rho}^\gamma \overline{\nabla}_\beta \xi^\rho \end{aligned}$$

- Consider a **uniform** ξ **field**, then we obtain the **identity**:

$$\sigma^{\alpha\beta} \left(g_{\alpha\gamma} \mathcal{T}_{\rho\beta}^\gamma + g_{\gamma\beta} \mathcal{T}_{\rho\alpha}^\gamma \right) + \Sigma_\gamma^{\alpha\beta} \overline{\nabla}_\rho \mathcal{T}_{\alpha\beta}^\gamma + \Xi_\gamma^{\lambda\alpha\beta} \overline{\nabla}_\rho \overline{\nabla}_\lambda \mathcal{T}_{\alpha\beta}^\gamma = 0$$

GCM: Proof (2)

- Use of same method as previously for **non-uniform** ξ, we factorize the remaining term by $\overline{\nabla}_\beta \xi^\rho$, and obtain :

$$
\begin{aligned}
\delta_\xi \mathcal{L} &= \sigma^{\alpha\beta} \left(\overline{\nabla}_\alpha \xi_\beta + \overline{\nabla}_\beta \xi_\alpha \right) \\
&+ \Sigma_\gamma^{\alpha\beta} \left(-\mathcal{T}_{\alpha\beta}^\rho \overline{\nabla}_\rho \xi^\gamma + \mathcal{T}_{\rho\beta}^\gamma \overline{\nabla}_\alpha \xi^\rho + \mathcal{T}_{\alpha\rho}^\gamma \overline{\nabla}_\beta \xi^\rho \right) \\
&- \Xi_\rho^{\lambda\alpha\gamma} \, \overline{\nabla}_\lambda \mathcal{T}_{\alpha\gamma}^\beta \, \overline{\nabla}_\beta \xi^\rho + \Xi_\gamma^{\beta\alpha\lambda} \, \overline{\nabla}_\rho \mathcal{T}_{\alpha\lambda}^\gamma \, \overline{\nabla}_\beta \xi^\rho \\
&+ \Xi_\gamma^{\lambda\beta\alpha} \, \overline{\nabla}_\lambda \mathcal{T}_{\rho\alpha}^\gamma \, \overline{\nabla}_\beta \xi^\rho + \Xi_\gamma^{\lambda\alpha\beta} \, \overline{\nabla}_\lambda \mathcal{T}_{\alpha\rho}^\gamma \, \overline{\nabla}_\beta \xi^\rho
\end{aligned}
$$

- The proof of the theorem follows. □

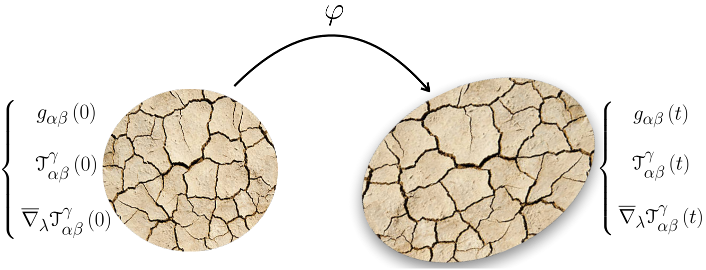
Note: Use of other vector field ξ allows us to obtain other conservation laws or identities (linear, angular momentum, ...).

GCM: Comment on Generalized Stress

Roughly speaking, the **generalized stress** includes three contributions:

1. the classical **Cauchy stress** / energy-momentum $\sigma^{\alpha\beta}$ deforms the underlying continuum shape.
2. the **micro-stress** / hypermomentum current $\Sigma_{\gamma}^{\alpha\beta}\mathcal{T}_{\rho\beta}^{\gamma}$ (and $\Sigma_{\gamma}^{\alpha\beta}\mathcal{T}_{\alpha\rho}^{\gamma}$) induces change of density of dislocations / defects at each point (discontinuity of scalar field)
3. the **polar stress** / (mass) quadrupole moment : $\Xi_{\gamma}^{\lambda\alpha\beta}\overline{\nabla}_{\lambda}\mathcal{T}_{\rho\beta}^{\gamma}$ (and $\Xi_{\gamma}^{\lambda\alpha\beta}\overline{\nabla}_{\lambda}\mathcal{T}_{\alpha\rho}^{\gamma}$) drives the relative motions of grains / continuum microcosms (discontinuity of vector field).

VI. Generalized Continuum Model and Example of Constitutive Laws



In this **particular case**, the two variables $\mathcal{T}^{\gamma}_{\alpha\beta}$, and $\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\beta}$ capturing evolutions of density of dislocations are considered as **internal variables**.

GCM: Overall Thermodynamic Process

A **Thermodynamical Process** of a particular GCM is assumed to be defined by the set (Coleman & Gurtin 1967):

1. spatial position of each point $M \in \mathscr{B}$: $\mathbf{x} = \varphi(\mathbf{X}, t)$;
2. temperature $\theta(M, t)$;
3. stress tensor $\sigma(M, t)$ (micro-stress $\Sigma(M, t)$ and polar stress $\Xi(M, t)$) and external body force $\rho\mathbf{b}(M, t)$;
4. entropy $s(M, t)$;
5. Helmholtz free energy $\phi(M, t)$;
6. heat flux $\mathbf{q}(M, t)$;
7. heat source $r(M, t)$
8. **Extended** set of **Internal Variables** : contortion $\mathcal{T}$ and its LC-covariant derivative $\overline{\nabla}\mathcal{T}$ (Ramaniraka & R 2000, R 2003).

Definition

($\mathscr{B}$-derivative) Let $\mathbf{A}$ be a tensor of type (p, q) on $\mathscr{B}$. The time derivative of $\mathbf{A}$ with respect to $\mathscr{B}$ is a tensor of the same type as $\mathbf{A}$, which satisfies for any p-uplet of vectors $(\mathbf{f}_1, ..., \mathbf{f}_p)$ and for any q-uplet of 1-forms $(\omega^1, ..., \omega^q)$, macroscopically embedded in $\mathscr{B}$,:

$$\frac{d^{\mathscr{B}}\mathbf{A}}{dt}\left(\mathbf{f}_1, ..., \mathbf{f}_p, \omega^1, ..., \omega^q\right) \equiv \frac{d}{dt}[\mathbf{A}(\mathbf{f}_1, \cdots, \mathbf{f}_p, \omega^1, \cdots, \omega^q)]$$

Generalized Continuum Model: The $\mathscr{B}$-derivatives of the primal / internal variables are given by:

$$\zeta_g := \frac{1}{2}\frac{d^{\mathscr{B}}\mathbf{g}}{dt}, \qquad \zeta_{\mathcal{T}} := \frac{d^{\mathscr{B}}\mathcal{T}}{dt}, \qquad \zeta_{\overline{\nabla}\mathcal{T}} := \frac{d^{\mathscr{B}}\overline{\nabla}\mathcal{T}}{dt}$$

Definition includes all so-called objective rate of tensor of classical continuum mechanics (and Lie derivatives on RC manifolds) (R 2022)

Theorem

A Generalized Continuum Model of the rate type $\mathscr{B}$ is defined by **constitutive tensor functions**: $\Im = \{\sigma, \Sigma, \Xi, \phi, s\}$:

$$\Im = \tilde{\Im}\left(\mathbf{g}, \mathcal{T}, \overline{\nabla}\mathcal{T}, \zeta_g, \zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}}\right)$$

Then the **free energy** ϕ *takes necessarily the form* $\phi = \tilde{\phi}\left(\mathbf{g}, \mathcal{T}, \overline{\nabla}\mathcal{T}\right)$ *and the* **entropy inequality** *as:*

$$\mathbf{J}_g : \zeta_g + \mathbf{J}_{\mathcal{T}} : \zeta_{\mathcal{T}} + \mathbf{J}_{\overline{\nabla}\mathcal{T}} : \zeta_{\overline{\nabla}\mathcal{T}} \geq 0$$

with worth dual dissipation variables $(\mathbf{J}_g, \mathbf{J}_{\mathcal{T}}, \mathbf{J}_{\overline{\nabla}\mathcal{T}})$.

Proof Extension of Coleman & Noll theorem (R 2003) $\square$. To ensure the **Entropy Inequality**, introduce a **Dissipation Potential** ψ which is positive, convex and zero when the rates are equal to zero (Moreau 1970, Germain 1973).

Evolution of defects may be derived by means of the Principle of Maximum Dissipation. (e.g. Hackl et al. 2007, 2011, 2024)

- Principle of **Maximum Dissipation** (constrained optimization):

$$\mathrm{Max}\left\{\psi\left(\zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}}\right) \quad \text{such that} \quad \psi - \left(\mathbf{J}_{\mathcal{T}} : \zeta_{\mathcal{T}} + \mathbf{J}_{\overline{\nabla}\mathcal{T}} : \zeta_{\overline{\nabla}\mathcal{T}}\right) \equiv 0\right\}$$

- Introduce a **extended dissipation function** to maximize:

$$\psi_{\text{ext}} := \psi + \lambda\left[\psi - \left(\mathbf{J}_{\mathcal{T}} : \zeta_{\mathcal{T}} + \mathbf{J}_{\overline{\nabla}\mathcal{T}} : \zeta_{\overline{\nabla}\mathcal{T}}\right)\right]$$

- Necessary stationarity conditions from **derivatives** to obtain three (in)-equations:

$$\left\{\begin{array}{rcll}
(1+\lambda)\dfrac{\partial\psi}{\partial\zeta_{\mathcal{T}}} - \lambda\,\mathbf{J}_{\mathcal{T}} & = & 0 & [\ \ni 0\ ] \\
(1+\lambda)\dfrac{\partial\psi}{\partial\zeta_{\overline{\nabla}\mathcal{T}}} - \lambda\,\mathbf{J}_{\overline{\nabla}\mathcal{T}} & = & 0 & [\ \ni 0\ ] \\
\psi - \left(\mathbf{J}_{\mathcal{T}} : \zeta_{\mathcal{T}} + \mathbf{J}_{\overline{\nabla}\mathcal{T}} : \zeta_{\overline{\nabla}\mathcal{T}}\right) & = & 0 &
\end{array}\right.$$

GCM: Non-smooth Evolution Rule

- Define a new Lagrange multiplier $\Lambda = (\lambda + 1)/\lambda$, and assume a **homogeneous function** ψ **of degree** 1 to deduce:

$$\begin{cases} \mathbf{J}_{\mathcal{T}} & = & \Lambda \dfrac{\partial \psi}{\partial \zeta_{\mathcal{T}}} \\ \mathbf{J}_{\overline{\nabla}\mathcal{T}} & = & \Lambda \dfrac{\partial \psi}{\partial \zeta_{\overline{\nabla}\mathcal{T}}} \end{cases} \quad \text{extended to} \quad \begin{cases} \mathbf{J}_{\mathcal{T}} & \in & \partial \psi_{\zeta_{\mathcal{T}}} \left(\zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}} \right) \\ \mathbf{J}_{\overline{\nabla}\mathcal{T}} & \in & \partial \psi_{\zeta_{\overline{\nabla}\mathcal{T}}} \left(\zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}} \right) \end{cases}$$

- Invert by using **Legendre-Fenchel** transform (e.g. Rockafellar, 1970):

$$\psi^* \left(\mathbf{J}_{\mathcal{T}}, \mathbf{J}_{\overline{\nabla}\mathcal{T}} \right) := \mathrm{Sup}_{\left(\zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}} \right)} \left[\mathbf{J}_{\mathcal{T}} : \zeta_{\mathcal{T}} + \mathbf{J}_{\overline{\nabla}\mathcal{T}} : \zeta_{\overline{\nabla}\mathcal{T}} - \psi \left(\zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}} \right) \right]$$

- Use conjugate function ψ^* to obtain **Defects Evolution Rule**:

$$\begin{cases} \zeta_{\mathcal{T}} & = & \dfrac{\partial \psi^*}{\partial \mathbf{J}_{\mathcal{T}}} \\ \zeta_{\overline{\nabla}\mathcal{T}} & = & \dfrac{\partial \psi^*}{\partial \mathbf{J}_{\zeta_{\overline{\nabla}\mathcal{T}}}} \end{cases} \quad \text{extended to} \quad \boxed{\begin{cases} \zeta_{\mathcal{T}} & \in & \partial \psi^*_{\mathbf{J}_{\mathcal{T}}} \left(\mathbf{J}_{\mathcal{T}}, \mathbf{J}_{\overline{\nabla}\mathcal{T}} \right) \\ \zeta_{\overline{\nabla}\mathcal{T}} & \in & \partial \psi^*_{\mathbf{J}_{\overline{\nabla}\mathcal{T}}} \left(\mathbf{J}_{\mathcal{T}}, \mathbf{J}_{\overline{\nabla}\mathcal{T}} \right) \end{cases}}$$

GCM: Normal Dissipative Material (from R 2003)

1. **Indicator function** ($\simeq$ rate-independent plasticity): A choice of convex, homogeneous of degree one **Dissipation Potential** $\psi = \hat{\psi}\left(\zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}}\right)$ that vanishes when rates are zero:

$$\begin{aligned} \hat{\psi}^*\left(\mathbf{J}_{\mathcal{T}}, \mathbf{J}_{\overline{\nabla}\mathcal{T}}\right) &:= \mathrm{Sup}_{\zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}}}\left[\mathbf{J}_{\mathcal{T}} : \zeta_{\mathcal{T}} + \mathbf{J}_{\overline{\nabla}\mathcal{T}} : \zeta_{\overline{\nabla}\mathcal{T}} - \hat{\psi}\left(\zeta_{\mathcal{T}}, \zeta_{\overline{\nabla}\mathcal{T}}\right)\right] \\ &= \begin{cases} 0 & \text{if} \quad \left(\mathbf{J}_{\mathcal{T}}, \mathbf{J}_{\overline{\nabla}\mathcal{T}}\right) \in \mathscr{C} \\ \infty & \text{if} \quad \left(\mathbf{J}_{\mathcal{T}}, \mathbf{J}_{\overline{\nabla}\mathcal{T}}\right) \notin \mathscr{C} \end{cases} \end{aligned}$$

allows us to satisfy the **entropy inequality** !

2. The **Helmholtz Free Energy** ϕ takes the form of:

$$\phi = \hat{\phi}\left(\mathbf{g}, \mathcal{T}, \overline{\nabla}\mathcal{T}\right)$$

Remarque

$\zeta_{\mathcal{T}}$ and $\zeta_{\overline{\nabla}\mathcal{T}}$ are the ***covariant rate of defects evolution*** *analogous to plastic deformation rates $\mathbf{D}^p := \dot{\mathbf{F}}^p \, \mathbf{F}^{p-1}$ and $\overline{\nabla}\mathbf{D}^p$ used in the framework of GND e.g. (Cermelli & Gurtin 2001).*

VIII. Concluding Remark

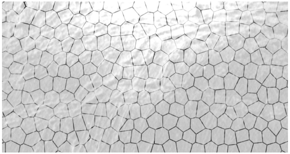
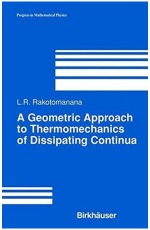
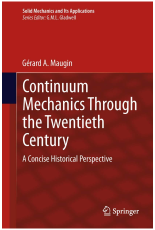
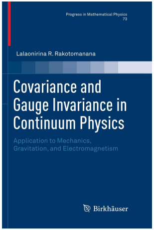



Figure: GCM of continuum dislocations and way of Life

Strain Gradient Plasticity including **density of dislocations** may be worthily modeled with **Generalized Continuum Model** (*i.e.* Riemann-Cartan manifold) with metric $g_{\alpha\beta}$ for shape change, torsion $\aleph^{\gamma}_{\alpha\beta}$ (resp. $\mathcal{T}^{\gamma}_{\alpha\beta}$) for translational dislocations, and curvature $\Re^{\gamma}_{\alpha\beta\lambda}$ (resp. $\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\beta}$) for rotational dislocations.







Lalaonirina Rakotomanana Ravelonarivo

Some Thoughts Concerning the Vacuum Spacetime and the Cosmological Constant: Gravitation and Electromagnetism

April 9, 2025

"We end our panoramic tour of **Generalized Continuum Mechanics** by mentioning an original geometric solution as presented in the book of Rakotomanana (2003), which offers a representation of a material manifold - that is **everywhere dislocated** - with the appropriate generalized **gradient operator**." (Maugin, 2013)

Merci pour votre attention !

Some papers and books

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