Mesoscopic plasticity : from the discrete to the continuum

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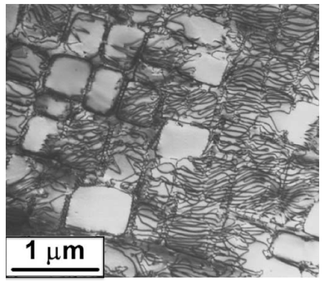
Plasticity : discrete or continuous?

Discrete dislocations :

- if distance between dislocations $\geq$ internal length scale (i.e. distance between interfaces…)

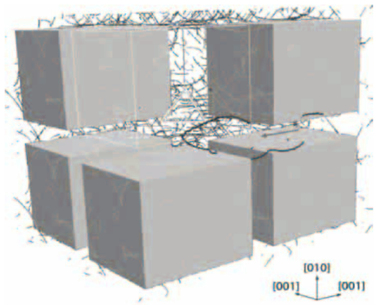
Creep in superalloys :

Exp.



Liprendy et al. (2012)

Dislocation Dynamique



A. Vattré et al. (2009)

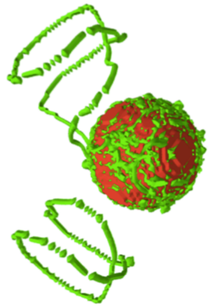
Coherency loss :

Exp.



Makenas *et al.*, (1980)

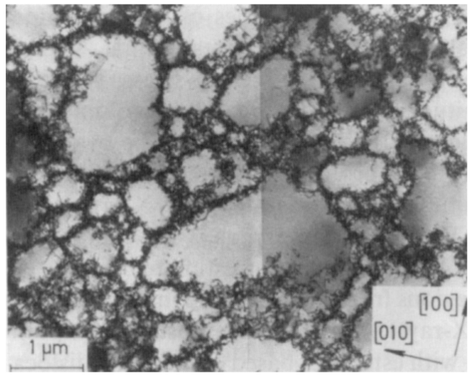
Phase Field method with dislocations



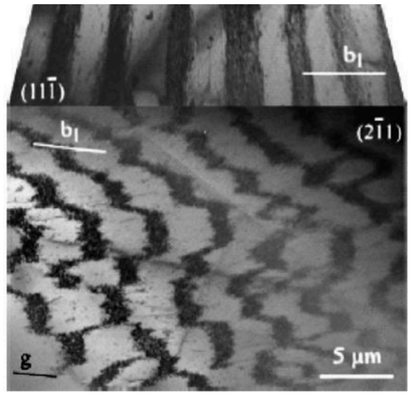
Geslin et al., (2014)

Continuous modelling :

- if distance between dislocations $\ll$ internal length scale
- huge number of dislocations $\rightarrow$ collective behaviour (flow and back stresses, patterning…)



(Mughrabi, 1986)



(Legros, 2004)

continuous modelling needed !

Preliminaries : continuum plasticity

Nye (53), Kröner (58), Kosevitch (62), Mura (63)…

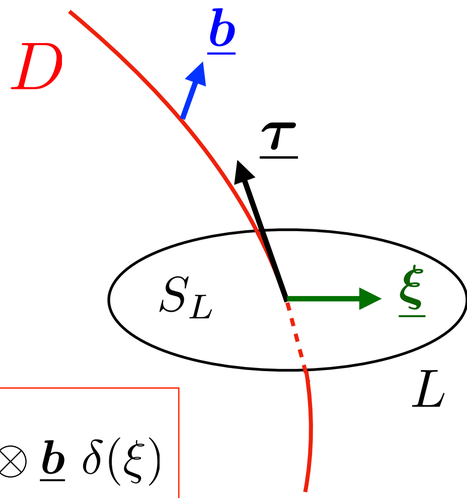
Deformation gradient :

gradient of the (total) displacement field

$$\beta_{ki} = \frac{\partial u_k}{\partial x_i} \qquad \underset{\sim}{\boldsymbol{\beta}} = \underset{\sim}{\boldsymbol{\beta}}^e + \underset{\sim}{\boldsymbol{\beta}}^p$$

$\underset{\sim}{\boldsymbol{\beta}}^e$: elastic distortion (reversible)

$\underset{\sim}{\boldsymbol{\beta}}^p$: plastic distortion (irreversible)



Dislocation and Burger's vector :

$$\oint_L \underset{\sim}{\boldsymbol{\beta}}^p . \underline{\boldsymbol{dx}} = \underline{\boldsymbol{b}}$$

Nye dislocation density tensor :

$$\underline{\boldsymbol{b}} = \oint_L \underset{\sim}{\boldsymbol{\beta}}^p . \underline{\boldsymbol{dx}} = \int_{S_L} \mathrm{curl}\, \underset{\sim}{\boldsymbol{\beta}}^p . dS = \int_{S_L} \underset{\sim}{\boldsymbol{\alpha}} . dS$$

$$\underset{\sim}{\boldsymbol{\alpha}} = \mathrm{curl}\, \underset{\sim}{\boldsymbol{\beta}}^p$$

Discrete dislocation : $\underset{\sim}{\boldsymbol{\alpha}} = \underline{\boldsymbol{\tau}} \otimes \underline{\boldsymbol{b}}\, \delta(\xi)$

Plastic distortion rate $\dot{\underset{\sim}{\boldsymbol{\beta}}}^p$:

Discrete dislocation with glide velocity $\underline{\boldsymbol{V}}$:

$$\dot{\underset{\sim}{\boldsymbol{\beta}}}^p = -(\underline{\boldsymbol{\tau}} \times \underline{\boldsymbol{V}}) \otimes \underline{\boldsymbol{b}}\, \delta(\xi) = -\underset{\sim}{\boldsymbol{\alpha}} \times \underline{\boldsymbol{V}}$$

Kinetics :

Transport equation :

$$\dot{\underset{\sim}{\boldsymbol{\alpha}}} - \mathrm{curl}\, \dot{\underset{\sim}{\boldsymbol{\beta}}}^p = 0$$

$$\dot{\underset{\sim}{\boldsymbol{\alpha}}} + \mathrm{curl}\, (\underset{\sim}{\boldsymbol{\alpha}} \times \underline{\boldsymbol{V}}) = 0$$

Dislocation velocity :

Dissipative kinetics : $\underline{\boldsymbol{V}} \sim$ resolved Peach-Koelher force :

$$\underline{\boldsymbol{V}} = M\, \underline{\boldsymbol{f}}^{PK} \qquad f^{PK} = (\underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{b}}) : \sigma$$

$\underline{\boldsymbol{n}}$: normal to glide plane

Mechanical equilibrium : displacement field $\underline{\boldsymbol{u}}$ such that :

$$\mathrm{div}\, \underset{\sim}{\boldsymbol{\sigma}} = 0 \qquad \underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{\lambda}} : (\underset{\sim}{\boldsymbol{\beta}} - \underset{\sim}{\boldsymbol{\beta}}^p)$$

at the micro-scale (i.e. dislocations are resolved) : problem is closed

From the micro- to the mesoscale : coarse-graining

- Coarse-graining :

$$< \dot{\underset{\sim}{\boldsymbol{\alpha}}} > + \mathrm{curl} \; < \underset{\sim}{\boldsymbol{\alpha}} \times \underline{\boldsymbol{V}} >= 0$$

$< ... >$: spatio-temporal average

- Correlations :

$$< \dot{\underset{\sim}{\boldsymbol{\alpha}}} > + \mathrm{curl} \left(< \underset{\sim}{\boldsymbol{\alpha}} > \times < \underline{\boldsymbol{V}} > + < \underset{\sim}{\boldsymbol{\alpha}} \times \underline{\boldsymbol{V}} >_{\mathrm{corr}} \right) = 0$$

$$< \underset{\sim}{\boldsymbol{\alpha}} \times \underline{\boldsymbol{V}} >_{\mathrm{corr}} = < \underset{\sim}{\boldsymbol{\alpha}} \times \underline{\boldsymbol{V}} > - < \underset{\sim}{\boldsymbol{\alpha}} > \times < \underline{\boldsymbol{V}} >$$

→ need to say something on the correlations to close the theory…

- Mesoscale Field Dislocation Mechanics (MFDM) : Acharya et al (2001, 2006, 2011….)

  - constitutive law :

$$L^P = < \underset{\sim}{\boldsymbol{\alpha}} \times \underline{\boldsymbol{V}} >_{\mathrm{corr}} \sim \underline{\boldsymbol{V}}^s \qquad \underline{\boldsymbol{V}}^s = V_0 \, \mathrm{sgn}(\sigma_s) \left(\frac{|\sigma_s|}{\sigma_0 + \sigma_h} \right)^n$$

- Continuum Dislocation Dynamics (CDD) : El-Azab et al (2000, 2006, 2015….)

  - Mean field theory (+ bundles) :

$$< \underset{\sim}{\boldsymbol{\alpha}} \times \underline{\boldsymbol{V}} >_{\mathrm{corr}} = 0$$

  - velocity field :

$$V = M \left\langle |\sigma_s| > -(\sigma_0 + \sigma_h) \right\rangle \mathrm{sgn}(\sigma_s)$$

Outline

A coarse-graining procedure (2D)

- clarify the coarse-graining procedure : how de we define *out-of-equilibrium* mesoscale densities ?
- what is the physical meaning and properties of the *emerging* mesoscopic driving forces ?

Valdenaire, Le Bouar, Appolaire, Finel, PRB (2016)

Coarse graining (1)

Initial model

- 2D, single glide system along X : N parallel edge dislocations with Burgers vectors +/-(b,0)
- N^+ dislocations with +b (s_i = +1), N^- dislocations with -b (s_i = -1)
- no sources, no annihilation, no cross-slip
- dissipative dynamics :

$$\frac{d\vec{r}_i}{dt} = B^{-1} s_i \left(\sum_{j \neq i}^{N} s_j \tau_{ind}(\vec{r_i} - \vec{r_j}) + \tau_{ext} \right)$$

$$\tau_{ind}(\vec{r}) = \frac{\mu b}{2\pi(1-\nu)} \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

Discrete dislocation densities

$$\rho^+_{dis}(r,t,\{r^0_k\}) = \sum_{i=1}^{N} \delta_{s_i,+1}\ \delta(r - r_i(t,\{r^0_k\}))$$

$$\rho^-_{dis}(r,t,\{r^0_k\}) = \sum_{i=1}^{N} \delta_{s_i,-1}\ \delta(r - r_i(t,\{r^0_k\}))$$

- discrete densities kinetics :

$$\dot{\underset{\sim}{\alpha}} + \mathrm{curl}\,(\underset{\sim}{\alpha} \times \underline{\boldsymbol{V}}) = 0 \quad \rightarrow$$

$$-\frac{\partial}{\partial t}\rho^+_{dis}(r) = B^{-1}\ \vec{b}\frac{\partial}{\partial \vec{r}} \{ \int_{r' \neq r} d^2r'\ \tau_{ind}(r-r')\ [\rho^+_{dis}(r')\rho^+_{dis}(r) - \rho^-_{dis}(r')\rho^+_{dis}(r)] + \tau_{ext}\ \rho^+_{dis}(r)\ \}$$

.....

- next step : averaging scheme

Coarse graining (2)

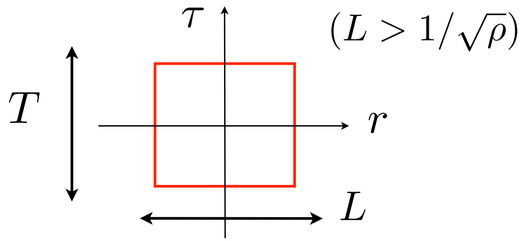
Space and time averaging by convolution :

- convolution window : $w(r,t) = w_L(r)\; w_{T(L)}(t)$
- coarse-grained densities :

one-body densities

$$\rho^+(r,t) = \iint d^2r''\, d\tau\; w(r'',\tau)\; \rho^+_{dis}(r+r'', t+\tau, \{r^0_k\})$$

$[\rho^-(r,t)...]$



two-body densities

$$\rho^{++}(r',r,t) = \iint d^2r''\, d\tau\, w(r'',\tau)\; \rho^+_{dis}(r'+r'', t+\tau)\; \rho^+_{dis}(r+r'', t+\tau, \{r^0_k\})$$

$[\rho^{+-}(r,t)...]$

Not enough :

- need to average on all the initial discrete configurations that are compatible with a given set of initial one-boby dislocation densities $\rho(r, t=0)$
- also, need two-body densities to be continuous

$\longrightarrow$ introduce a statistical ensemble of initial conditions, defined by a probability density :

$P(r^0_1, r^0_2, ..., r^0_N; s_1, s_2, ..., s_N)$ = probability density to have an initial configuration with a dislo. of sign s_1 at r^0_1.....

$\longrightarrow$ Space-time convolution + ensemble averaging on initial discrete conditions :

one-particle density :

$$\rho^+(r,t) = \langle\!\langle\, \rho^+_{dis}(r+r'', t+\tau, \{r^0_k\}, \{s_i\})\, \rangle\!\rangle_P$$

two-particle density :

$$\rho^{++}(r',r,t) = \langle\!\langle\, \rho^+_{dis}(r'+r'', t+\tau, \{r^0_k\}, \{s_i\}) \rho^+_{dis}(r+r'', t+\tau, \{r^0_k\}, \{s_i\})\, \rangle\!\rangle_P$$

with

$$\langle\!\langle\, X(r,t,\{r^0_i\},\{s_i\})\, \rangle\!\rangle_P = \prod_{k=1}^{N} \int d^2r^0_k \prod_{k=1}^{N} \sum_{s_k=\pm 1} P(\{r^0_i\},\{s_i\}) \iint d^2r" d\tau\, w(r",\tau) X(r+r", t+\tau, \{r^0_i\}, \{s_i\})$$

Coarse graining (3)

Kinetic equations on coarse-grained densities :

- kinetic equations keep the same form :

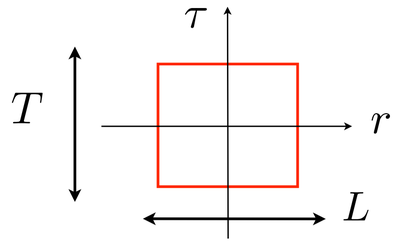
$$-\frac{\partial}{\partial t}\rho^{+}(r) = B^{-1}\,\vec{b}\frac{\partial}{\partial \vec{r}}\,\{\int_{r'\neq r} d^2r'\;\tau_{ind}(r-r')\,[\rho^{++}(r',r)-\rho^{-+}(r',r)]\;+\;\tau_{ext}\;\rho^{+}(r)\;\}$$

$$-\frac{\partial}{\partial t}\rho^{-}(r) = -B^{-1}\,\vec{b}\frac{\partial}{\partial \vec{r}}\,\{\int_{r'\neq r} d^2r'\;\tau_{ind}(r-r')\,[\rho^{+-}(r',r)-\rho^{--}(r',r)]\;+\;\tau_{ext}\;\rho^{-}(r)\;\}$$

At this stage :

- no approximations...
- one- and two-body densities : continuous
- coarse-grained densities are *scale-dependent* (w-dependent)
- problem of closure : need to say something on the 2-body densities $\rho^{++}(r',r)$...
- define correlations : $\rho^{s's}(r',r,t) = \rho^{s'}(r',t)\rho^{s}(r,t)\,(1+d^{s's}(r',r,t))$
- key point : approximate 2-body correlations in terms of 1-body densities…

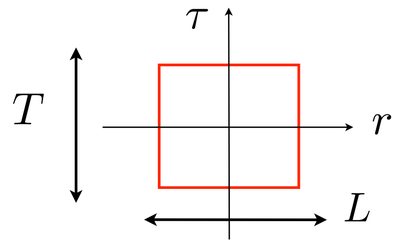
$$w(r,t) = w_L(r)\;w_{T(L)}(t)$$



Coarse graining (4)

Local approximation for the correlations : needed to close the theory !

$$w(r,t) = w_L(r)\; w_{T(L)}(t)$$



→ spatial convolution :

- correlations are short-ranged : $\text{correlation length} \sim 1/\sqrt{\rho}$
- spatial convolution window width : $L \gg 1/\sqrt{\rho}$
- then we get :

$$d^{ss'}(r,r',t) = d^{ss'}(r-r', \{\rho^s(r,t)\}, t)$$

→ time convolution :

- we want $d^{ss'}$ to depend only on densities $\{\rho^s(r,t)\}$, defined at scale L
- we must get rid of the explicit time dependance !
- $T(L)$ such that it incorporates all the time scales associated to kinetics up to scale L

  i.e. such that dislocations reach a steady state at scale L

→ *L* finite $\Longrightarrow$ simulations show that *T(L)* is finite !

ex. in 2D : around critical stress, power law relaxation $\dot{\gamma} \sim t^{-2/3}$ to a steady-state, with a cut-off time $t_c(L)$ that is system-size dependent (Ispanovity et al., 2011, 2014; Valdenaire, 2016)

$\rightarrow$ select convolution time $T(L) \sim$ relaxation time $t_c(L)$

→ local density approximation : $d^{ss'}(r,r',t) \simeq d^{ss'}(r-r', \{\rho^s(r,t)\})$

Coarse graining (5)

With local density approximation, kinetic equations become :

$$\begin{aligned} -\frac{\partial}{\partial t}\rho^{+}(r,t) &= B^{-1}\,\vec{b}\frac{\partial}{\partial \vec{r}}\left[\{\tau_{ext}+\tau_{sc}(r)-\tau_{f}^{+}(r)-\tau_{b}^{+}(r)\}\;\rho^{+}(r,t)\right] \\ -\frac{\partial}{\partial t}\rho^{-}(r,t) &= -B^{-1}\,\vec{b}\frac{\partial}{\partial \vec{r}}\left[\{\tau_{ext}+\tau_{sc}(r)-\tau_{f}^{-}(r)-\tau_{b}^{-}(r)\}\;\rho^{-}(r,t)\right] \end{aligned}$$

→ “mean-field” stress, i.e stress generated by the “one-body” densities (long-ranged dislocation interactions) :

$$\tau_{sc}(r)=\int d^{2}r'\;\tau_{ind}(r-r')\;[\;\rho^{+}(r')-\rho^{-}(r')]$$

→ depends only on the GND density $\kappa(r)=\rho^{+}(r)-\rho^{-}(r)$

→ correlation-induced local stresses :

- depend on the local dislocation densities :

$$s=\pm 1\;:\quad \tau_{f}^{s}(r)=-\sum_{s'}s'\rho^{s'}(r)\int_{r'\neq r}d^{2}r'\;\tau_{ind}(r-r')\;\;d^{ss'}(r-r',\{\rho^{s}\})$$

- depend on the gradients of the local dislocation densities :

$$s=\pm 1\;:\quad \tau_{b}^{s}(r)=-\sum_{s'}s'\frac{\partial\rho^{s'}(r)}{\partial\vec{r}}\int_{r'\neq r}d^{2}r'\;(\vec{r}\,'-\vec{r}\,)\tau_{ind}(r-r')\;\;d^{ss'}(r-r',\{\rho^{s}\})$$

Physical meaning of the correlation-induced stresses

Symmetry properties of the correlation-induced stresses :

$\longrightarrow$ def. of the “low-order” stress (i.e. local stress if we neglect dislocation correlations) : $\tau_{lo}(\vec{r}) = \tau_{ext} + \tau_{sc}(\vec{r})$

$\longrightarrow$ we can show that we have the following properties :

- *correlation-induced stresses that depend on the local dislocation densities :*

$$s = \pm 1 \ :$$

$$\tau_f^s(\vec{r}, \{\rho^s(\vec{r},t)\}, -\tau_{lo}(\vec{r},t)) = -\tau_f^s(\vec{r}, \{\rho^s(\vec{r},t)\}, \tau_{lo}(\vec{r},t)) \Longrightarrow$$

τ_f^s same sign as τ_{lo}

it always opposes τ_{lo}

$\Longrightarrow$ friction stresses

- *correlation-induced stresses that depend on the gradient of the local dislocation densities :*

$$s = \pm 1 \ :$$

$$\tau_b^s(\vec{r}, \{\frac{\partial \rho^s(\vec{r},t)}{\partial \vec{r}}\}, -\tau_{lo}(\vec{r},t)) = \tau_b^s(\vec{r}, \{\frac{\partial \rho^s(\vec{r},t)}{\partial \vec{r}}\}, \tau_{lo}(\vec{r},t)) \Longrightarrow$$

τ_b^s even in τ_{lo}

generate a translation of elastic domain (Bauschinger effect)

$\Longrightarrow$ back stresses

Next : correlation-induced stresses depend on the sign of the dislocation !

Symmetry breaking in coarse-grained transport equations

Correlation-induced stresses depend on the sign of the dislocation :

$$\tau_f^+ \neq \tau_f^- \qquad \tau_b^+ \neq \tau_b^-$$

$$\begin{aligned} \tau_f(r) &= (\tau_f^+(r) + \tau_f^-(r))/2 \\ \tau_b(r) &= (\tau_b^+(r) + \tau_b^-(r))/2 \end{aligned}$$

sign-independent components

$$\begin{aligned} \tilde{\tau}_f(r) &= (\tau_f^+(r) - \tau_f^-(r))/2 \\ \tilde{\tau}_b(r) &= (\tau_b^+(r) - \tau_b^-(r))/2 \end{aligned}$$

“symmetry-breaking” stresses

Positive and negative dislocations do not experience the same local stress :

$$\begin{aligned} -\frac{\partial \rho^+(r,t)}{\partial t} &= +Mb\frac{\partial}{\partial r}\left[\rho^+(r)\left\{\tau_{ext} + \tau_{sc}(r) - \tau_f(r) - \tau_b(r) - \tilde{\tau}_f(r) - \tilde{\tau}_b(r)\right\}\right] \\ -\frac{\partial \rho^-(r,t)}{\partial t} &= -Mb\frac{\partial}{\partial r}\left[\rho^-(r)\left\{\tau_{ext} + \tau_{sc}(r) - \tau_f(r) - \tau_b(r) + \tilde{\tau}_f(r) + \tilde{\tau}_b(r)\right\}\right] \end{aligned}$$

⟶ sign-independent components :

$$\begin{aligned} \tau_f(r) &= \frac{1}{2}\rho(r)\int \tau_{ind}(r-r')\; d^{+-}(r-r')\; d^2r' \\ \tau_b(r) &\simeq -\frac{1}{4}\frac{\partial \kappa}{\partial r}\int (r'-r)\; \tau_{ind}(r-r')\left\{d^{++}(r'-r) + d^{--}(r'-r) + d^{-+}(r'-r) + d^{+-}(r'-r)\right\} d^2r' \end{aligned}$$

$$\kappa(r) = \rho^+(r) - \rho^-(r)$$

$$\rho(r) = \rho^+(r) + \rho^-(r)$$

⟶ “symmetry-breaking” stresses :

“new” local stresses

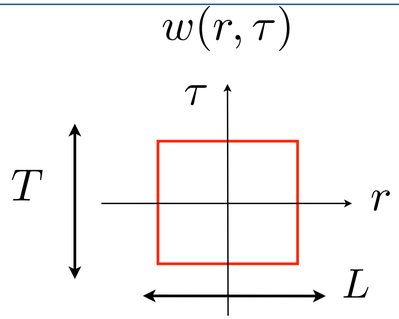
$$\begin{aligned} \tilde{\tau}_f(r) &= -\frac{1}{2}\kappa(r)\int \tau_{ind}(r-r')\; d^{+-}(r-r')d^2r' \\ \tilde{\tau}_b(r) &\simeq -\frac{1}{4}\frac{\partial \rho}{\partial r}\int (r'-r)\; \tau_{ind}(r-r')\left\{d^{++}(r'-r) + d^{--}(r'-r) - d^{+-}(r'-r) - d^{-+}(r'-r)\right\} d^2r' \end{aligned}$$

$\tilde{\tau}_b(r)$ generates lenght scale selection !!! (see below “Patterning”)

How to compute the correlation-induced stresses ?

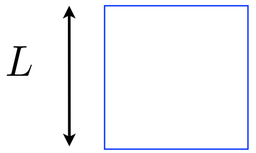
Correlations depend on the coarse-graining procedure :

$$d^{ss}(r',r,t) = \frac{\langle\langle\ \rho^{s'}_{dis}(r'+r'',t+\tau...)\,\rho^{s}_{dis}(r+r'',t+\tau...)\ \rangle\rangle_P}{\langle\langle\ \rho^{s'}_{dis}(r+r'',t+\tau...)\ \rangle\rangle_P\ \langle\langle\ \rho^{s}_{dis}(r+r'',t+\tau...)\ \rangle\rangle_P}$$



⟶ how to choose the convolution window :

- space averaging :



D : length scale we want to investigate

⟶ choose $1/\sqrt{\rho} \ll L < D$

- time averaging :

$T(L)$: characteristic time for dislocations to explore scale L

⟶ then : correlations become functions of the local one-body densities and local stress :

$$d^{s's}(r',r,t) = d^{s's}(r'-r\,;\,\kappa(r,t),\rho(r,t),\tau_{\rm local}(r,t),L) \quad \text{with} \quad \tau_{\rm local}(r) = \tau_{ext} + \tau_{sc}(r)$$

Scaling analysis of kinetic equations shows that :

$$d^{s's}(r'-r\,;\,\rho,\kappa,\tau,L) = f^{s's}((r'-r)\sqrt{\rho},\frac{\kappa}{\rho},\frac{\tau}{G\,b\sqrt{\rho}},L\sqrt{\rho})$$

How to compute the scaling functions :

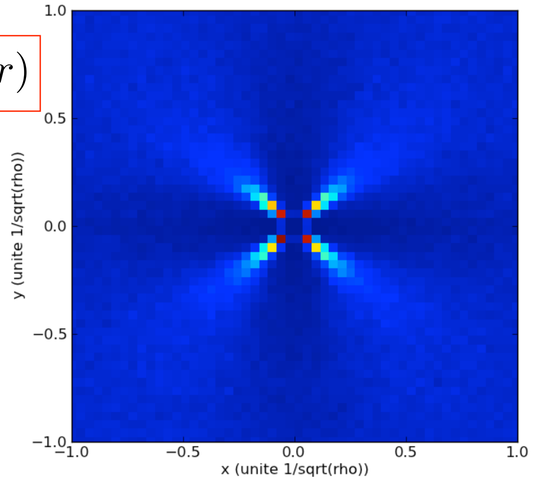
- 2D discrete dislocation dynamics in a box of size *L*

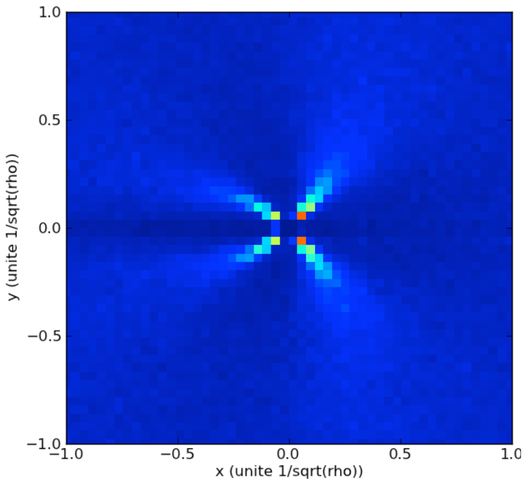
Coarse-graining with 2D dislocation dynamics (1)

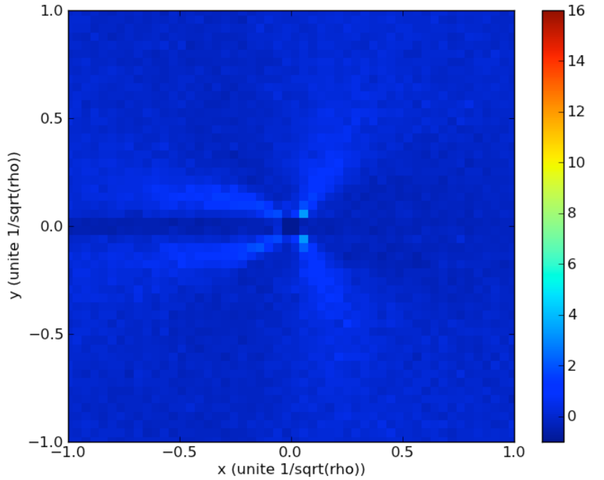
Correlation maps $d_{+-}(r)$ and $d_{++}(r)$ as a function of applied stress

$\tau_{ext}/\mu b\sqrt{\rho} = 0$ $\tau_{ext}/\mu b\sqrt{\rho} = 0.30$ $\tau_{ext}/\mu b\sqrt{\rho} = 0.45$

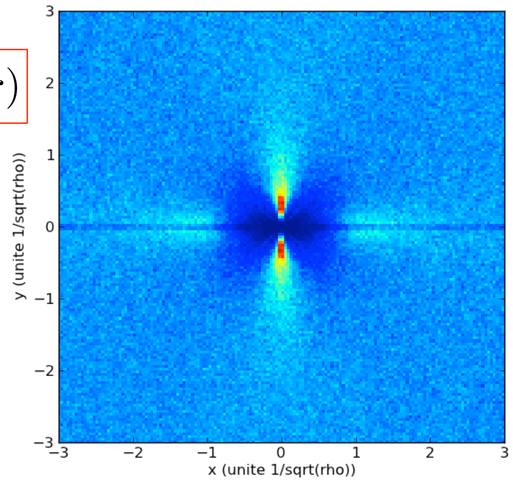
$d_{+-}(r)$

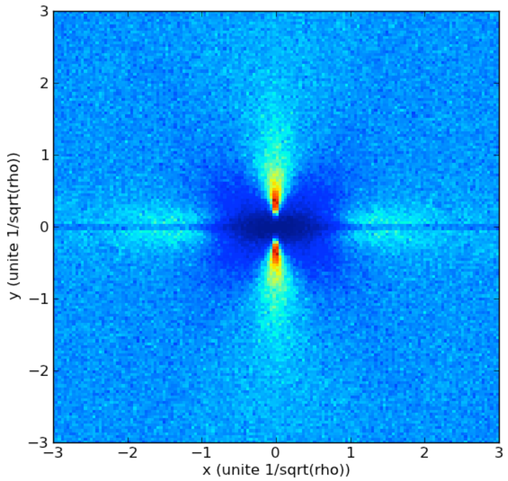


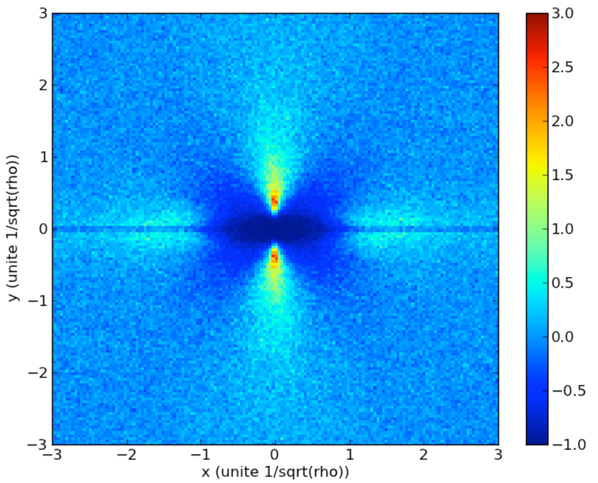




$d_{++}(r)$







Coarse-graining with 2D dislocation dynamics (2)

Investigation of the "flow stress" term $\tau_f(r)$:

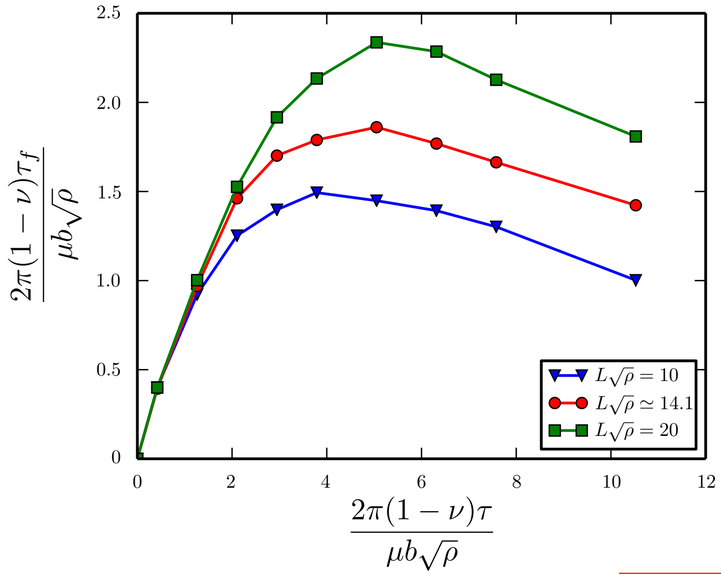
- scaling analysis leads to :

$$\tau_f = Gb\sqrt{\rho}\; f(\frac{\kappa}{\rho}, \frac{\tau_{ext}}{G\,b\sqrt{\rho}}, L\sqrt{\rho}) \quad \text{with}$$

$$f(\frac{\kappa}{\rho}, \frac{\tau_{ext}}{G\,b\sqrt{\rho}}, L\sqrt{\rho}) = \int d^2\eta \frac{\eta_x(\eta_x^2 - \eta_y^2)}{(\eta_x^2+\eta_y^2)^2}\; f_{+-}(\eta, \frac{\tau_{ext}}{G\,b\sqrt{\rho}}, \frac{\kappa}{\rho}, L\sqrt{\rho})$$

- correlations computed by 2D dislocation dynamics (convolution $\{L, T(L) \sim t_c(L)\}$ + ensemble averaging) :

$$\tau(r) = \tau_{ext} + \tau_{sc}(r)$$



$$|\tau| > \tau_c \;:\; \frac{\tau - \tau_f(\vec{r})}{G\,b\sqrt{\rho}} = \text{sign}(\tau)\; B \left(\frac{|\tau| - \tau_c}{G\,b\sqrt{\rho}}\right)^{\alpha}$$

$$|\tau| \le \tau_c \;:\; \frac{\tau - \tau_f(\vec{r})}{G\,b\sqrt{\rho}} = 0$$

$$\tau_c \simeq 0.3\, Gb\sqrt{\rho}$$

$L\sqrt{\rho}$	10	14.1	20
α	1.655	1.719	1.765
B	0.156	0.197	0.246

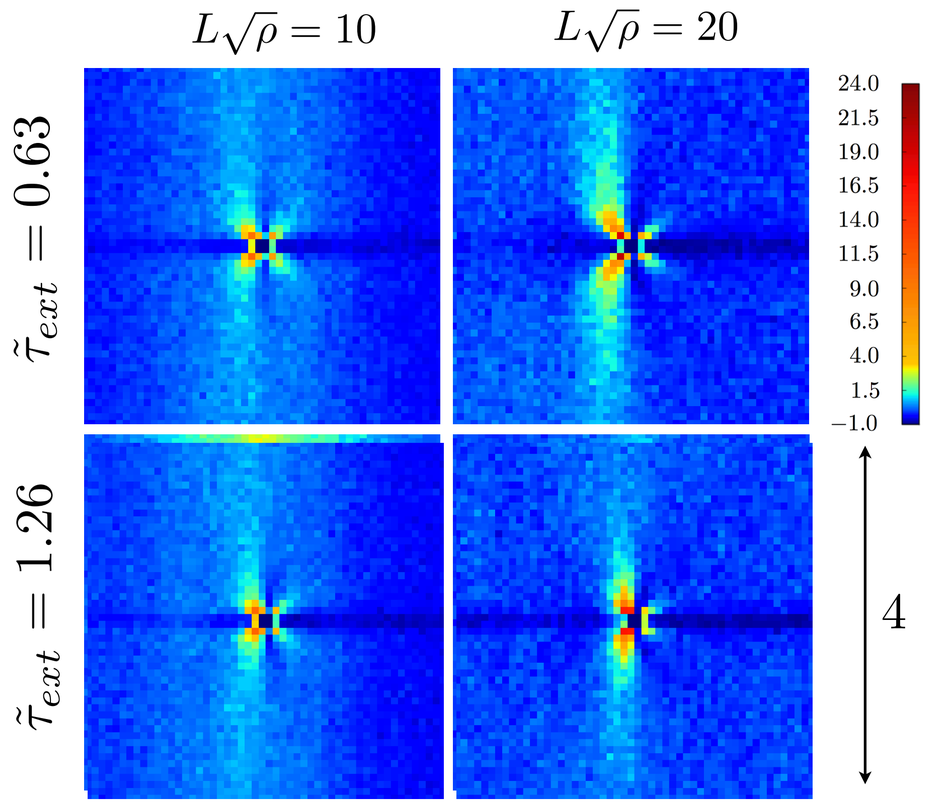
$$\alpha(L\sqrt{\rho}) \sim 1.57 + 0.01\, L\sqrt{\rho}$$

⟶ conclusion : "correlation-induced" stresses depend on the coarse-graining scale *L*

Correlations are scale-dependent

→ two different simulation boxes with the same density ρ :

d^{+-} maps :



→ dynamical formation of short-range dipoles

$$\tau_f = Gb\sqrt{\rho}\; f(\frac{\kappa}{\rho}, \frac{\tau_{ext}}{G\,b\sqrt{\rho}}, L\sqrt{\rho})$$

with $$f(\frac{\kappa}{\rho}, \frac{\tau_{ext}}{G\,b\sqrt{\rho}}, L\sqrt{\rho}) = \int d^2\eta \frac{\eta_x(\eta_x^2 - \eta_y^2)}{(\eta_x^2 + \eta_y^2)^2}\; f_{+-}(\eta, \frac{\tau_{ext}}{G\,b\sqrt{\rho}}, \frac{\kappa}{\rho}, L\sqrt{\rho})$$

$\Longrightarrow$ τ_f increases with L !

Patterning (1) ?

Kinetics :

$$
\begin{aligned}
-\frac{\partial \rho^{+}(r,t)}{\partial t} &= +Mb\frac{\partial}{\partial r}\left[\rho^{+}(r)\left\{\tau_{ext}+\tau_{sc}(r)-\tau_f(r)-\tau_b(r)-\tilde{\tau}_f(r)-\tilde{\tau}_b(r)\right\}\right] \\
-\frac{\partial \rho^{-}(r,t)}{\partial t} &= -Mb\frac{\partial}{\partial r}\left[\rho^{-}(r)\left\{\tau_{ext}+\tau_{sc}(r)-\tau_f(r)-\tau_b(r)+\tilde{\tau}_f(r)+\tilde{\tau}_b(r)\right\}\right]
\end{aligned}
$$

- friction stress $\tau_f(r)$

  function of “low order stress” $\tau(r)=\tau_{ext}+\tau_{sc}(r)$ $\longrightarrow$

$$
\begin{aligned}
&|\tau|>\tau_c \;:\; \frac{\tau-\tau_f(\vec{r})}{G\,b\sqrt{\rho}} = \mathrm{sign}(\tau)\;B\left(\frac{|\tau|-\tau_c}{G\,b\sqrt{\rho}}\right)^{\alpha} \\
&|\tau|\le\tau_c \;:\; \frac{\tau-\tau_f(\vec{r})}{G\,b\sqrt{\rho}} = 0 \\
&\tau_c \simeq 0.3\,Gb\sqrt{\rho}
\end{aligned}
$$

- back stress $\tau_b(r)$
- symmetry-breaking back stress $\tilde{\tau}_b(r)$
- symmetry-breaking friction stress $\tilde{\tau}_f(r)$

$$
\begin{aligned}
\tau_b(r) &\simeq (C_{++}-C_{+-})\;G\;b\;\frac{1}{\rho}\;\frac{\partial\kappa}{\partial x} \\
\tilde{\tau}_b(r) &\simeq (C_{++}+C_{+-})\;G\;b\;\frac{1}{\rho}\;\frac{\partial\rho}{\partial x} \\
\tilde{\tau}_f(r) &= -\frac{\kappa(r)}{\rho(r)}\;\tau_f(r)
\end{aligned}
$$

- coefficient C^{++} and C^{+-} computed by 2D DDD :

$$
\begin{aligned}
C_{++} &\simeq 1.09 \\
C_{+-} &\simeq -0.23
\end{aligned}
\qquad (L\sqrt{\rho}=20, \tau_{ext}=0)
$$

Patterning (2) ?

Linear stability analysis :

- initial state with small fluctuations :

$$\rho(r) = \rho_0 + \delta\rho(r)$$
$$\kappa(r) \neq 0$$

$$\partial\rho(r), \kappa(r) \ll \rho_0$$

- Fourier analysis :

$$\partial\rho(r) = \sum_q e^{\lambda(q)t} \exp(iqr)\, \delta\rho(q)$$
$$\kappa(r) = \sum_q e^{\lambda(q)t} \exp(iqr)\, \kappa(q)$$

Results :

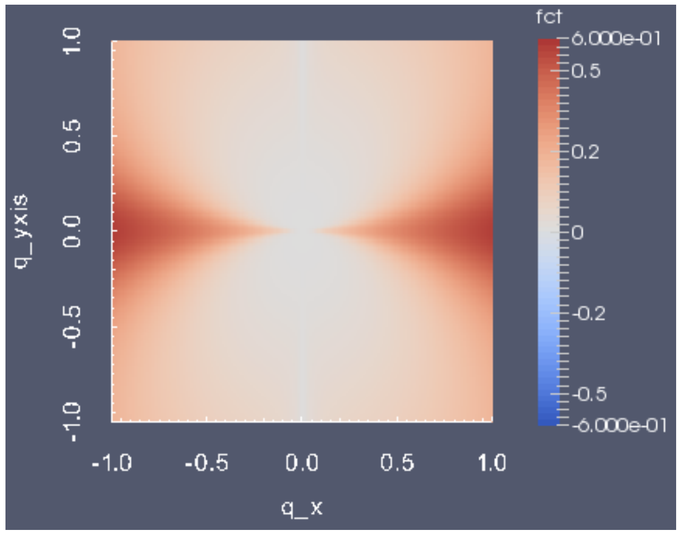
⟶ instability if applied stress such that : $\tau_c < \tau_{ext} < \dfrac{3}{3-\alpha}\tau_c$

with (see above) :

$$\alpha \simeq 1.7 \pm 0.05$$
$$\tau_c \simeq 0.3\, Gb\sqrt{\rho_0}$$

- if symmetry-breaking back-stress $\tilde{\tau}_b(r) \sim G\, b\, \dfrac{1}{\rho}\dfrac{\partial\rho}{\partial x}$ neglected :

$\lambda(q_x, q_y)$



⟶ all q-vector are instable

⟶ no length scale selection !

Patterning (2) ?

Linear stability analysis :

- initial state with small fluctuations :

$$\rho(r) = \rho_0 + \delta\rho(r)$$
$$\kappa(r) \neq 0$$

$$\partial\rho(r), \kappa(r) \ll \rho_0$$

- Fourier analysis :

$$\partial\rho(r) = \sum_q e^{\lambda(q)t} \exp(iqr)\, \delta\rho(q)$$
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Results :

⟶ instability if applied stress such that : $\tau_c < \tau_{ext} < \frac{3}{3-\alpha}\tau_c$

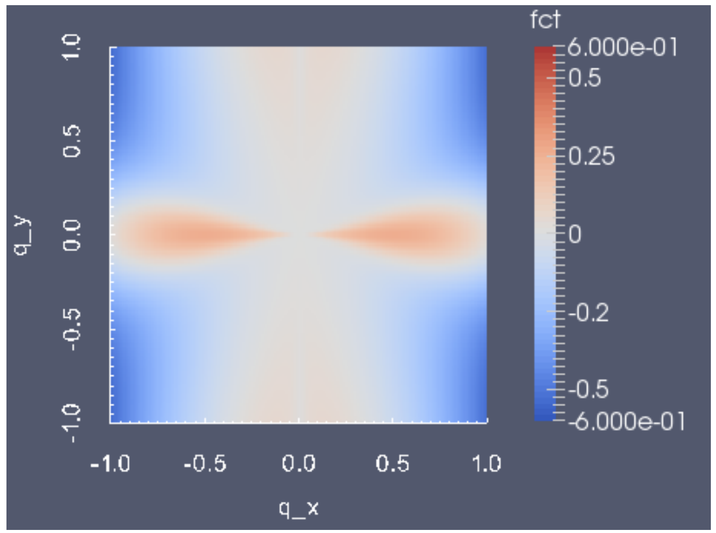
with (see above) :

$\alpha \simeq 1.7 \pm 0.05$

$\tau_c \simeq 0.3\, Gb\sqrt{\rho_0}$

- if symmetry-breaking back-stress $\tilde{\tau}_b(r) \sim G\, b\, \frac{1}{\rho}\frac{\partial\rho}{\partial x}$ is taken into account :

$\lambda(q_x, q_y)$



⟶ instability in a finite q-domain

⟶ length scale selection !

Conclusion

From the discrete to the continuum (2D) :

- critical analysis of the coarse-graining procedure
- role of the coarse-graining length
- scaling forms of the correlation-induced stress terms
- correlation-induced local stresses are functions of local stress (applied and mean-field stress) and *coarse graining length*
- new correlation-induced stress components
- Taylor-type flow stress without any junction / reaction

→ “symmetry-breaking” friction and back- stresses...

$$\tilde{\tau}_f = -Gb \frac{\kappa}{\rho} \sqrt{\rho} \; f(...)$$

$$\tilde{\tau}_b \simeq Gb \frac{1}{\rho} \frac{\partial \rho}{\partial x} \{C_{++}(...) + C_{+-}(...)\}$$

- length scale selection due to “symmetry-breaking” back-stress (even though there is only a single glide system...)

Next :

- multiple glide
- 2D gives hints to go to 3D...