Cohomology of Lie algebroids: from *Der* to non-abelian extensions

Vladimir Roubtsov

LAREMA, UMR 6093 du CNRS, Université d'Angers

Institute of Geometry and Physics, Trieste

Annual Meeting 2022

GDR "Géométrie et Mécanique"

Bordeaux, Juillet, 1, 2022

GEOMETRY OF JET SPACES AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

I. S. Krasil'shchik
V. V. Lychagin
A. M. Vinogradov

Department of Mathematics and Mechanics
Moscow State University
Moscow, USSR

Translated from the Russian
by A. B. Sosinskii

GORDON AND BREACH SCIENCE PUBLISHERS
New York London Paris Montreux Tokyo

§1. Derivations in Fibre Bundles

6.1.1 Choose a module P over a K-algebra A, where K is a commutative ring with unit. By a derivation of the module P over the derivation $X \in D(A)$ we mean a linear map $\bar{X}: P \to P$ such that

$$\bar{X}(ap) = X(a)p + a\bar{X}(p) \tag{6.1}$$

for all $a \in A$, $p \in P$.

The set of all derivations of P is a module (over A) which shall be denoted by $\mathrm{Der}(P)$. It follows from (6.1) that $\delta_a(X) \in \mathrm{End}\, P$ is a scalar: $[\delta_a(\bar{X})](p) = X(a)p$, $a \in A$, $p \in P$; thus $\mathrm{Der}\, P \subset \mathrm{Diff}_1(P, P)$.

Let us give another description of the module $\mathrm{Der}\, P$. To do this consider the homomorphism $\zeta: \mathrm{Diff}_1(P, P) \to D(\mathrm{End}\, P)$, where $\zeta(\Delta)(a) = \delta_a(\Delta)$.

In the module $\mathrm{End}\, P$, consider the submodule consisting of scalar endomorphisms; this submodule will be identified with the algebra A.

This can always be done when A is a $\mathbb{Q}$-algebra. It follows from the definition of the module $\mathrm{Diff}_1(P, P)$ that the sequence

$$0 \to \mathrm{End}\, P \to \mathrm{Diff}_1(P, P) \xrightarrow{\zeta} D(\mathrm{End}\, P) \tag{6.2}$$

is exact.

Note that the definition of the submodule $\mathrm{Der}\, P \subset \mathrm{Diff}_1(P, P)$ implies the existence of the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Diff}_1(P, P) & \xrightarrow{\zeta} & D(\mathrm{End}\, P) \\ 0 \to \mathrm{End}\, P & \nearrow & \uparrow & & \uparrow \\ & \searrow & \mathrm{Der}\, P & \xrightarrow{\zeta} & D(A \subset \mathrm{End}\, P) = D(A) \end{array} \tag{6.3}$$

Here, if we use the notation of (6.1), $\zeta(\bar{X}) = X$.

Uspekhi Mat. Nauk 35:4 (1980), 209–210 Russian Math. Surveys 35:4 (1980), 190–191

THE COHOMOLOGY OF THE Der COMPLEX

V. N. Rubtsov

The present note is devoted to the study of a certain complex that is connected with projective modules, and which we call the Der complex. Complexes of this kind were discussed in [1] and [2]. As is shown in [1], Der complexes are useful in the algebraic theory of differential equations.

The cohomology of a Der complex carries all the cohomological invariants of projective modules. In particular, the Chern and Euler classes defined in [2] and [3] lie in it. These classes are obtained by means of transgressions in spectral sequences (see §2). Using the definition of the Chern classes in [2], and also some information on the cohomology of the Der complex, we can determine the secondary characteristic classes of projective modules, which are analogues of the Chern–Simons classes.

Let A be an algebra with a unit element over a commutative ring with a unit element, and suppose that the A-module $\Lambda^1(A)$ is projective. All the modules discussed below are assumed to be finitely generated and projective, although some results remain valid under weaker assumptions. $D(A)$ denotes the A-module of derivations of A.

1. DEFINITION. Let P be an A-module. A derivation of a module $\overline{X}$ over a derivation $X \in D(A)$ is a k-linear map $\overline{X}: P \to P$ such that $\overline{X}(ap) = X(a)p + a\overline{X}(p)$ for $a \in A$ and $p \in P$. The set of all derivations P over derivations of A is equipped with an obvious A-module structure, denoted by Der P. By End P we denote the submodule of Der P consisting of the A-homomorphisms of P into itself. It is easy to see that there is a short exact sequence of A-modules:

$$0 \to \text{End } P \to \text{Der } P \xrightarrow{\pi} D(A) \to 0. \tag{1.1}$$

We also note the A-modules $D(A)$, Der P, and End P have natural structures of Lie k-algebras, that End P is an ideal in Der P, and that (1.1) is a sequence of Lie algebra homomorphisms.

THEOREM 1. *The spectral sequence constructed from this filtration converges to the cohomology* $H^*(P, Q)$, *and*: 1°. $E_0^{p,q} \cong \Lambda^p(A) \otimes_A \Omega^q(\text{End } P, Q)$. 2°. $E_1^{p,q} \cong \Lambda^p(A) \otimes_A H^q(\text{End } P, Q)$. 3°. $E_2^{p,q} \cong H^p(A, H^q)\text{End } P, Q))$, *where on the right-hand side we have de Rham cohomology of the algebra A in the system of local coefficients defined by the flat connection in the A-module* $H^q(\text{End } P, Q)$. *This connection is specified by the differential* $d_1^{0,q}$, *and* $H^q(\text{End } P, Q)$ *is the cohomology of the Lie algebra* End P; Q *is equipped with the structure of an* End P *module by means of* φ.

Lie algebroids

X: a differentiable manifold, or complex manifold, or a smooth noetherian separated scheme over an algebraically closed field $\Bbbk$ of characteristic zero.

Lie algebroid: a vector bundle/coherent sheaf $\mathscr{C}$ with a morphism of $\mathscr{O}_X$-modules $a\colon \mathscr{C} \to \Theta_X$ and a $\Bbbk$-linear Lie bracket on the sections of $\mathscr{C}$ satisfying

$$[s, ft] = f[s, t] + a(s)(f)\, t$$

for all sections s, t of $\mathscr{C}$ and f of $\mathscr{O}_X$.

- a is a morphism of sheaves of Lie $\Bbbk$-algebras
- $\ker a$ is a bundle of Lie $\mathscr{O}_X$-algebras

Examples of Lie algebroids

Example

- A sheaf of Lie algebras, with $a = 0$
- Θ_X, with $a = \mathrm{id}$
- More generally, foliations, i.e., a is injective
- Poisson structures $\Omega^1_X \xrightarrow{\pi} \Theta_X$,

  Poisson-Nijenhuis bracket

$$\{\omega, \tau\} = \mathrm{Lie}_{\pi(\omega)}\tau - \mathrm{Lie}_{\pi(\tau)}\omega - d\pi(\omega, \tau)$$

  Jacobi identity $\Leftrightarrow [\![\pi, \pi]\!] = 0$

Example

- Atiyah algebroid of a vector bundle/coherent sheaf $\mathscr{E}$

$$0 \longrightarrow End(\mathscr{E}) \longrightarrow \mathscr{D}_{\mathscr{E}} \xrightarrow{\sigma} \Theta_X \dashrightarrow 0$$

$\mathscr{D}_{\mathscr{E}}$: sheaf of 1-st order differential operators on $\mathscr{E}$ with scalar symbol. If $\mathscr{E}$ is locally free:

$$D(s)^\alpha = \sum_{i,\beta} A(z)^{\alpha i}_{\beta} \frac{\partial s^\beta}{\partial z^i} + \sum_{\beta} B(z)^{\alpha}_{\beta} s^\beta$$

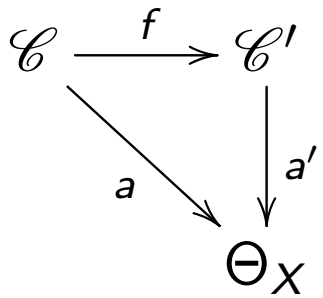
D has scalar symbol if

$$A(z)^{\alpha i}_{\beta} = \delta^{\alpha}_{\beta} v^i(z)$$

$$\sigma(D) = v \quad \text{or} \quad \sigma_\xi(D) = \xi(v)$$

Lie algebroid morphisms

$f : \mathscr{C} \to \mathscr{C}'$ a morphism of $\mathscr{O}_X$-modules & sheaves of Lie $\Bbbk$-algebras



$\Rightarrow$ ker f is a bundle of Lie algebras

Lie-Rinehart algebras

A a finitely generated commutative, associative unital algebra over a field $\Bbbk$

Lie-Rinehart algebra over $(\Bbbk, A)$: a pair (L, a) where

- L is an A-module equipped with a $\Bbbk$-linear Lie algebra bracket $\{\,,\,\}$
- $a\colon L \to \mathrm{D}_{\Bbbk}(A)$ is a representation of L in $\mathrm{D}_{\Bbbk}(A)$ (the anchor) that satisfies the Leibniz rule

$$\{s, ft\} = f\{s, t\} + a(s)(f)\, t$$

where $s, t \in L$ and $f \in A$.

Lie-Rinehart algebras and "other (???) animals"

The list of sources for these concepts is a considerable one:

- 1953-Herz –*pseudo-algèbre de Lie*
- 1955-Hochschild–*regular restricted Lie algebra extension*

- 1961-Palais –*Lie d—ring*
- 1963-Rinehart –(L, A)— *Lie algebra*
- 1964- de Barros –(L, A)—*éspace d'Elie Cartan régulier et sans courbure*
- 1967-Hermann –*Lie algebra with an associative module structure*
- 1967-Nelson –*Lie module*
- 1971-Kamber and Tondeur –*sheaf of twisted Lie algebras*
- 1972-Illusie –*algèbre de Lie sur* (A/L)

- 1985-Kastler and Stora –*Lie-Cartan pair*
- 1988-Beilinson and Schechtmann –*Atiyah algebra*
- 1989-Huebschmann–*Lie-Rinehart algebra*
- 1990-Kostant and Sternberg –$(\mathscr{A}, \mathscr{C})$—*system*

Lie-Rinehart algebras and "other (???) animals"

The list of sources for these concepts is a considerable one:

- 1953-Herz –*pseudo-algèbre de Lie*
- 1955-Hochschild–*regular restricted Lie algebra extension*

- 1961-Palais –*Lie d—ring*
- 1963-Rinehart –(L, A)— *Lie algebra*
- 1964- de Barros –(L, A)—*éspace d'Elie Cartan régulier et sans courbure*
- 1967-Hermann –*Lie algebra with an associative module structure*
- 1967-Nelson –*Lie module*
- 1971-Kamber and Tondeur –*sheaf of twisted Lie algebras*
- 1972-Illusie –*algèbre de Lie sur* (A/L)

- 1985-Kastler and Stora –*Lie-Cartan pair*
- 1988-Beilinson and Schechtmann –*Atiyah algebra*
- 1989-Huebschmann–*Lie-Rinehart algebra*
- 1990-Kostant and Sternberg –$(\mathscr{A}, \mathscr{C})$—*system*

Lie-Rinehart algebras and "other (???) animals"

The list of sources for these concepts is a considerable one:

- 1953-Herz –*pseudo-algèbre de Lie*
- 1955-Hochschild–*regular restricted Lie algebra extension*
- 1957-Atiyah-*Complex–analytic connection, principal bundle Atiyah sequence*
- 1961-Palais –*Lie d—ring*
- 1963-Rinehart –(L, A)— *Lie algebra*
- 1964- de Barros –(L, A)—*éspace d'Elie Cartan régulier et sans courbure*
- 1967-Hermann –*Lie algebra with an associative module structure*
- 1967-Nelson –*Lie module*
- 1971-Kamber and Tondeur –*sheaf of twisted Lie algebras*
- 1972-Illusie –*algèbre de Lie sur* (A/L)
- 1978-79- Krasil'shchik, Lychagin and Vinogradov - *Der-module*
- 1980 - R. - *Cohomology for Der-modules*
- 1985-Kastler and Stora –*Lie-Cartan pair*
- 1988-Beilinson and Schechtmann –*Atiyah algebra*
- 1989-Huebschmann–*Lie-Rinehart algebra*
- 1990-Kostant and Sternberg –$(\mathscr{A}, \mathscr{C})$—*system*

Der as the Atiyah-Lie-Rinehart algebra

Example

Let M be an $A-$module. Given a derivation $\bar{D} \in D_{\Bbbk}(A)$, a derivation D of M over $\bar{D}$ is a $\Bbbk-$linear morphism $D : M \to M$ such that

$$D(xm) = xD(m) + \bar{D}(x)m, \forall x \in A, m \in M.$$

The set $Der(M)$ of derivations of M over derivations of A is an $A-$module. The morphism $\sigma : Der(M) \to D_{\Bbbk}(A)$, mapping D to $\bar{D}$, is a morphism of Lie $\Bbbk-$algebras, and $(Der(M), \sigma)$ is a $(\Bbbk, A)-$Lie-Rinehart algebra. Actually σ is a symbol map, so that $Der(M)$ could be called the space of derivations of M with scalar symbol. Moreover, the sequence of $A-$modules

$$0 \longrightarrow End_A(M) \longrightarrow \mathrm{Der}(M) \xrightarrow{\sigma} D_{\Bbbk}(A)$$

is exact, and the symbol morphism σ is surjective when M is free over A. The pair $(Der(M), \sigma)$ is the Atiyah-Lie-Rinehart algebra of the module M.

Derived functors

$\mathfrak{A}$ an abelian category, $A \in \mathrm{Ob}(\mathfrak{A})$

$$\mathrm{Hom}(-, A) \colon \to \mathfrak{Ab}$$

is a (contravariant) left exact functor, i.e., if

$$0 \to B' \to B \to B'' \to 0$$

is exact, then

$$0 \to \mathrm{Hom}(B'', A) \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(B', A)$$

is exact

Definition

$I \in \mathrm{Ob}(\mathfrak{A})$ *is injective if* $\mathrm{Hom}(-, I)$ *is exact, i.e., if*

$$0 \to \mathrm{Hom}(B'', I) \to \mathrm{Hom}(B, I) \to \mathrm{Hom}(B', I) \to 0$$

is exact

Definition

The category $\mathfrak{A}$ has enough injectives if every object in $\mathfrak{A}$ has an injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \dots$$

$\mathfrak{A}$ abelian category with enough injectives

$$F : \mathfrak{A} \to \mathfrak{B} \quad \text{left exact functor}$$

Derived functors $R^i F : \mathfrak{A} \to \mathfrak{B}$

$$R^i F(A) = H^i(F(I^\bullet))$$

Example: Sheaf cohomology. X topological space, $\mathfrak{A} = \mathfrak{Sh}_X$, $\mathfrak{B} = \mathfrak{Ab}$, $F = \Gamma$ (global sections functor)

$$R^i \Gamma(\mathscr{F}) = H^i(X, \mathscr{F})$$

Hyperfunctors

$\mathfrak{A}$ category with enough injectives, $F : \mathfrak{A} \to \mathfrak{B}$ left exact functor

$\mathscr{K}^\bullet$ complex of objects in $\mathfrak{A}$, $\mathscr{I}^\bullet$ quasi-isomorphic injective complex

(i.e. there is a morphism $\mathscr{K}^\bullet \to \mathscr{I}^\bullet$ which is an isomorphism in cohomology)

$$\boxed{\mathbb{R}^i F(\mathscr{K}^\bullet) = H^i(F(\mathscr{I}^\bullet))}$$

Example (Hypercohomology): $\mathfrak{A} = \mathfrak{Sh}_X$, $\mathfrak{B} = \mathfrak{Ab}$, $F = \Gamma$ (global sections functor)

$\mathscr{K}^\bullet \in K_+(\mathfrak{Sh}_X)$

$$\boxed{\mathbb{H}^i(X, \mathscr{K}^\bullet) = H^i(\Gamma(\mathscr{I}^\bullet))}$$

(Hyper)cohomology of a Lie algebroid

$\mathscr{C}$ Lie algebroid over a scheme, $(\rho, \mathscr{M})$ a representation

$$\Omega(\mathscr{C}, \mathscr{M})^\bullet = \mathscr{M} \otimes_{\mathscr{O}_X} \Lambda^\bullet_{\mathscr{O}_X} \mathscr{C}^*, \qquad \partial_{\mathscr{C},\mathscr{M}} : \Omega(\mathscr{C}, \mathscr{M})^\bullet \to \Omega(\mathscr{C}, \mathscr{M})^{\bullet+1}$$

$$\begin{aligned}(\partial_{\mathscr{C},\mathscr{M}}\xi)(s_1, \ldots, s_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} \rho(s_i)(\xi(s_1, \ldots, \hat{s}_i, \ldots, s_{p+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \xi([s_i, s_j], \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_{p+1})\end{aligned}$$

for $s_1, \ldots, s_{p+1}$ sections of $\mathscr{C}$, and ξ a section of $\Omega^p_{\mathscr{C}}$

$\Rightarrow$ hypercohomology $\mathbb{H}^\bullet(\Omega^\bullet_{\mathscr{C}}, \partial_{\mathscr{C},\mathscr{M}}) =: \mathbb{H}^\bullet(\mathscr{C}; \mathscr{M})$

In the previous examples this reduces to

- Cartain-Eilenberg Lie algebra cohomology
- de Rham cohomology
- foliated de Rham cohomology
- Lichnerowicz-Poisson cohomology

The Lie algebroid cohomology of the Atiyah algebroid of a vector bundle was studied in Ugo Bruzzo and V. R., Cent. Eur. J. Math. **10** (2012) 1442–1454.

The category Rep($\mathscr{C}$)

From now on, X will be a scheme (with the previous hypotheses)

Given a Lie algebroid $\mathscr{C}$ there is a notion of enveloping algebra $\mathfrak{U}(\mathscr{C})$

It is a sheaf of associative $\mathscr{O}_X$-algebras with a $\Bbbk$-linear augmentation $\mathfrak{U}(\mathscr{C}) \to \mathscr{O}_X$

$$\boxed{\mathrm{Rep}(\mathscr{C}) \simeq \mathfrak{U}(\mathscr{C})\text{-}\mathbf{mod}}$$

$\Rightarrow$ Rep($\mathscr{C}$) has enough injectives

A $\Bbbk$-algebra with an algebra monomorphism $\imath\colon A \to \mathfrak{U}(L)$ and a $\Bbbk$-module morphism $\jmath\colon L \to \mathfrak{U}(L)$, such that

$$[\jmath(s), \jmath(t)] - \jmath([s,t]) = 0\,, \quad s,t \in L\,,$$
$$[\jmath(s), \imath(f)] - \imath(a(s)(f)) = 0\,, \quad s \in L, f \in A \qquad (*)$$

Construction: standard enveloping algebra $U(A \rtimes L)$ of the semi-direct product $\Bbbk$-Lie algebra $A \rtimes L$

$$\mathfrak{U}(L) = U(A \rtimes L)/V, \qquad V = \langle f(g,s) - (fg, fs)\rangle$$

- $\mathfrak{U}(L)$ is an A-module via the morphism $\imath$
- due to (*) the left and right A-module structures are different
- morphism $\varepsilon\colon \mathfrak{U}(L) \to \mathfrak{U}(L)/I = A$ (the augmentation morphism) where I is the ideal generated by $\jmath(L)$. Note that ε is a morphism of $\mathfrak{U}(L)$-modules but not of A-modules, as $\varepsilon(fs) = a(s)(f)$ when $f \in A$, $s \in L$.

Lie alg. cohomology as derived functor

Given a representation $(\rho, \mathscr{M})$ of $\mathscr{C}$ define

$$\mathscr{M}^{\mathscr{C}}(U) = \{m \in \mathscr{M}(U) \mid \rho(\mathscr{C})(m) = 0\}$$

and a left exact functor

$$\begin{array}{rccc} I^{\mathscr{C}}: & \mathrm{Rep}(\mathscr{C}) & \to & \mathbb{k}\text{-}\mathbf{mod} \\ & \mathscr{M} & \mapsto & \Gamma(X, \mathscr{M}^{\mathscr{C}}) \end{array}$$

Theorem (Ugo Bruzzo 2016[1])

If $\mathscr{C}$ is locally free

$$\mathbb{H}^{\bullet}(\mathscr{C}; \mathscr{M}) \simeq R^{\bullet} I^{\mathscr{C}}(\mathscr{M})$$

([1]) J. of Algebra **483** (2017) 245–261

Proof

A δ-functor is a collection of functors $\{S^i : \mathfrak{A} \to \mathfrak{B}\}$ such that for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{A}$ there are morphisms $\sigma^i : S^i(C) \to S^{i+1}(A)$ giving rise to a long exact sequence

$$0 \to S^0(A) \to S^0(B) \to S^0(C) \xrightarrow{\sigma^0} S^1(A)$$
$$\to S^1(B) \to S^1(C) \xrightarrow{\sigma^1} S^1(A) \to \dots$$

functorial w.r.t. morphisms of exact sequences

Theorem

If $\{S^\bullet\}$, $\{T^\bullet\}$ are δ-functors $\mathfrak{A} \to \mathfrak{B}$ such that

- $S^i(I) = T^i(I) = 0$ *for all* $i > 0$ *when* I *is an injective object*
- $S^0 \simeq T^0$

then $S^i \simeq T^i$ *for all* $i \geq 0$.

We apply this to the functors $I^{\mathscr{C}}$ and

$$\mathbb{H}^i(\mathscr{C}; -)\colon \ \mathrm{Rep}(\mathscr{C}) \to \Bbbk\text{-}\mathbf{mod}$$

When $\mathscr{C}$ is not locally free this method only provides morphisms

$$R^i I^{\mathscr{C}}(\mathscr{M}) \to H^i(\mathscr{C}; \mathscr{M})$$

Grothendieck's thm about composition of derived functors

$$\mathfrak{A} \xrightarrow{F} \mathfrak{B} \xrightarrow{G} \mathfrak{C}$$

$\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, abelian categories

$\mathfrak{A}$, $\mathfrak{B}$ with enough injectives

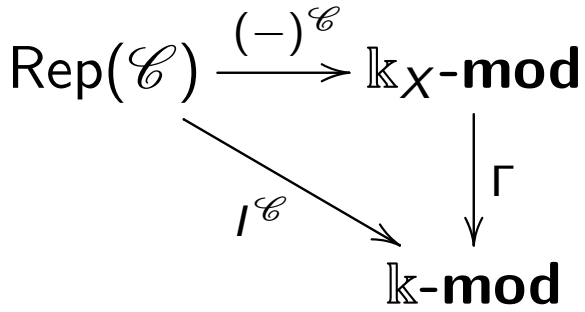
F and G left exact, F sends injectives to G-acyclics (i.e., $R^i G(F(I)) = 0$ for $i > 0$ when I is injective)

Theorem

For every object A in $\mathfrak{A}$ there is a spectral sequence abutting to $R^\bullet(G \circ F)(A)$ whose second page is

$$E_2^{pq} = R^p F(R^q G(A))$$

Local to global



Grothendieck's theorem on the derived functors of a composition of functors implies:

Theorem (Local to global spectral sequence)

There is a spectral sequence, converging to $\mathbb{H}^\bullet(\mathscr{C};\mathscr{M})$*, whose second term is*

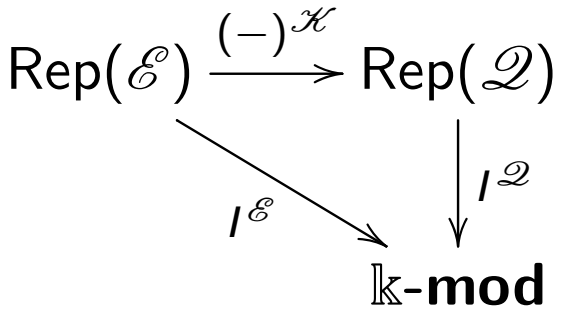
$$E_2^{pq} = H^p(X, \mathscr{H}^q(\mathscr{C};\mathscr{M}))$$

Hochschild-Serre

Extension of Lie algebroids

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$$

$\mathscr{K}$ is a sheaf of Lie $\mathscr{O}_X$-algebras



Moreover, the sheaves $\mathscr{H}^q(\mathscr{K};\mathscr{M})$ are representations of $\mathscr{Q}$

Theorem (Hochschild-Serre type spectral sequence)

For every representation $\mathscr{M}$ of $\mathscr{E}$ there is a spectral sequence E converging to $\mathbb{H}^\bullet(\mathscr{E};\mathscr{M})$, whose second page is

$$E_2^{pq} = \mathbb{H}^p(\mathscr{Q};\mathscr{H}^q(\mathscr{K};\mathscr{M})).$$

The extension problem

An extension

$$0 \to \mathscr{K} \to \mathscr{E} \xrightarrow{\pi} \mathscr{Q} \to 0 \tag{1}$$

defines a morphims

$$\begin{aligned} \alpha\colon \mathscr{Q} &\to \mathscr{O}ut(Z(\mathscr{K})) \\ \alpha(x)(y) &= \{y, x'\} \qquad \text{where} \quad \pi(x') = x \end{aligned} \tag{2}$$

The extension problem is the following:

Given a Lie algebroid $\mathscr{Q}$, a coherent sheaf of Lie $\mathscr{O}_X$-algebras $\mathscr{K}$, and a morphism α as in (2), does there exist an extension as in (1) which induces the given α?

We assume $\mathscr{Q}$ is locally free

Abelian extensions

If $\mathscr{K}$ is abelian, $(\mathscr{K}, \alpha)$ is a representation of $\mathscr{Q}$ on $\mathscr{K}$, and one can form the semidirect product

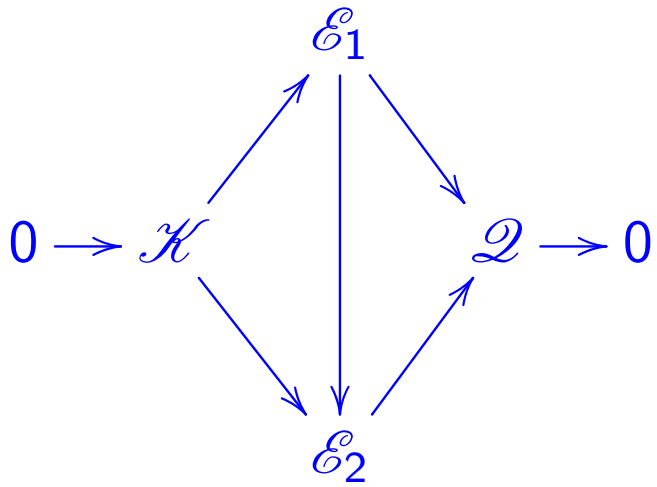
$$\mathscr{E} = \mathscr{K} \rtimes_\alpha \mathscr{Q},$$

$$\mathscr{E} = \mathscr{K} \oplus \mathscr{Q} \quad \text{as } \mathscr{O}_X\text{-modules},$$

$$\{(\ell, x), (\ell', x')\} = (\alpha(x)(\ell') - \alpha(x')(\ell), \{x, x'\})$$

Theorem ([2])

If $\mathscr{K}$ is abelian, the extension problem is unobstructed; extensions are classified up to equivalence by the hypercohomology group $\mathbb{H}^2(\mathscr{Q}; \mathscr{K})^{(1)}_\alpha$



([2]) U.Bruzzo, I. Mencattini, V. R., and P. Tortella, Nonabelian holomorphic Lie algebroid extensions, Internat. J. Math. **26** (2015) 1550040

$\mathscr{M}$ a representation of a Lie algebroid $\mathscr{C}$. Sharp truncation of the Chevalley-Eilenberg complex $\sigma^{\geq 1}\Lambda^\bullet\mathscr{C}^* \otimes \mathscr{M}$ defined by

$$0 \longrightarrow \mathscr{C}^* \otimes \mathscr{M} \longrightarrow \Lambda^2\mathscr{C}^* \otimes \mathscr{M} \longrightarrow \cdots$$

We denote $\mathbb{H}^i(\mathscr{C};\mathscr{M})^{(1)} := \mathbb{H}^i(X, \sigma^{\geq 1}\Lambda^\bullet\mathscr{C}^* \otimes \mathscr{M})$

Derivation of $\mathscr{C}$ in $\mathscr{M}$: morphism $d\colon \mathscr{C} \to \mathscr{M}$ such that

$$d(\{x,y\}) = x(d(y)) - y(d(x))$$

Proposition

The functors $\mathbb{H}^i(\mathscr{C};-)^{(1)}$ are, up to a shift, the derived functors of

$$\begin{array}{rccl} \mathrm{Der}(\mathscr{C};-)\colon & \mathrm{Rep}(\mathscr{C}) & \to & \Bbbk\textbf{-mod} \\ & \mathscr{M} & \mapsto & \mathrm{Der}(\mathscr{C},\mathscr{M}) \end{array}$$

i.e.,

$$R^i\,\mathrm{Der}(\mathscr{C};-) \simeq \mathbb{H}^{i+1}(\mathscr{C};-)^{(1)}$$

Realize the hypercohomology using Čech cochains: if $\mathscr{U}$ is an affine cover of X, and $\mathscr{F}^\bullet$ a complex of sheaves on X, then $\mathbb{H}^\bullet(X, \mathscr{F}^\bullet)$ is isom. to the cohomology of the total complex T of

$$K^{p,q} = \check{C}^p(\mathscr{U}, \mathscr{F}^q)$$

$$0 \longrightarrow \mathscr{K}_{|U_i} \longrightarrow \mathscr{E}_{|U_i} \underset{s_i}{\overset{\pi}{\rightleftarrows}} \mathscr{Q}_{|U_i} \longrightarrow 0 \qquad (3)$$

If $U_i \in \mathscr{U}$, $\mathrm{Hom}(\mathscr{Q}_{|U_i}, \mathscr{E}_{|U_i}) \to \mathrm{Hom}(\mathscr{Q}_{|U_i}, \mathscr{Q}_{|U_i})$ is surjective, so that one has splittings s_i

$$\{\phi_{ij} = s_i - s_j\} \in \check{C}^1(\mathscr{U}, \mathscr{K} \otimes \mathscr{Q}^*)$$

This is a 1-cocycle, which describes the extension as an extension of $\mathscr{O}_X$-modules

$$0 \to \mathscr{K}(U_i) \to \mathscr{E}(U_i) \to \mathscr{Q}(U_i) \to 0$$

is an exact sequence of Lie-Rinehart algebras (over $(\mathbb{k}, \mathscr{O}_X(U_i))$)

which is described by a 2-cocycle ψ_i in the Chevalley-Eilenberg (-Rinehart) cohomology of $\mathscr{Q}(U_i)$ with coefficients in $\mathscr{K}(U_i)$

$$(\phi, \psi) \in \check{C}^1(\mathscr{U}, \mathscr{K} \otimes \mathscr{Q}^*) \oplus \check{C}^0(\mathscr{U}, \mathscr{K} \otimes \Lambda^2 \mathscr{Q}^*) = T^2$$

$$\delta\phi = 0, \qquad d\phi + \delta\psi = 0, \qquad d\psi = 0$$

$\Rightarrow$ cohomology class in $\mathbb{H}^2(\mathscr{Q}; \mathscr{K})^{(1)}_\alpha$

The nonabelian case

Theorem ([2,3])

If $\mathscr{K}$ is nonabelian, the extension problem is obstructed by a class $\mathbf{ob}(\alpha)$ *in* $\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{K}))^{(1)}_{\alpha}$.

If $\mathbf{ob}(\alpha) = 0$, *the space of equivalence classes of extensions is a torsor on* $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{K}))^{(1)}_{\alpha}$.

Proof

$\mathscr{Q}$ can be written as a quotient of a free Lie algebroid $\mathscr{F}$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathscr{J} & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \mathscr{N} & \to & \mathfrak{U}(\mathscr{F}) & \to & \mathfrak{U}(\mathscr{Q}) & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathscr{O}_X & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

([3]) E. Aldrovandi, U.Bruzzo, V. R. , Lie algebroid cohomology and Lie algebroid extensions, J. of Algebra 2018

$$\widetilde{\mathscr{N}}^i = \mathscr{N}^i/\mathscr{N}^{i+1}, \qquad \widetilde{\mathscr{J}}^i = \mathscr{N}^i\mathscr{J}/\mathscr{N}^{i+1}\mathscr{J}, \quad \text{for } i = 0, \dots$$

Locally free resolution

$$\dots \to \widetilde{\mathscr{N}}^2 \to \widetilde{\mathscr{J}}^1 \to \widetilde{\mathscr{N}}^1 \to \widetilde{\mathscr{J}}^0 \to \mathscr{J} \to 0$$

As $\mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\mathscr{J}, Z(\mathscr{K})) \simeq \mathrm{Der}(\mathscr{Q}, Z(\mathscr{K}))$, applying the functor $\mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(-, Z(\mathscr{K}))$ we obtain

$$0 \to \mathrm{Der}(\mathscr{Q}, Z(\mathscr{K})) \to \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^0, Z(\mathscr{K})) \xrightarrow{d_1}$$
$$\mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{K}}^1, Z(\mathscr{K})) \xrightarrow{d_2} \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^1, Z(\mathscr{K})) \xrightarrow{d_3}$$
$$\mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{K}}^2, Z(\mathscr{K})) \to \dots$$

The cohomology of this complex is isomorphic to $\mathbb{H}^{\bullet+1}(\mathscr{Q}; Z(\mathscr{K}))$.

Pick a lift $\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}er(\mathscr{K})$ of α and get commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & Z(\mathscr{K}) & \longrightarrow & \mathscr{K} & \xrightarrow{\mathsf{ad}} & \mathscr{D}er(\mathscr{K}) & \longrightarrow & \mathscr{O}ut(\mathscr{K}) & \longrightarrow & 0 \end{array}$$

where β is the induced morphism.

Define a morphism

$$o\colon \widetilde{\mathscr{J}^1} \to Z(\mathscr{K}) \tag{4}$$

It is enough to define o on an element of the type yx, where x is a generator of $\mathscr{F}$, and y is a generator of $\mathscr{T}$

$$o(yx) = \beta(\{x, y\}) - \tilde{\alpha}(x)(\beta(y)).$$

Note that $o \in \mathrm{Hom}_{\mathscr{U}(\mathscr{Q})}(\widetilde{\mathscr{J}^1}, Z(\mathscr{K}))$.

Lemma

$d_3(o) = 0$. Moreover, the cohomology class of $[o] \in \mathbb{H}^3(\mathscr{Q}; Z(\mathscr{K}))^{(1)}$ only depends on α.

Part I of the proof: if an extension exists consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \| & & \\ 0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \end{array}$$

Define

$$\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}er(\mathscr{K}, \mathscr{K}), \qquad \tilde{\alpha} = -\operatorname{ad} \circ \gamma$$

Then $\tilde{\alpha}$ is a lift of α, and for all sections t of $\mathscr{T}$ and x of $\mathscr{F}$

$$\beta(\{x, t\}) - \tilde{\alpha}(x)(\beta(t)) = 0 \tag{5}$$

so that the obstruction class $\mathbf{ob}(\alpha)$ vanishes.

Conversely, assume that $\mathbf{ob}(\alpha) = 0$, and take a lift $\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}\mathsf{er}(\mathscr{K}, \mathscr{K})$. The corresponding cocycle lies in the image of the morphism d_2, so it defines a morphism $\beta\colon \mathscr{T} \to \mathscr{K}$, which satisfies the equation (5). Again, we consider the extension

$$0 \to \mathscr{T} \to \mathscr{F} \to \mathscr{Q} \to 0.$$

Note that $\mathscr{K}$ is an $\mathscr{F}$-module via $\mathscr{F} \to \mathscr{Q}$. The semidirect product $\mathscr{K} \rtimes \mathscr{F}$ contains the sheaf of Lie algebras

$$\mathscr{H} = \{(\ell, x) \,|\, x \in \mathscr{T},\ \ell = \beta(x)\}.$$

The quotient $\mathscr{E} = \mathscr{K} \rtimes \mathscr{F}/\mathscr{H}$ provides the desired extension

Part II of the proof: reduction to the abelian case

Proposition

Once a reference extension $\mathscr{E}_0$ has been fixed, the equivalence classes of extensions of $\mathscr{Q}$ by $\mathscr{K}$ inducing α are in a one-to-one correspondence with equivalence classes of extensions of $\mathscr{Q}$ by $Z(\mathscr{K})$ inducing α, and are therefore in a one-to-one correspondence with the elements of the group $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{K}))^{(1)}$

$\mathscr{C}_1$, $\mathscr{C}_2$ Lie algebroids with surjective morphisms $f_i : \mathscr{C}_i \to \mathscr{Q}$.
Assuming $Z(\ker f_1) \simeq Z(\ker f_2) = \mathscr{Z}$ define

$$\mathscr{C}_1 \star \mathscr{C}_2 = \mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2 / \mathscr{Z},$$

where $\mathscr{Z} \to \mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2$ by $z \mapsto (z, -z)$

Fix a reference extension $\mathscr{E}_0$ of $\mathscr{Q}$ by $\mathscr{K}$

Lemma

(1) Any extension $\mathscr{E}$ of $\mathscr{Q}$ by $\mathscr{K}$ is equivalent to a product $\mathscr{E}_0 \star \mathscr{D}$ where $\mathscr{D}$ is an extension of $\mathscr{Q}$ by $Z(\mathscr{K})$

(2) Given two extensions $\mathscr{D}_1$, $\mathscr{D}_2$ of $\mathscr{Q}$ by $Z(\mathscr{K})$, the extensions $\mathscr{E}_1 = \mathscr{E}_0 \star \mathscr{D}_1$ and $\mathscr{E}_2 = \mathscr{E}_0 \star \mathscr{D}_2$ are equivalent if and only if $\mathscr{D}_1$ and $\mathscr{D}_2$ are equivalent

Thank you!!