Algorithms for Fundamental Invariants and Equivariants of a finite group

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Joint work with **Erick Rodriguez Bazan**
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Algorithms for Fundamental Invariants and Equivariants

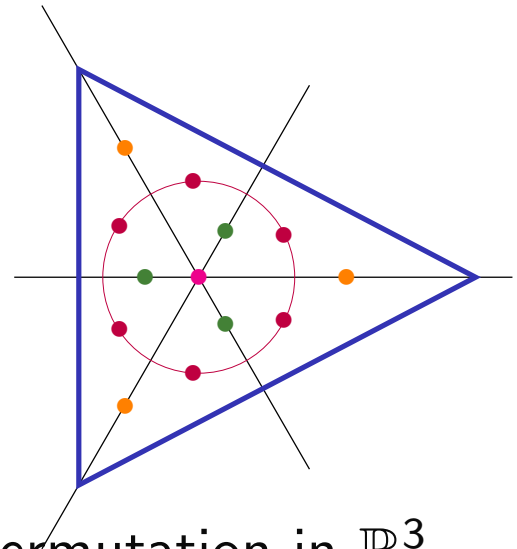
1. Invariants and equivariants
2. Symmetry adapted basis
3. Simultaneous computation of invariants and equivariants

The group of order 6 with generators s_1 and s_2 and relationships

$$s_1^2 = s_2^2 = (s_1 s_2)^3 = 1.$$

$\mathfrak{D}_3$: the group of symmetry of the triangle

$$\tau(s_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tau(s_2) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$



Also represented as the subgroup of O_3 of coordinate permutation in $\mathbb{R}^3$

$$\rho(s_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\mathfrak{S}_3$: the symmetric group of all permutations on 3 elements.

$$\sigma(s_1) = (12), \quad \sigma(s_2) = (23).$$

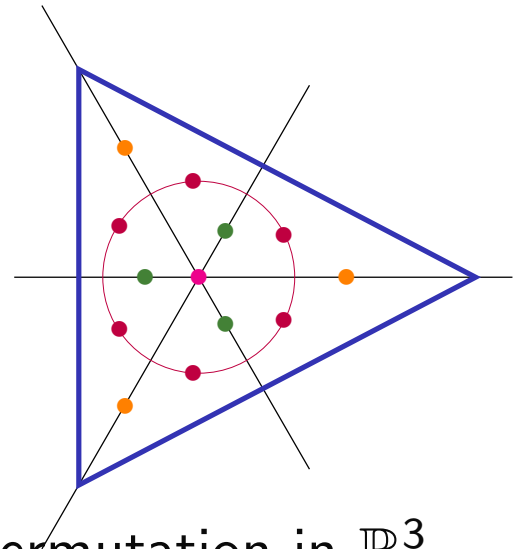
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irreducible



Also represented as the subgroup of O_3 of coordinate permutation in $\mathbb{R}^3$

$$\rho(s_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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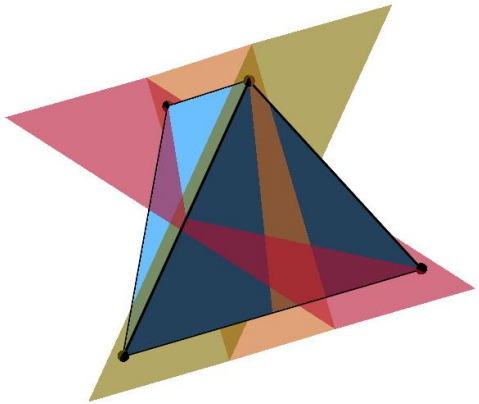
$$\sigma(s_1) = (12), \quad \sigma(s_2) = (23).$$

The group of order 24 with generators s_1, s_2 and s_3 and relationships

$$s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = 1.$$

$\mathfrak{T}_h$: the group of symmetry of the tetrahedron

$$\tau(s_1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \tau(s_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tau(s_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



Also represented as the subgroup of O_4 of coordinate permutation in $\mathbb{R}^4$

$$\rho(s_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\mathfrak{S}_4$: the symmetric group of all permutations on 4 elements.

$$\sigma(s_1) = (12), \quad \sigma(s_2) = (23), \quad \sigma(s_3) = (34).$$

- The ρ-**invariants** of the representation of $\mathfrak{S}_3$ by permutations of coordinates are the polynomials $h \in \mathbb{R}[x, y, z]$ s.t. $h \circ \rho(s) = h$. They are *generated* by

$$h_1 = x + y + z, \quad h_2 = yz + zx + xy, \quad h_3 = xyz$$

to form the **ring** $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3} = \mathbb{R}[h_1, h_2, h_3]$

- The $(\rho : \tau)$-**equivariants** are the row vectors of polynomials

$$\mathrm{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \quad \text{s.t.} \quad \mathrm{q} \circ \rho(s) = \mathrm{q}\,\tau(s)$$

They are *generated* by

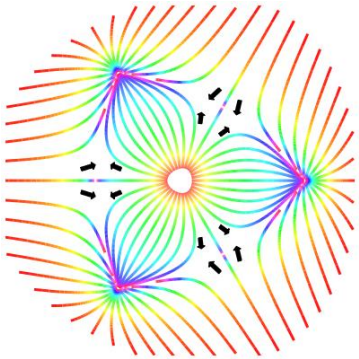
$$\mathrm{q}_1 = \begin{bmatrix} \sqrt{3}(y - z) & 2x - y - z \end{bmatrix}, \quad \mathrm{q}_2 = \begin{bmatrix} \sqrt{3}(y^2 - z^2) & 2x^2 - y^2 - z^2 \end{bmatrix}$$

to form the $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}$-**module** $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}_{\tau}$

Equivariance expresses symmetry

Dynamical systems with symmetry [Chossat Lauterbach 00, Gatermann 01]

$$\dot{\mathrm{x}} = \mathrm{p}(\mathrm{x})$$



$$\mathrm{p}\,(\,\tau(g)\,\mathrm{x}\,) = \tau(g)\,\mathrm{p}(\,\mathrm{x}\,)$$

Sum of squares: [Gatermann & Parillo 06]

$$x^2 + y^2 + z^2 - (yz + zx + xy) = \tfrac{3}{4}\,(y - z)^2 + \tfrac{1}{4}\,(2x - y - z)^2$$

Symmetry adapted bases of the polynomial ring : main motivation.

a.k.a *integrity bases*

Global optimization [Gatermann Parillo], [Riener *et al.*] Approximation theory [Rodriguez & H.],[Collowald & H.] Cryptography, combinatorics, and other areas of mathematics. . . Physics, chemistry [Fässler Stiefels], [Muggli], [Cassam Chennai *et al.*], . . .

Irreducible representations and fundamental equivariants

Consider a finite group $\mathfrak{G}$ and a representation $\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{R})$

$$\text{i.e.,} \quad \varrho(\mathrm{id}_{\mathfrak{G}}) = \mathrm{I}_n \quad \text{and} \quad \varrho(g_1 g_2) = \varrho(g_1)\varrho(g_2) \text{ for any } g_1, g_2 \in \mathfrak{G}$$

- $\mathfrak{r}^{(1)}, \ldots, \mathfrak{r}^{(t)}$ the irreducible representations of $\mathfrak{G}$:

$$\mathfrak{r}^{(1)}(g) = [1] \qquad \mathfrak{r}^{(\ell)} : \mathfrak{G} \to \mathrm{GL}_{n_\ell}(\mathbb{C})$$

- A $\mathfrak{r}^{(\ell)}$-equivariant is a row vector $\mathrm{q} = \begin{bmatrix} q_1 & \ldots & q_{n_\ell} \end{bmatrix} \in \mathbb{R}[\mathrm{x}]^{n_\ell}$ s.t.

$$\mathrm{q}\left(\rho(g)x\right) = \mathrm{q}(x)\,\mathfrak{r}^{(\ell)}(g) \quad \text{for all } g \in \mathfrak{G}$$

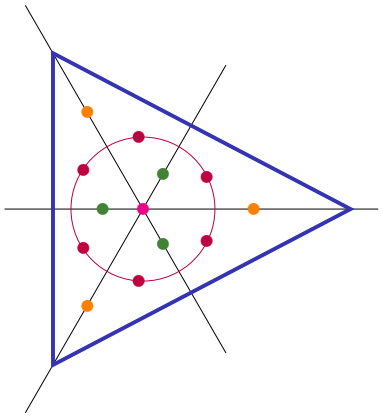
$\mathfrak{r}^{(\ell)}$-equivariants form the module $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$

The fundamental invariants and equivariants consist of

- the generators of the ring $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}$
- the generators of the $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}$-modules $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$, $2 \leq \ell \leq \mathrm{t}$

Fundamental invariants and equivariants of $\mathfrak{S}_3$

$$\varrho : \mathfrak{S}_3 \to \mathrm{GL}_3(\mathbb{R}) \qquad \text{s.t.} \qquad \rho(s_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



	s_1	s_2
$\mathfrak{r}^{(1)}$	$[1]$	$[1]$
$\mathfrak{r}^{(2)}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$
$\mathfrak{r}^{(3)}$	$[-1]$	$[-1]$

Fundamental invariants and equivariants:

- $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3} = \mathbb{R}[x_1 + x_2 + x_3,\ x_2x_3 + x_3x_1 + x_1x_2,\ x_1x_2x_3]$
- $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}_{\mathfrak{r}^{(2)}} = \mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}\, \mathrm{q}_1 \oplus \mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}\, \mathrm{q}_2$ where

$$\mathrm{q}_1 = \begin{bmatrix} \sqrt{3}(y-z) & 2x-y-z \end{bmatrix}, \quad \mathrm{q}_2 = \begin{bmatrix} \sqrt{3}(y^2-z^2) & 2x^2-y^2-z^2 \end{bmatrix}$$

- $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}_{\mathfrak{r}^{(3)}} = \mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}\, (y-z)(z-x)(x-y)$

Previously, for finite, or compact, groups

Computing invariants: for a given degree can be done thanks to the Reynolds operator: $\pi^{(1)} : \mathbb{R}[\mathrm{x}] \to \mathbb{R}[\mathrm{x}]^{\mathfrak{G}}$

$$\pi^{(1)}(f) = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} f \circ \rho(g)$$

- Using the Hilbert series of the invariant ring, known from Molien's formula, for termination [Gatermann 00]
- Computing a homogeneous system of parameters and then using the Hilbert series of the invariant ring [Derksen & Kemper 15]
- Using the Hilbert ideal. [Derksen 99], [King 13]

Computing equivariants: the module of $(\rho : \tau)$-equivariants can be identified to a submodule of the ring of $(\rho \oplus \tau)$-invariants. [Gatermann 00].

Algorithms for fundamental invariants and equivariants

finite groups

- Reflection groups : ideal interpolation along an orbit

  The invariants are read on a H-basis of the ideal J of a generic orbit.
  The equivariants on the s.a.b. of the least interpolation space,
  which is also the orthogonal complement of J^0. [Rodriguez Bazan, H. 23]

- Free module generators over primary invariants

  from the s.a.b. of an invariant complement of the ideal generated by the primary invariants [Rodriguez Bazan, H. 21]

- Minimal set of generating invariants and equivariants

  Computing invariants and equivariants degree by degree.
  Constructing the H-basis of the Hilbert ideal N and a s.a.b. of its orthogonal complement.

Algorithms for Fundamental Invariants and Equivariants

1. Invariants and equivariants
2. Symmetry adapted basis
3. Simultaneous computation of invariants and equivariants

Symmetry adapted bases

$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{R})$ $\quad$ $\mathfrak{r}^{(1)}, \ldots, \mathfrak{r}^{(t)}$ the irreducible representations of $\mathfrak{G}$

$$\varrho(g) = Q \begin{bmatrix} \mathrm{I}_{m_1} \otimes \mathfrak{r}^{(1)}(g) & & \\ & \ddots & \\ & & \mathrm{I}_{m_t} \otimes \mathfrak{r}^{(t)}(g) \end{bmatrix} Q^{-1} = P \begin{bmatrix} \mathfrak{r}^{(1)}(g) \otimes \mathrm{I}_{m_1} & & \\ & \ddots & \\ & & \mathfrak{r}^{(t)}(g) \otimes \mathrm{I}_{m_t} \end{bmatrix} P^{-1}$$

P provides a symmetry adapted basis. It can be computed thanks to the linear maps

$$\pi_{ij}^{(\ell)} = \sum_{g \in \mathfrak{G}} \mathfrak{r}_{ij}^{(\ell)}(g^{-1})\, \varrho(g)$$

$$\mathcal{Q} = \bigcup_{\ell=1}^{t} \mathcal{Q}^{(\ell)} \text{ with } \mathcal{Q}^{(\ell)} = \left\{ \left[q_{i1}^{(\ell)}, \ldots, q_{im_\ell}^{(\ell)} \right] \mid 1 \leq i \leq m_\ell \right\}$$

- $q_{11}^{(\ell)}, \ldots, q_{m_\ell 1}^{(\ell)}$ a basis of $\pi_{11}^{(\ell)}(\mathbb{R}^n)$
- $q_{ij}^{(\ell)} = \pi_{j1}^{(\ell)}\left(q_{i1}^{(\ell)}\right)$

Block diagonalisation of equivariant linear maps

Equivariant linear map

$$\Phi : U \to V \qquad \begin{array}{rcl} \mu : \mathfrak{G} & \to & \mathrm{GL}(U) \\ \nu : \mathfrak{G} & \to & \mathrm{GL}(V) \end{array} \qquad \Phi(\mu(g)\, u) = \nu(g)\, \Phi(u)$$

In s.a.b. of U and V the matrix of Φ is

$$\mathrm{diag}\left(\mathrm{I}_{\mathfrak{n}_\ell} \otimes \Phi^{(\ell)} \,|\, 1 \leq \ell \leq \mathfrak{t}\right) = \begin{bmatrix} \Phi^{(1)} & & & & & & \\ & \ddots & & & & & \\ & & \Phi^{(1)} & & & & \\ & & & \ddots & & & \\ & & & & \Phi^{(\mathfrak{t})} & & \\ & & & & & \ddots & \\ & & & & & & \Phi^{(\mathfrak{t})} \end{bmatrix}$$

Symmetry adapted bases of $\mathbb{R}[x]$ and basic equivariants

$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{R})$ $\qquad$ $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$

$$\rho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{R}[x]_d) \qquad \rho(g)(f) = f \circ \varrho(g^{-1})$$

$$\mathbb{R}[x]_d = \mathbb{R}[x]_d^{(1)} \oplus \ldots \oplus \mathbb{R}[x]_d^{(t)}$$

$\mathbb{R}[x]_d^{(\ell)}$ spanned by $q_1^{(\ell)}, \ldots, q_{m_\ell}^{(\ell)}$ where $q_k^{(\ell)} = \begin{bmatrix} q_1 & \ldots & q_{n_\ell} \end{bmatrix} \in \mathbb{R}[x]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$

i.e., $q_k^{(\ell)} \circ \varrho(g) = q_k^{(\ell)}\, \mathfrak{r}^{(\ell)}(g)$ (matrix multiplication)

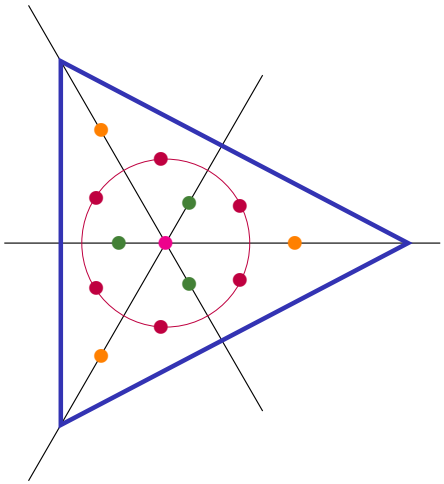
The $\mathbb{R}[x]^{\mathfrak{G}}$-modules $\mathbb{R}[x]^{\mathfrak{G}}_{\mathfrak{r}^{(1)}}, \ldots, \mathbb{R}[x]^{\mathfrak{G}}_{\mathfrak{r}^{(t)}}$ provide **s.a.b.** for $\mathbb{R}[x]$

Our contributions: Fundamental invariants and equivariants

From a s.a.b of $\mathbb{R}[x]_{\leq d}$ compute minimal generators for $\mathbb{R}[x]^{\mathfrak{G}}$ and $\mathbb{R}[x]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$

and these provide generators of $\mathbb{R}[x]^{\mathfrak{G}}_{\tau}$ for any representation τ.

$\mathbb{R}[x]$ as a $\mathbb{R}[x]^{\mathfrak{S}_3}$-module



	s_1	s_2
$\mathfrak{r}^{(1)}$	$[1]$	$[1]$
$\mathfrak{r}^{(2)}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$
$\mathfrak{r}^{(3)}$	$[-1]$	$[-1]$

$$\varrho : \mathfrak{S}_3 \to \mathrm{GL}_3(\mathbb{R}) \qquad \text{s.t.} \qquad \rho(s_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbb{R}[x] = \underbrace{\mathbb{R}[x]^{\mathfrak{G}}}_{\mathbb{R}[x]^{(1)}} \oplus \underbrace{\mathbb{R}[x]^{\mathfrak{G}} q_{11}^{(2)} \oplus \mathbb{R}[x]^{\mathfrak{G}} q_{12}^{(2)} \oplus \mathbb{R}[x]^{\mathfrak{G}} q_{21}^{(2)} \oplus \mathbb{R}[x]^{\mathfrak{G}} q_{22}^{(2)}}_{\mathbb{R}[x]^{(2)}} \oplus \underbrace{\mathbb{R}[x]^{\mathfrak{G}} q^{(3)}}_{\mathbb{R}[x]^{(3)}}$$

$$q_1^{(2)} = [\sqrt{3}(y-z) \quad 2x-y-z], \quad q_2^{(2)} = [\sqrt{3}(y^2-z^2) \quad 2x^2-y^2-z^2] \qquad q^{(3)} = (y-z)(z-x)(x-y)$$

Algorithms for Fundamental Invariants and Equivariants

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Hilbert ideal and covariant algebra : the key concepts

$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{R})$ $\qquad$ $\rho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{R})$

Hilbert ideal: $N = \langle h \mid h \in \mathbb{R}[\mathrm{x}]^{\mathfrak{G}} \setminus \mathbb{R} \rangle$

[Hilbert 1990] $h_1, \ldots, h_k$ generate $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}$ as a ring iff $H = \langle h_1, \ldots, h_k \rangle$

Covariant algebra: $\mathbb{R}[\mathrm{x}]/N$ a $\mathbb{R}$-vector space

$\mathfrak{G}$ finite $\Rightarrow$ $\mathbb{R}[\mathrm{x}]/H$ finite dimensional

If $\mathbb{R}[\mathrm{x}] = N \oplus Q$ then $Q \cong \mathbb{R}[\mathrm{x}]/N$. We can choose Q invariant.

$$\mathcal{Q} = \bigcup_{\ell=1}^{\mathrm{t}} \mathcal{Q}^{(\ell)} \text{ with } \mathcal{Q}^{(\ell)} = \left\{ \left[q_{i1}^{(\ell)}, \ldots, q_{i\mathrm{n}_\ell}^{(\ell)} \right] \mid 1 \leq i \leq m_\ell \right\} \text{ s.a.b. of } Q$$

$$\text{[Nakayama] } \mathbb{R}[\mathrm{x}] = \mathbb{R}[\mathrm{x}]^{\mathfrak{G}} \oplus \bigoplus_{\ell=2}^{\mathrm{t}} \bigoplus_{j=1}^{\mathrm{n}_\ell} \sum_{i=1}^{m_\ell} \mathbb{R}[\mathrm{x}]^{\mathfrak{G}} \, q_{ij}^{(\ell)}$$

$\Rightarrow$ $\mathcal{Q}^{(\ell)}$ is a basis of $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}(\ell)}$ as a $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}$-module.

Basic ideas of the algorithm

Compute degree by degree

- an orthogonal H-basis H of the Hilbert ideal $N = \langle\, h \,|\, h \in \mathbb{R}[\mathrm{x}]^{\mathfrak{G}} \setminus \mathbb{R} \rangle$
- a s.a.b. $Q = \bigcup_\ell Q^{(\ell)}$ of the orthogonal complement of N in $\mathbb{R}[x]$

Then

- $H = \{h_1, \ldots, h_k\}$ is a minimal generating set of invariants
- $Q^{(\ell)}$ is a basis of $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{t}(\ell)}$ as a $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}$-module .

Basically

- $\mathbb{R}[\mathrm{x}]_d = \Psi_d(H_{d-1}) \overset{\perp}{\oplus} \langle K_d \rangle_{\mathbb{R}} \overset{\perp}{\oplus} \langle R_d \rangle_{\mathbb{R}}$
  $\Psi_d(H) = \sum_{h \in H} \langle\, p\, h \mid \deg(p) + \deg(h) = d \,\rangle$
- $H_d \leftarrow H_{d-1} \cup K_d$; $\quad Q_d \leftarrow Q_{d-1} \cup R_d$.

taking into account the $\rho - \tau_d$ equivariance of Ψ

Fundamental invariants and equivariants

Algorithm [H. & Rodriguez Bazan]

$d := 0$; $R_0^{(1)} = \{1\}$;

do $d \leftarrow d + 1$

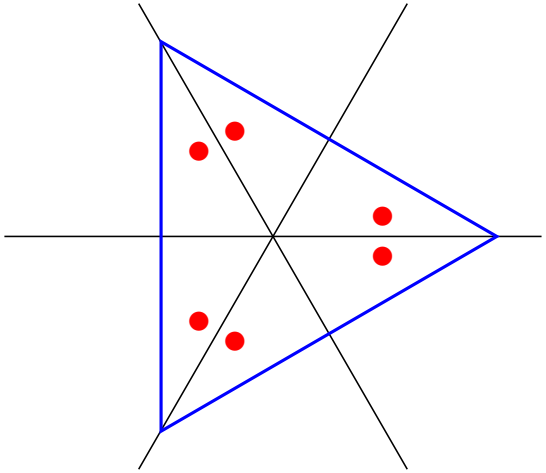
- $\mathbb{R}[\mathrm{x}]_d^{(1)} = \psi_d^{(1)}(H_{d-1}) \overset{\perp}{\oplus} \langle K_d \rangle_{\mathbb{R}}$
- $\mathbb{R}[\mathrm{x}]_d^{(\ell,1)} = \psi_d^{(\ell,1)}(H_{d-1}) \overset{\perp}{\oplus} \langle R_d^{(\ell,1)} \rangle_{\mathbb{R}} \overset{\text{lemma}}{=} \psi_d^{(1)}\left(Q_{d-1}^{(\ell,1)}\right) \overset{\perp}{\oplus} \langle R_d^{(\ell,1)} \rangle_{\mathbb{R}}$
- $H_d \leftarrow H_{d-1} \cup K_d, \quad Q_d^{(\ell)} \leftarrow Q_{d-1}^{(\ell)} \cup R_d^{(\ell)}$

until $R_d^{(\ell)} = \emptyset$

Output:

- $H = \{h_1, \ldots, h_k\}$ is a minimal generating set of invariants
- $Q^{(\ell)}$ is a minimal basis of $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}(\ell)}$ as a $\mathbb{R}[\mathrm{x}]^{\mathfrak{G}}$-module

$\mathfrak{D}_3$



	s_1	s_2
$\mathfrak{r}^{(1)}$	$[1]$	$[1]$
$\mathfrak{r}^{(2)}$	$[-1]$	$[-1]$
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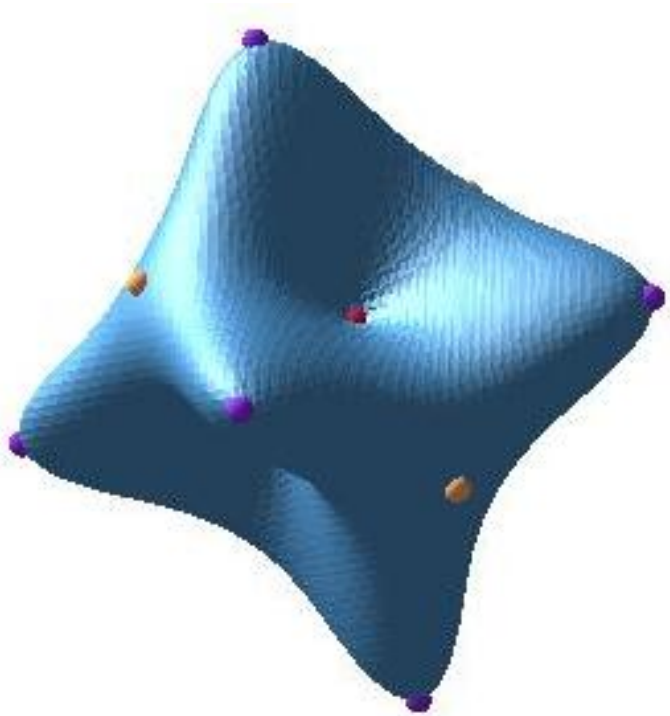
$$H = \{x^2+y^2,\ x(x^2-3y^2)\}$$

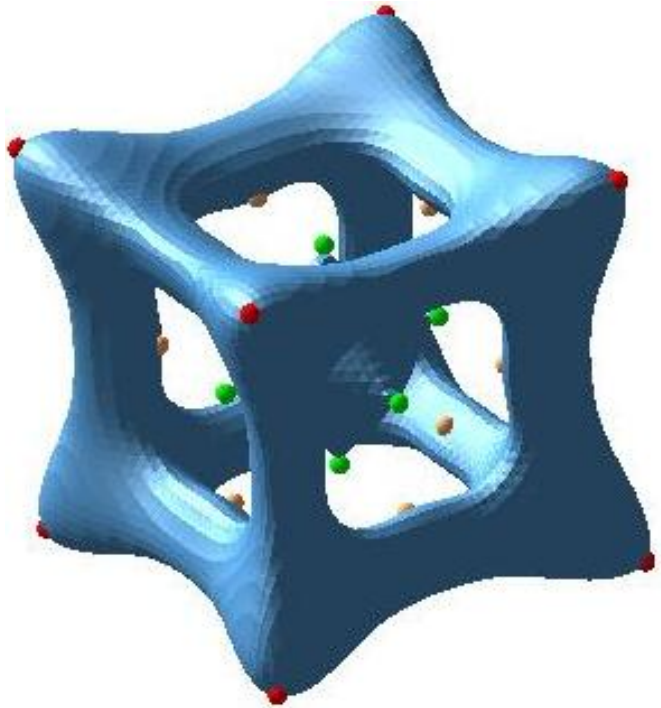
$$Q^{(1)} = \{1\},\ Q^{(2)} = \{[x,y], [y^2-x^2, 2xy]\},\ Q^{(3)} = \{y(y^2-3x^2)\}$$

$$\mathbb{R}[x,y]^{\mathfrak{D}_3} = \mathbb{R}[x^2+y^2, x(x^2-3y^2)], \qquad \mathbb{R}[x,y]^{\mathfrak{D}_3}_{\mathfrak{r}^{(3)}} = y(y^2-3x^2)\mathbb{R}[x,y]^{\mathfrak{D}_3}$$

$$\mathbb{R}[x,y]^{\mathfrak{D}_3}_{\mathfrak{r}^{(2)}} = [x,y]\mathbb{R}[x,y]^{\mathfrak{D}_3} \oplus [y^2-x^2, 2xy]\mathbb{R}[x,y]^{\mathfrak{D}_3}$$

Thanks





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https://hal.inria.fr/hal-03209117