Applications moment, connexions du groupe affine et factorisation de la forme symplectique

Géry de Saxcé

LaMcube UMR CNRS 9013 Université Lille

GDR-GDM 2022 Bordeaux

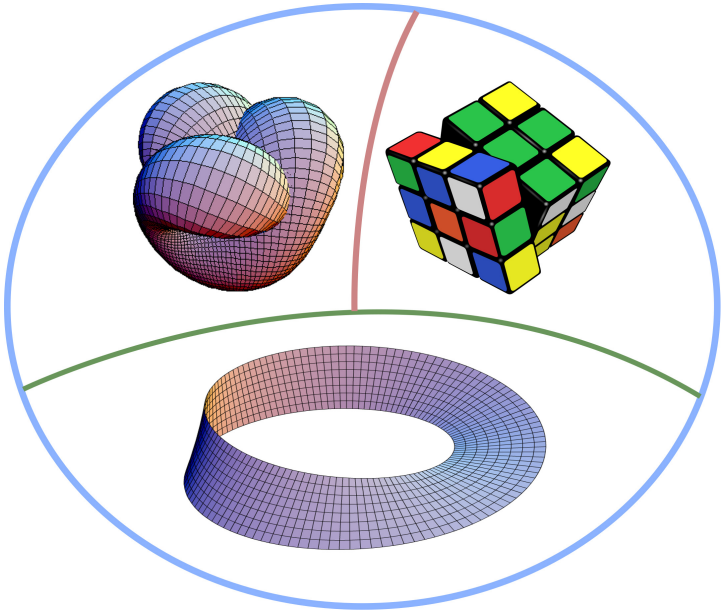




Prologue



Prologue



State of the Art

Closer to the present work :

- **Mechanical connection**
  - Smale 1970 Topology and mechanics
  - Abraham & Marsden 1978 Foundation of Mechanics
  - Kummer 1981, Guichardet 1984, Shapere & Wilczeck 1989
  - Simo, Lewis & Marsden 1991, Montgomery 1990
  - Marsden 1992 Lectures on Mechanics
- **Papers in relation with the theory of Yang-Mills fields :**
  - Sternberg & Ungar 1978
  - Duval & Horváthy 1982
  - Guillemin & Sternberg 1984 Symplectic technics in Physics
  - Curtis & Miller 1985

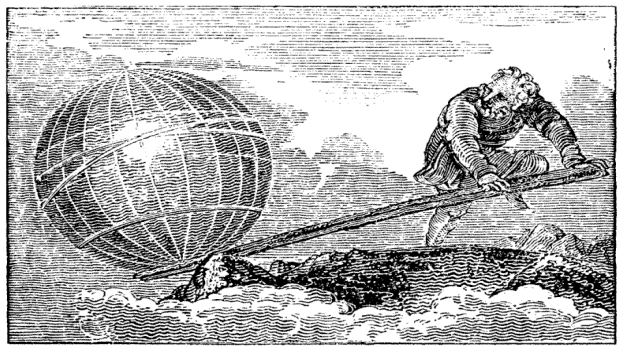
The present work :

Algorithm of putting into duality the moments and the connections

Deducing the equations of motion in a covariant form

The momentum as affine tensor





The affine tensors

- $\mathcal{M}$ is a differential manifold
- $AT_{\boldsymbol{X}}\mathcal{M}$ is the affine space associated to the tangent vector space $T_{\boldsymbol{X}}\mathcal{M}$
- $A^*T_{\boldsymbol{X}}\mathcal{M}$ is the vector space of the affine functions $\boldsymbol{\Psi}$ on $AT_{\boldsymbol{X}}\mathcal{M}$ (denominated affine forms)
- We call **momentum** a bilinear form $\boldsymbol{\mu}$

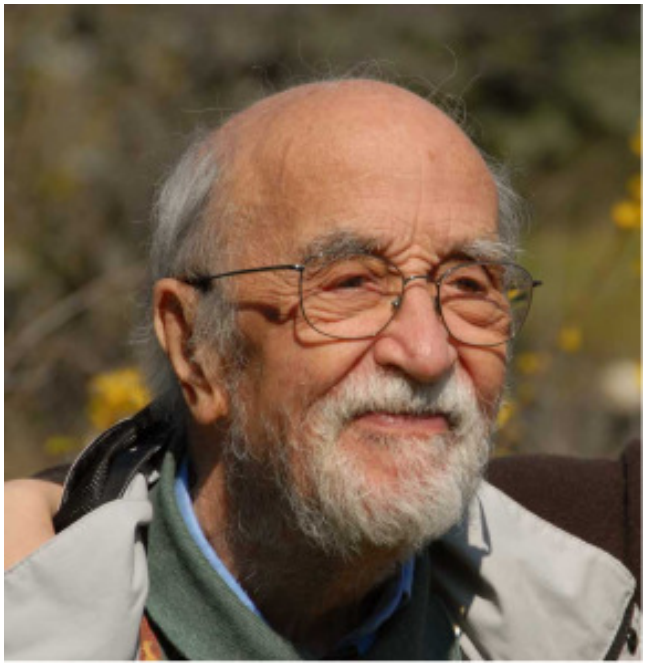
$$\boldsymbol{\mu} : T_{\boldsymbol{X}}\mathcal{M} \times A^*T_{\boldsymbol{X}}\mathcal{M} \to \mathbb{R} : (\overrightarrow{\boldsymbol{V}}, \boldsymbol{\Psi}) \mapsto \boldsymbol{\mu}(\overrightarrow{\boldsymbol{V}}, \boldsymbol{\Psi})$$

  It is a 1-covariant and 1-contravariant mixed affine tensor.
- It is represented in an affine frame f compound of an origin and a basis by

$$\boldsymbol{\mu}(\overrightarrow{\boldsymbol{V}}, \boldsymbol{\Psi}) = (\chi\, K_\beta + \Phi_\alpha L^\alpha_\beta)\, V^\beta$$

  where K_β and L^α_β are the components of $\boldsymbol{\mu}$ in f

A fragance of symplectic mechanics



A fragance of symplectic mechanics

- The **configuration space** $\mathcal{N}$ is equipped with a field of skew-symmetric 2-covariant tensors $\omega(\delta\eta, d\eta)$, the **presymplectic form** (or Lagrange's brackets)

- It allows to recover the **canonical equations** as

$$d\eta \in Ker(\omega) \qquad \Leftrightarrow \qquad \forall \delta\eta, \quad \omega(\delta\eta, d\eta) = 0$$

A fragance of symplectic mechanics

- In Mechanics, a powerful tool is the one of **symmetry group**, a Lie group G acting on $\mathcal{N}$ by $a \mapsto a \cdot \eta$

  Let $\mathfrak{g}^*$ be the dual of its Lie algebra $\mathfrak{g}$

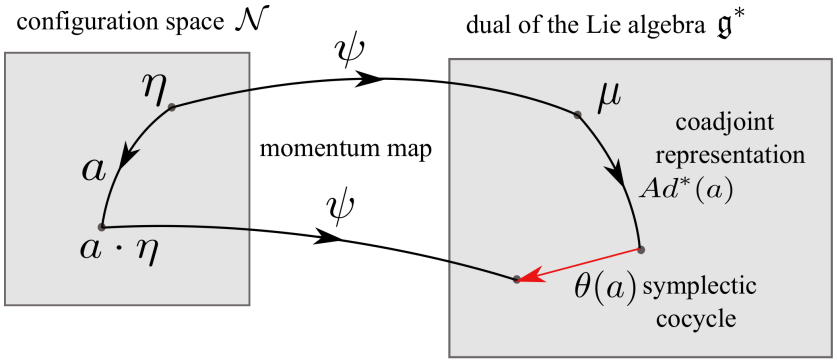
- $\eta \mapsto \mu = \psi(\eta) \in \mathfrak{g}^*$ is a **momentum map** (Souriau) if

$$\omega(Z \cdot \eta, d\eta) = -d(\psi(\eta)\, Z)$$

  that allows to recover the **integrals of motion**
  (modern form of Noether's theorem)

- G naturally acts on $\mathfrak{g}$ by the adjoint representation $Ad(a)\, Z = a\, Z\, a^{-1}$ and on $\mathfrak{g}^*$ by the induced action Ad^* (**co-adjoint representation**)

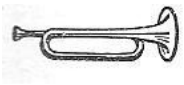
A fragance of symplectic mechanics



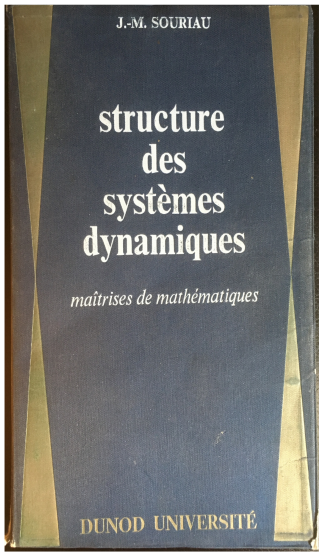
- **Théorème** (Souriau)

  There exists a map $\theta : G \mapsto \mathfrak{g}^*$ called **symplectic cocycle** such that $\psi(a \cdot \eta) - Ad^*(a)\,\psi(\eta) = \theta(a)$

- modulo a coboundary, it defines a **class of symplectic cohomology** $[\theta] \in H^1(G; \mathfrak{g}^*)$, generally null

-  A noticeable exception is **Galileo's group**, the symmetry group of the classical mechanics

Momentum tensor and momentum map





Momentum tensor and momentum map

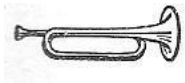
- The transformation law of $\boldsymbol{\mu}$
  is given by the induced action of the affine group $\mathbb{A}ff(n)$

$$K' = K\,P^{-1} \qquad L' = (P\,L + C\,K)\,P^{-1}$$

- Let $a = (C, P) \in \mathbb{A}ff(n)$ be the transformation $dX \mapsto C + P\,dX$
  and $Z \in \mathfrak{g}$ be the infinitesimal generator $da = (dC, dP)$

- If the space of μ is identified to $\mathfrak{g}^*$ thanks to the dual pairing

$$\mu\,Z = \mu\,da = (K, L)\,(dC, dP) = K\,dC + Tr(L\,dP)$$



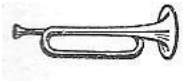
The transformation law of μ is nothing else the coadjoint representation !

Momentum tensor and momentum map

- This mathematical construction is not relevant for classical mechanics and we extend it by considering a generalized transformation law

$$\mu = a \cdot \mu' = Ad^*(a)\,\mu' + \theta(a)$$

which is an affine representation of G in $\mathfrak{g}^*$
(because we wish the momentum to be an affine tensor)

-  Comparing to the Formula of the Theorem on symplectic cocycles, restated as

$$\psi(\eta) = Ad^*(a)\,\psi'(\eta) + \theta(a)$$

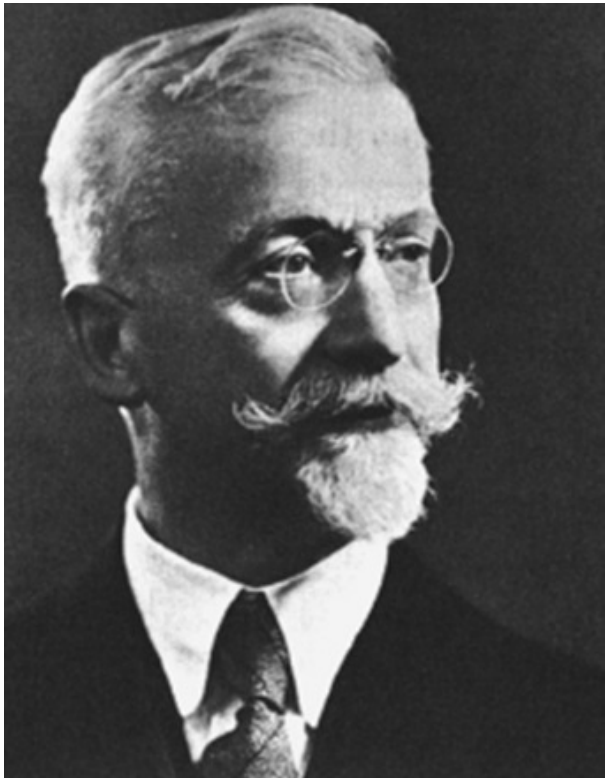
where $\psi \mapsto \psi' = a \cdot \psi$ is the induced action,
it turns out that **the values of the momentum map are just the component systems $\mu = (K, L)$ of the momentum tensors !**

Kirillov-Kostant-Souriau theorem (KKS)

Theorem

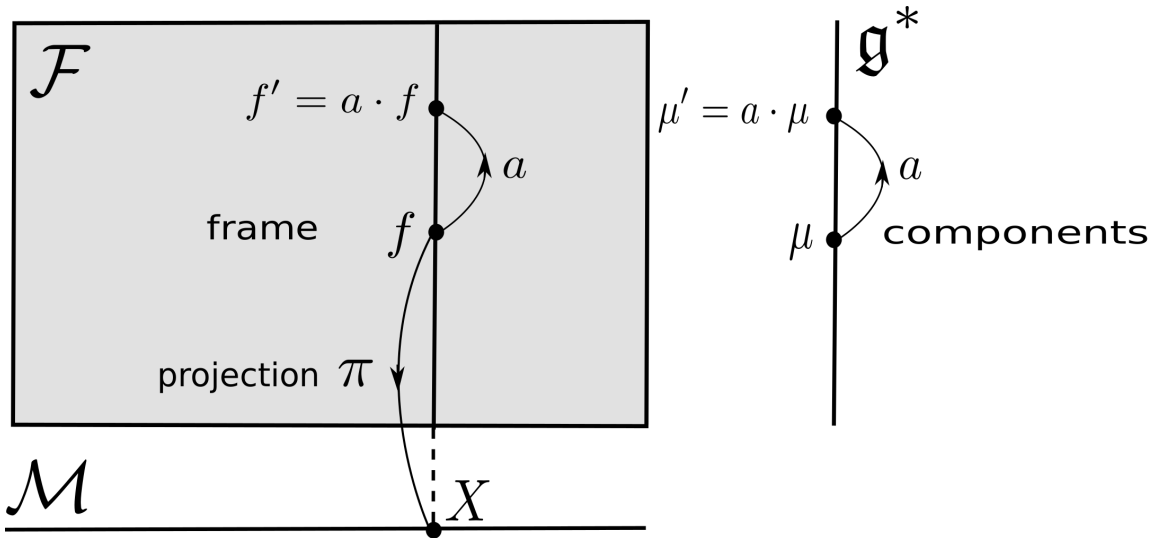
For the action $\mu = a \cdot \mu' = Ad^*(a)\,\mu' + \theta(a)$ on the orbit $orb(\mu) \subset \mathfrak{g}^*$, there exists a symplectic structure invariant by action of G and the identity map is a momentum map.

Factorized 2-form

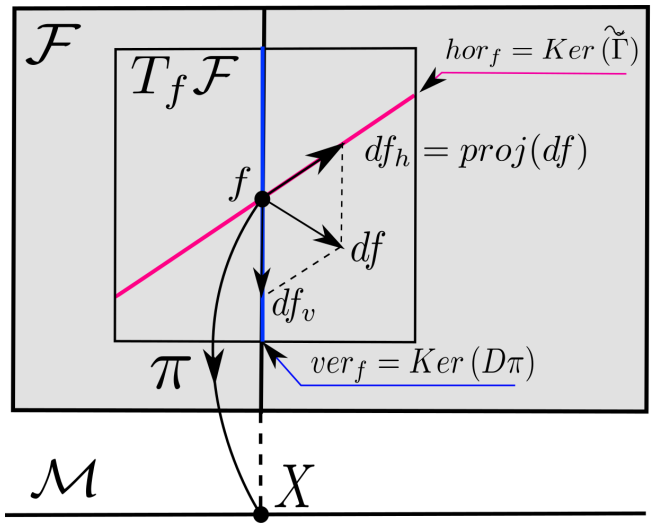


Factorized 2-form

Principal G-bundle of affine frames



Factorized 2-form



Erhesmann's connection is defined by :

- a field of supplementary subspaces hor_f in $T_f\mathcal{F}$
- alternatively, a field of $\mathfrak{g}$-valued 1-forms $\tilde{\Gamma}$ on $\mathcal{F}$ such that $hor_f = Ker\,\tilde{\Gamma}$

Factorized 2-form

Algorithm of putting into duality the moments and the connections :

- $\mu = (K, L) \in \mathfrak{g}^*$ components of $\boldsymbol{\mu}$ in the frame f
- $\tilde{\Gamma} = (\Gamma_A, \Gamma) \in \mathfrak{g}$ affine connection [Élie Cartan 1923]
  field of connection 1-forms on the bundle $\mathcal{F}$ of frames defining :
  - the infinitesimal motion of the origin through Γ_A (observer's motion)
  - the infinitesimal motion of the basis through Γ (gravitation in Relativity)
- dual pairing :
$$d\mu\,\tilde{\Gamma} = dK\,\Gamma_A + Tr(dL\,\Gamma)$$
- **factorized** 2-**form**
$$\omega = \frac{1}{2}\, d\mu \bigwedge \tilde{\Gamma}$$
(field of 2-forms on the principal bundle $\mathfrak{g}^* \times \mathcal{F}$ of the couples (μ, f))
- The orbit of a couple (μ, f) can be identified to the tensor $\boldsymbol{\mu}$

Factorized 2-form

Theorem

- the restriction $\psi_{\boldsymbol{\mu}}$ to the orbit $\boldsymbol{\mu}$ of the projection $\eta = (\mu, f) \mapsto \mu$ is a submersion
- on each orbit, ω is the pull-back by $\psi_{\boldsymbol{\mu}}$ of the symplectic form of Kirillov-Kostant-Souriau theorem and it is invariant by left action of G
- $\psi_{\boldsymbol{\mu}}$ is a momentum map and $\psi_{\boldsymbol{\mu}} \circ L_a = Ad^*(a)\,\psi_{\boldsymbol{\mu}} + \theta(a)$
- The covariant equations of motion are

$$d\eta \in Ker(\omega) \quad \Leftrightarrow \quad \tilde{\nabla}_{dX}\mu = d\,\mu + ad^*(\tilde{\Gamma})\mu - \iota(\tilde{\Gamma})\,(D\theta(e)) = 0 \qquad (1)$$

- if the last term is null, (1) is nothing else Euler-Poincaré equation [Poincaré 1901]
- In fact, (1) generalizes it when the class of symplectic cohomology of the group is not null, especially for the important case of Galileo's group

Application to the Galilean Mechanics

- Event $X = \begin{pmatrix} t \\ x \end{pmatrix} \in$ **space-time** $\mathcal{M}$
- The **Galilean transformations** $X' \mapsto X$ are the transformations preserving the **Uniform Straight Motion**, the **durations**, the **distances** and the **oriented volumes**,
  then affine of the form $X = P\,X' + C$ with

$$P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

  where $u \in \mathbb{R}^3$ is the **Galilean boost** and R is a rotation
- Their set is **Galileo's group**, $\mathbb{GAL}$
  a Lie group of dimension 10

Application to the Galilean Mechanics

For Galileo's group

- The dual pairing reads $\mu\, Z = l \cdot d\varpi - q \cdot du + p \cdot dk - e\, d\tau_0$
- We deduce the action onto the momentum tensor components
  - the energy $e = e' + u \cdot (R\, p') + \frac{1}{2}\, m \parallel u \parallel^2$
  - the linear momentum $p = R\, p' + m\, u$
  - the passage $q = R\,(q' - \tau_0\, p') + m\,(k - \tau_0\, u)$
  - the angular momentum $l = R\, l' - u \times (R\, q') + k \times (R\, p') + m\, k \times u$

  where the terms containing m are those of the symplectic cocycle
- The Galilean gravitation has two components :
  - the **gravity** g
  - the **spinning** Ω

Application to the Galilean Mechanics

the **covariant equations of motion**

$$\tilde{\nabla}_{dX}\mu = d\,\mu + ad^*(\tilde{\Gamma})\mu - \iota(\tilde{\Gamma})\,(D\theta(e)) = 0 \qquad (1)$$

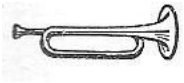
provides the **conservation of**

- the **energy** $\dot{e} = g \cdot p$
- the **linear momentum** $\dot{p} = m\,(g - 2\,\Omega \times v)$

  allowing to explain Foucault's pendulum
- the **passage** $\dot{q} = p$
- the **angular momentum** $\dot{l} + \Omega \times I_0 = x \times \; m\,(g - 2\,\Omega \times v)$

  where $I_0 = I - \frac{1}{m}\, q \times p$ is the spin,

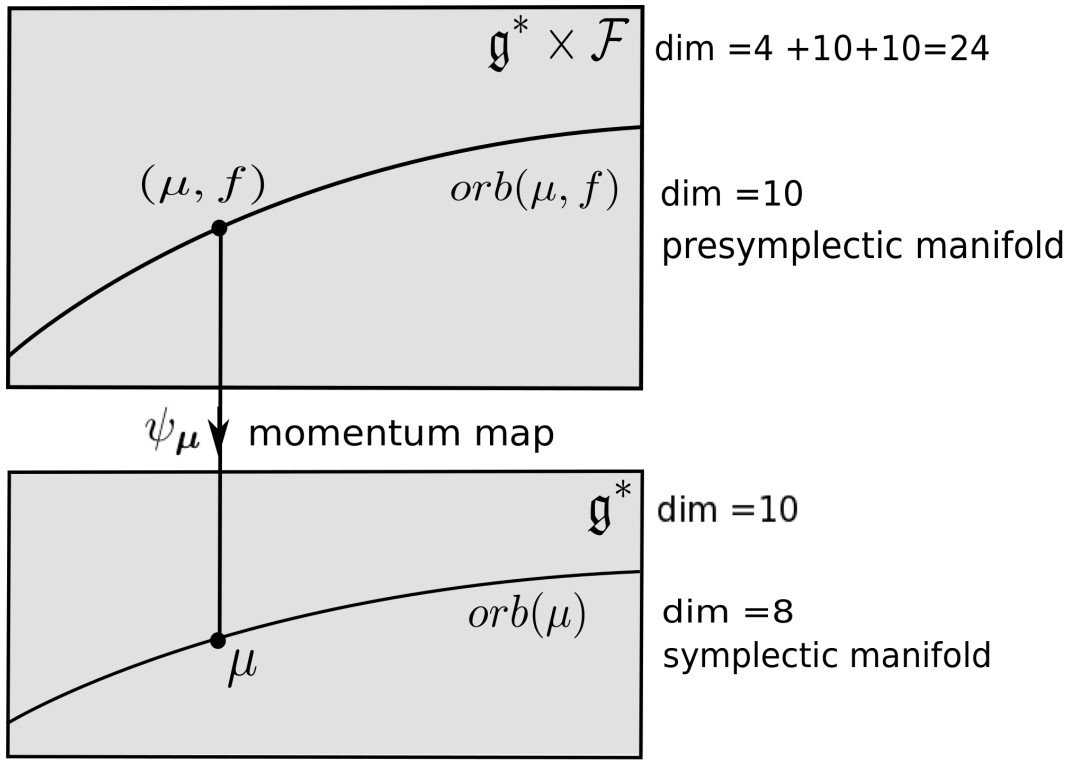
  allowing to explain Lagrange's top



the last term in (1) resulting from the symplectic cocycle is required to get the 2nd and 3rd conservation equations

Application to the Galilean Mechanics

Momentum map in classical mechanics



Epilogue : three loopholes

1. **Add an extra dimension** GDR-GDM La Rochelle 2019
   Bargmann's group (dimension 11)
   Momentum components : p, q, l, e, m (mass)
2. **Working in Relativity**
   Lorentz-Poincaré's group (dimension 10)

$$\tilde{\nabla} K = \nabla K = 0, \qquad \tilde{\nabla} L = \nabla L + \frac{1}{2}\left(\Gamma_A K - K^* \Gamma_A^*\right) = 0$$

   Momentum components : $p, q, l, e = mc^2$

3. **Using the torsor tensor** GDR-GDM Cachan 2019
   a skew-symmetric 2-contravariant affine tensor

$$\boldsymbol{\tau} : A^* T_{\boldsymbol{X}} \mathcal{M} \times A^* T_{\boldsymbol{X}} \mathcal{M} \to \mathbb{R} : (\hat{\boldsymbol{\Psi}}, \boldsymbol{\Psi}) \mapsto \boldsymbol{\tau}(\hat{\boldsymbol{\Psi}}, \boldsymbol{\Psi})$$

   Torsor components : p, q, l, m

Thank you !



