Théorie des invariants et lois de comportement

Elasticité[1] : classes de symétrie, covariants, bases fonctionnelles

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[1] un champ de vecteur sur la variété des métriques Riemanniennes (Rougée, 2006).

Elasticité linéaire unidimensionnelle

Ressort de raideur k (Hooke, 1676):

$$F = k\Delta l, \qquad \Delta l = l - l_0$$

Barreau de section S_0, de longueur initiale l_0, de module d'Young E:

$$\sigma = \frac{F}{S_0} = E\frac{\Delta l}{l_0} = E\epsilon, \qquad E = \frac{kl_0}{S_0}$$

Contrainte σ et (petite) déformation ϵ

$$\sigma = \frac{F}{S_0}, \qquad \epsilon = \frac{\Delta l}{l_0}$$

Essai de traction

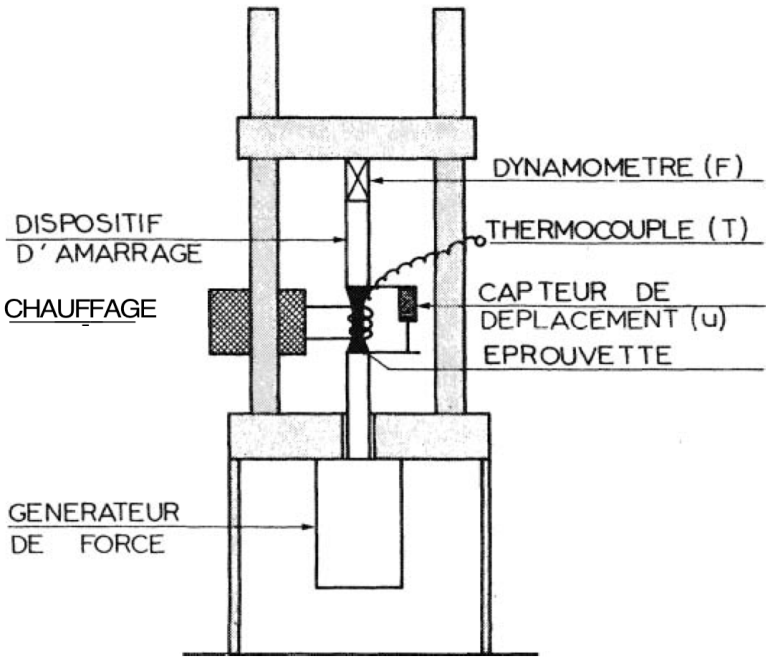


Figure: Machine de traction (schéma).

Essai de traction



Figure: Machine de traction (machine réelle 5t).

Outline

Linear elasticity

Harmonic decomposition – Covariant characterization of elasticity

Functional bases and separating sets

Functional bases on symmetry strata of elasticity tensors

Conclusion

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Configurations – plongements

- Le body $\mathcal{B}$ (Truesdell et Noll, 1965) est une variété à bord compacte et orientable, munie d'une forme volume μ, la mesure de masse (telle que $m = \int_{\mathcal{B}} \mu$).
- Une configuration est un plongement p du body $\mathcal{B}$ dans l'espace Euclidien $(\mathcal{E}, \mathbf{q})$.
- L'espace des configurations est noté $\mathrm{Emb}(\mathcal{B}, \mathcal{E})$.

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- Une configuration est un plongement p du body $\mathcal{B}$ dans l'espace Euclidien $(\mathcal{E}, \mathbf{q})$.
- L'espace des configurations est noté $\mathrm{Emb}(\mathcal{B}, \mathcal{E})$.
- Une configuration de référence (un plongement p_0 fixé) est nécessaire en mécanique du solide.

Plongement $p = \varphi \circ p_0 : \mathcal{B} \to \Omega_p$

$$\underbrace{\mathcal{B}}_{\text{variété à bord}} \xrightarrow{p_0} \underbrace{\Omega_0 \xrightarrow{\varphi} \Omega_p}_{\text{plongées dans } (\mathcal{E}, \mathbf{q})},$$

φ est appelée la transformation

Déplacement

- Lorsque l'espace $\mathcal{E}$ est considéré comme une variété Riemannienne, on introduit l'application exponentielle

$$T\mathcal{E} \to \mathcal{E} \times \mathcal{E}$$
$$(\mathbf{x}_0, \delta\mathbf{x}_0) \mapsto (\mathbf{x}_0, \exp_{\mathbf{x}_0} \delta\mathbf{x}_0)$$

  qui est un difféomorphisme d'un voisinage de la section nulle sur un voisinage de la diagonale.
- On définit alors le déplacement $\boldsymbol{\xi}$, un champ de vecteur sur $\Omega_0 \subset \mathcal{E}$, de manière univoque par la relation $\varphi(\mathbf{x}_0) = \exp_{\mathbf{x}_0} \boldsymbol{\xi}(\mathbf{x}_0)$ tant que $d(\varphi, \mathrm{id})$ reste assez petite (Grubic et al, 2014).

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- Dans un espace affine euclidien $(\mathcal{E}, \mathbf{q})$, l'application exponentielle est un difféomorphisme global et s'écrit $\exp_{\mathbf{x}_0} \delta\mathbf{x}_0 = \mathbf{x}_0 + \delta\mathbf{x}_0$, de sorte que

$$\varphi(\mathbf{x}_0) = \exp_{\mathbf{x}_0} \boldsymbol{\xi}(\mathbf{x}_0) = \mathbf{x}_0 + \boldsymbol{\xi}(\mathbf{x}_0)$$

Le déplacement $\boldsymbol{\xi} = \varphi - \mathrm{id}$ est un champ de vecteur sur $\Omega_0 \subset \mathcal{E}$.
L'application linéaire tangente s'écrit alors

$$T\varphi = \mathbf{Id} + \nabla\boldsymbol{\xi}$$

Métriques – Petites déformations

Métrique sur Ω_0 (tenseur de Cauchy Green-droit)

$$\mathbf{C} = \varphi^* \mathbf{q} = \mathbf{F}_\varphi^\star \mathbf{q}\, \mathbf{F}_\varphi = \mathbf{q}\, \mathbf{F}_\varphi^t \mathbf{F}_\varphi, \qquad \mathbf{F}_\varphi = T\varphi = \mathbf{Id} + \nabla \boldsymbol{\xi}$$

On a alors classiquement

$$\mathbf{C} = \frac{1}{2}\left(\mathbf{F}_\varphi^\star \mathbf{q}\, \mathbf{F}_\varphi - \mathbf{q}\right) = \frac{1}{2} \underbrace{\left(\mathbf{q}\, \nabla \boldsymbol{\xi} + (\nabla \boldsymbol{\xi})^\star \mathbf{q}\right)}_{\mathrm{L}_{\boldsymbol{\xi}}\, \mathbf{q}} + \underbrace{\frac{1}{2} (\nabla \boldsymbol{\xi})^\star \mathbf{q}\, \nabla \boldsymbol{\xi}}_{o(\|\nabla \boldsymbol{\xi}\|)}$$

soit

$$\mathbf{C} \approx \boldsymbol{\epsilon} := \frac{1}{2} \mathrm{L}_{\boldsymbol{\xi}}\, \mathbf{q}$$

Tenseur des petites déformations

$$\boldsymbol{\epsilon} := \frac{1}{2} \mathrm{L}_{\boldsymbol{\xi}}\, \mathbf{q} = \frac{1}{2} \left((\nabla \boldsymbol{\xi}^\flat + (\nabla \boldsymbol{\xi}^\flat)^t \right)$$

The stress tensor

Internal Forces

Internal forces (that ensure cohesion of the material) are traditionally described by the stress tensor $\boldsymbol{\sigma} = (\sigma^{ij})$ introduced by Cauchy in 1822.

Elastic work (first gradient theory)

From a modern point of view, these internal forces are described by their work, which is represented by the linear functional

$$W(\boldsymbol{w}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\epsilon} \, \mathrm{vol}_{\mathbf{q}}, \qquad \boldsymbol{\epsilon} = \frac{1}{2} \mathrm{L}_{\boldsymbol{w}} \, \mathbf{q}$$

Refer to Epstein et Segev (1980, 1986...) for a geometrization (and KD-2021), to Dell'Isola–Madeo–Seppecher (2016) for a generalization.

Hooke's Law – Linear elasticity

Hooke's law (initialy for springs, 1676), nowadays linear elasticity law $\sigma = E\epsilon$, is an example of a constitutive law: an empirical model $\boldsymbol{\sigma}(\boldsymbol{\epsilon})$ which describes the mechanical behavior of a material.

Generalized Hooke's Law

In linear elasticity, we assume a linear relation between the stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\epsilon}$,

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi^e}{\partial \boldsymbol{\epsilon}} = \mathbf{E} : \boldsymbol{\epsilon}, \qquad \left(\sigma^{ij} = \rho \frac{\partial \psi^e}{\partial \epsilon_{ij}} = E^{ijkl} \epsilon_{kl} \right).$$

that derives from an energy density $\rho\psi^e = \frac{1}{2}\boldsymbol{\epsilon} : \mathbf{E} : \boldsymbol{\epsilon}$.

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Isotropic elasticity

$$\boldsymbol{\sigma} = 2\mu\, \boldsymbol{\epsilon}^\sharp + \lambda\, \mathrm{tr}(\mathbf{q}^{-1}\boldsymbol{\epsilon})\, \mathbf{q}^{-1}, \qquad \boldsymbol{\epsilon}^\sharp = \mathbf{q}^{-1}\boldsymbol{\epsilon}\, \mathbf{q}^{-1}$$

λ, μ: Lamé constants.

Action of SO(3) on $\mathbb{E}$la

Let

$$\mathbb{E}\text{la} := \left\{ \mathbf{E} \in \otimes^4(\mathbb{R}^3);\ E^{ijkl} = E^{klij} = E^{jikl} \right\} \qquad (\dim \mathbb{E}\text{la} = 21)\,,$$

be the vector space of three-dimensional elasticity tensors.

Natural linear action (linear representation)

$\mathbb{E}$la is endowed with the natural SO(3) representation given by

$$(g \star \mathbf{E})^{ijkl} := g^i{}_p\, g^j{}_q\, g^k{}_r\, g^l{}_s E^{pqrs}, \quad g \in \mathrm{SO}(3).$$

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From now on we consider an orthonormal basis $(\boldsymbol{e}_i)$ and that we do not have to distinguish between covariant and contravariant tensors.

Symmetry groups / classes / strata

Let X be a set of points x (for example $x = \mathbf{E}$ in vector space $\mathsf{V} = \mathbb{E}\mathrm{la}$) and let $\star$ denote the action of a group G on X.

- The orbit of the point $x \in \mathsf{X}$ is defined as the set

$$\mathrm{Orb}(x) := \{g \star x, \quad g \in G\}\,.$$

- The symmetry group of x is defined as $G_x := \{g \in G,\, g \star x = x\}$.

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- The symmetry class of x, noted $[G_x]$, is defined as the conjugacy class of G_x in G, *i.e.*

$$[G_x] := \left\{g G_x g^{-1},\, g \in G\right\}.$$

- A symmetry stratum $\Sigma_{[H]}$ is the set of all elements x with symmetry group G_x conjugate to H:

$$\Sigma_{[H]} := \{x \in X, \quad G_x \in [H]\}\,.$$

All x in $\mathrm{Orb}(x)$ belong to the same symmetry stratum, as $G_{g\star x} = g G_x g^{-1}$.

Quelques questions

- Quand est ce que 2 tenseurs $\mathbf{E}_i$ représentent le même matériau ? (Boehler-Kirillov–Onat 1994, ...)
- Quelles sont les classes de symétrie de l'élasticité ? (Forte–Vianello, 1996)
- Comment les caractériser ? (tentative par Bona–Bucataru–Slawinski, 2004, ...)
- Quelle est la distance d'un tenseur expérimental à une classe de symétrie donnée ? (Gazis, 1963, M. François, 1995, ...)
- Quels sont les invariants pertinents du tenseur d'élasticité ? (base d'intégrité par Olive, 2014, ...)
- Extension aux lois de comportement nonlinéaires ? Aux grandes transformations ?

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When are two elastic materials "the same"

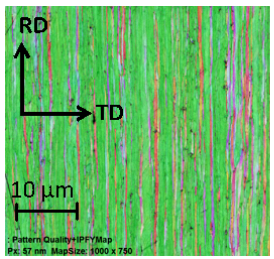
Does there exists a rotation $g \in \mathrm{SO}(3)$ such that

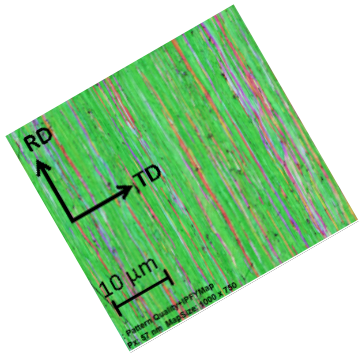
$$(\mathbf{E}_2)_{ijkl} = g_{ip} g_{jq} g_{kr} g_{ls} (\mathbf{E}_1)_{pqrs} \, ?$$

a relation that we shall denote by

$$\mathbf{E}_2 = g \star \mathbf{E}_1$$

and we say then that the two tensors are in the same orbit.





Acier 14%-Cr ODS (CEA)

When such a rotation does not exist, the two tensors describe different elastic materials.

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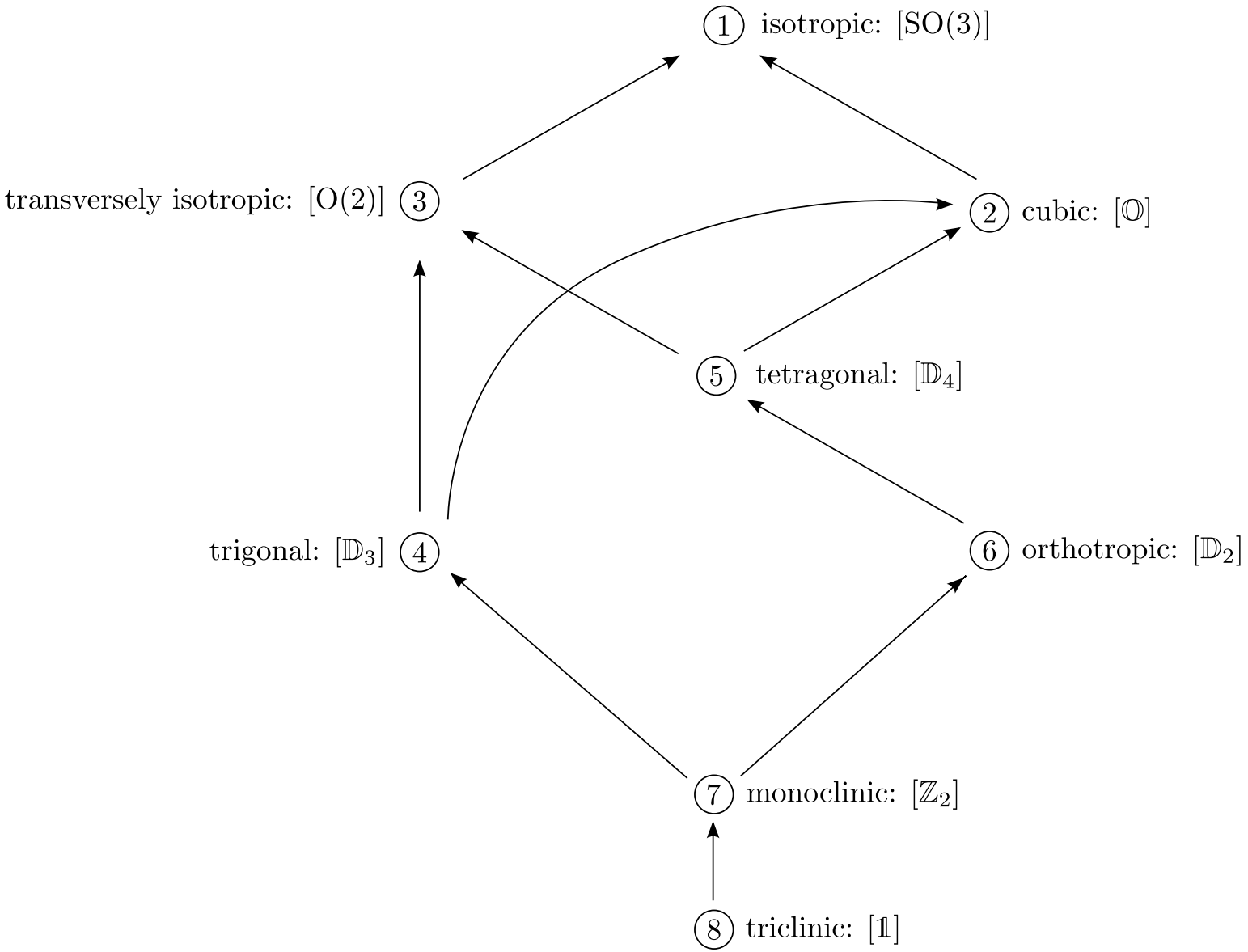


Figure: Partial ordering set for symmetry classes of $\mathbb{E}$la (Forte–Vianello, 1996).

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Usefull tensor products

Let $\mathbf{S}_1 \in \mathbb{S}^{n_1}(\mathbb{R}^3)$ and $\mathbf{S}_2 \in \mathbb{S}^{n_2}(\mathbb{R}^3)$ be two totally symmetric tensors.

Symmetric tensor product

$$\mathbf{S}_1 \odot \mathbf{S}_2 := (\mathbf{S}_1 \otimes \mathbf{S}_2)^s \ \in \mathbb{S}^{n_1+n_2}(\mathbb{R}^3)$$

Generalized (Lie–Poisson) cross product

$$\mathbf{S}_1 \times \mathbf{S}_2 := (\mathbf{S}_2 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{S}_1)^s \in \mathbb{S}^{n_1+n_2-1}(\mathbb{R}^3),$$

where $\boldsymbol{\varepsilon} = (\det(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k))$ is the third-order Levi-Civita tensor.

We have the property

$$\mathbf{S} \times \mathbf{q} = 0, \quad \forall \mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3),$$

where $\mathbf{q} = (\delta_{ij})$ is the Euclidean metric.

Harmonic decomposition

Every three-dimensional tensor space V decomposes into irreducible representations $\mathbb{H}^k = \mathbb{H}^k(\mathbb{R}^3)$ for $\mathrm{O}(3)$ (harmonic decomposition).

For the elasticity tensor, $\mathsf{V} = \mathbb{E}\mathrm{la}$ (Backus, 1970)

$$\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H}) \in \mathbb{H}^0 \times \mathbb{H}^0 \times \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^4$$

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$$\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H}) \in \mathbb{H}^0 \times \mathbb{H}^0 \times \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^4$$

with the dilatation and Voigt tensors

$$\mathbf{d} := \operatorname{tr}_{12}\mathbf{E}, \qquad \mathbf{v} := \operatorname{tr}_{13}\mathbf{E},$$

$\mathbf{d}' := \mathbf{d} - \frac{1}{3}(\operatorname{tr}\mathbf{d})\,\mathbf{q}$, $\mathbf{v}' := \mathbf{v} - \frac{1}{3}(\operatorname{tr}\mathbf{v})\,\mathbf{q}$ and with

$$\mathbf{H} := (\mathbf{E}^s)' = \mathbf{E}^s - \mathbf{q} \odot \mathbf{a}' - \frac{7}{30}(\operatorname{tr}\mathbf{a})\,\mathbf{q} \odot \mathbf{q}, \quad \text{with } \mathbf{a} := \frac{2}{7}(\mathbf{d} + 2\mathbf{v})$$

($\mathbf{q}$ is the euclidian tensor and $\mathbf{a} \odot \mathbf{b} = (\mathbf{a} \otimes \mathbf{b})^s$)

Harmonic components as linear covariants of the elasticity tensor

The harmonic decomposition of $\mathbf{E}$ is equivariant. For any $g \in \mathrm{SO}(3)$:

$$g\star\mathbf{E} = (g\star\mathrm{tr}\,\mathbf{d}, g\star\mathrm{tr}\,\mathbf{v}, g\star\mathbf{d}', g\star\mathbf{v}', g\star\mathbf{H}) = (\underbrace{\mathrm{tr}\,\mathbf{d}, \mathrm{tr}\,\mathbf{v}}_{\text{invariants}}, \underbrace{g\star\mathbf{d}', g\star\mathbf{v}', g\star\mathbf{H}}_{\text{covariants}})$$

The harmonic components $\mathrm{tr}\,\mathbf{d} = \mathrm{tr}(\mathbf{d}(\mathbf{E}))$, $\mathrm{tr}\,\mathbf{v} = \mathrm{tr}(\mathbf{v}(\mathbf{E}))$ and

$$\mathbf{d}' = \mathbf{d}'(\mathbf{E}), \quad \mathbf{v}' = \mathbf{v}'(\mathbf{E}), \quad \mathbf{H} = \mathbf{H}(\mathbf{E}),$$

are linear covariants $\mathbf{C}(\mathbf{E})$ of $\mathbf{E}$.

Covariants of the elasticity tensor

The covariants of a tensor $\mathbf{E}$ satisfy the rule

$$\mathbf{C}(g \star \mathbf{E}) = g \star \mathbf{C}(\mathbf{E}), \qquad \forall g \in \mathrm{SO}(3).$$

There exist nonlinear covariants. For instance, the quadratic covariant

$$\mathbf{d}_2(\mathbf{H}) := \mathrm{tr}_{13}\, \mathbf{H}^2 = \mathbf{H} \vdots \mathbf{H}, \qquad ((\mathbf{d}_2)_{ij} = H_{ipqr} H_{pqrj})$$

and the cubic covariant

$$\mathbf{d}_3(\mathbf{H}) := \mathrm{tr}_{13}\, \mathbf{H}^3, \qquad ((\mathbf{d}_3)_{ij} = H_{kipq} H_{pqrs} H_{rskj})$$

introduced by Boehler, Kirillov and Onat in 1994.

Covariant characterization of elasticity symmetry classes

Theorem (Olive et al, MMS, 2021)

Let $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H}) \in \mathbb{E}\mathrm{la}$ *be an elasticity tensor. Then*

1. $\mathbf{E}$ *is isotropic if and only if* $\mathbf{d}' = \mathbf{v}' = \mathbf{d}_2 = 0$.
2. $\mathbf{E}$ *is cubic if and only if* $\mathbf{d}' = \mathbf{v}' = \mathbf{d}_2' = 0$ *and* $\mathbf{d}_2 \neq 0$.
3. $\mathbf{E}$ *is transversely isotropic if and only if* $(\mathbf{d}_2, \mathbf{d}, \mathbf{v})$ *is transversely isotropic and*

$$\mathbf{H} \times \mathbf{d}_2 = \mathbf{H} \times \mathbf{d} = \mathbf{H} \times \mathbf{v} = 0.$$

4. $\mathbf{E}$ *is tetragonal if and only if* $(\mathbf{d}_2, \mathbf{d}, \mathbf{v})$ *is transversely isotropic,*

$$\operatorname{tr}(\mathbf{H} \times \mathbf{d}_2) = \operatorname{tr}(\mathbf{H} \times \mathbf{d}) = \operatorname{tr}(\mathbf{H} \times \mathbf{v}) = 0, \quad \textit{and}$$

$$\mathbf{H} \times \mathbf{d}_2 \neq 0, \quad \textit{or} \quad \mathbf{H} \times \mathbf{d} \neq 0, \quad \textit{or} \quad \mathbf{H} \times \mathbf{v} \neq 0.$$

5. $\mathbf{E}$ *is trigonal if and only if* $(\mathbf{d}_2, \mathbf{d}, \mathbf{v})$ *is transversely isotropic,*

$$\mathbf{d}_2 \times (\mathbf{H} : \mathbf{d}_2) = \mathbf{d} \times (\mathbf{H} : \mathbf{d}) = \mathbf{v} \times (\mathbf{H} : \mathbf{v}) = 0, \quad \textit{and}$$

$$\operatorname{tr}(\mathbf{H} \times \mathbf{d}_2) \neq 0, \quad \textit{or} \quad \operatorname{tr}(\mathbf{H} \times \mathbf{d}) \neq 0, \quad \textit{or} \quad \operatorname{tr}(\mathbf{H} \times \mathbf{v}) \neq 0.$$

The characterization of orthotropic and monoclinic symmetry classes involves more covariants.

Interest of a characterization by covariants

Case of trigonal harmonic fourth order tensors[2] $D = \mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ (Auffray–Kolev–Petitot, 2014)

Proposition 5.8. *A harmonic tensor $D \in \mathbb{H}^4$ is in the closed stratum $\overline{\Sigma}_{[\mathbb{D}_3]}$ if and only if the invariants $J_2(D) \cdots J_{10}(D)$ satisfy the following polynomial relations*

$$
\begin{aligned}
192 J_6 &= -51 {J_2}^3 + 216 J_2 J_4 + 10 {J_3}^2 \\
36 J_7 &= -2 {J_2}^2 J_3 + 6 J_3 J_4 + 27 J_2 J_5 \\
768 {J_4}^2 &= -99 {J_2}^4 + 552 {J_2}^2 J_4 + 10 J_2 {J_3}^2 + 240 J_3 J_5 \\
240 J_8 &= -33 {J_2}^4 + 96 {J_2}^2 J_4 + 30 J_2 {J_3}^2 + 40 J_3 J_5 \\
576 J_4 J_5 &= -41 {J_2}^3 J_3 + 120 J_2 J_3 J_4 + 216 {J_2}^2 J_5 + 30 {J_3}^3 \\
1152 J_9 &= -99 {J_2}^3 J_3 + 296 J_2 J_3 J_4 + 648 {J_2}^2 J_5 + 10 {J_3}^3 \\
1440 {J_5}^2 &= -11 {J_2}^5 + 32 {J_2}^3 J_4 - 70 {J_2}^2 {J_3}^2 + 240 J_2 J_3 J_5 + 240 {J_3}^2 J_4 \\
8640 J_{10} &= -891 {J_2}^5 + 2592 {J_2}^3 J_4 + 730 {J_2}^2 {J_3}^2 + 2160 J_2 J_3 J_5 + 240 {J_3}^2 J_4
\end{aligned}
\tag{21}
$$

together with inequality (20). D is in the trigonal class $[\mathbb{D}_3]$ if moreover $3J_4 - J_2^2 \neq 0$ and $98 J_4 - 41 J_2^2 \neq 0$. In that case, it admits the normal form (18) where

$$\delta = -\frac{1}{4} \frac{J_5}{{J_2}^2 - 3 J_4}$$

and where σ is the positive root of $J_2 = 280\,\delta^2 + 16\,\sigma^2$.

[2] Inequality (20) $= 2 J_2 (J_2^2 - 3 J_4)^2 - 35 J_5^2 \geq 0$.

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Notions from Invariant Theory

There are three concepts in Invariant Theory

- the concept of integrity basis (generating set of the invariant algebra, here $\mathbb{R}[\mathbb{E}\mathrm{la}]^{\mathrm{SO}(3)}$)
- the concepts of functional basis and separating set, which are moreover equivalent (Wineman and Pipkin, 1964).

Remark

The notion of functional basis, introduced in Weyl's classical book, has become a key notion in the mechanical science literature related to Invariant Theory (Smith (1971), Boehler (1987), Smith (1994), Zheng (1994)).

The algebra of G-invariant functions

- The action of G on a set X induces a linear action of G on the algebra $\mathcal{F}(\mathsf{X})$ of real-valued functions on X, which writes

$$(g \star f)(x) := f(g^{-1} \star x),$$

  where $f \in \mathcal{F}(\mathsf{X})$ and $g \in G$.

- The subalgebra of G-invariant functions, noted $\mathcal{F}(\mathsf{X})^G$, is defined by

$$\mathcal{F}(\mathsf{X})^G := \{f \in \mathcal{F}(\mathsf{X}), \quad g \star f = f, \quad \forall g \in G\}.$$

Functional bases

Definition (Functional basis)

A finite set $\mathscr{F} := \{\varphi_1, \ldots, \varphi_s\}$ of G-invariant functions is a functional basis of $\mathcal{F}(\mathsf{X})^G$ if for any G-invariant function $f \in \mathcal{F}(\mathsf{X})^G$ there exists a function $F : \mathbb{R}^s \to \mathbb{R}$ such that

$$f(x) = F(\varphi_1(x), \ldots, \varphi_s(x)), \quad \forall x \in \mathsf{X}.$$

A functional basis $\mathscr{F}$ is said to be minimal if no proper subset $\mathscr{F}'$ of $\mathscr{F}$ is a functional basis.

Remark

As pointed out by Weyl the word function has to be understood in its widest scope. Such a function F may not even be continuous.

Separating set

Definition (Separating set)

A finite set $\mathscr{S} := \{\kappa_1, \ldots, \kappa_r\}$ of G-invariant functions is a separating set if for any $x, \overline{x}$ in X

$$\mathrm{Orb}(x) = \mathrm{Orb}(\overline{x}) \iff \kappa_i(x) = \kappa_i(\overline{x}), \quad i = 1, \ldots, r.$$

A separating set $\mathscr{S}$ is said to be minimal if no proper subset $\mathscr{S}'$ of $\mathscr{S}$ is a separating set.

Remark

It is an answer to the question "Are two (elastic) materials the same?"

Remarks

- The cardinal of a minimal separating set/functional basis may vary from one minimal set to another.
- A lower bound on the cardinal of such a set depends drastically on the class of functions (continuous, differentiable, ...).
- By dropping off continuity, it is always possible to construct a separating set of only one element (Wang, 1970).
- When the orbit space X/G is a topological manifold of dimension d, the cardinal r of any separating set of continuous functions is bigger than d ($r \geq d$).

Isotropy stratification

Consider a compact Lie group G acting linearly on a vector space V.

Then, there exists only a finite number of symmetry classes $[H_1], \ldots, [H_l]$ and V splits into a disjoint union of strata (Bredon, 1960, 1972, Abud–Sartori, 1983).

$$\mathsf{V} = \Sigma_{[H_1]} \cup \ldots \cup \Sigma_{[H_l]},$$

where each stratum $\Sigma_{[H]}$ is a G-stable smooth submanifold of V.

Isotropy stratification

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where each stratum $\Sigma_{[H]}$ is a G-stable smooth submanifold of V.

$d = \dim \Sigma_{[H]}/G$ for some elasticity strata

$$\dim(\Sigma_{[\mathbb{1}]}/\mathrm{SO}(3)) = 18, \quad \dim(\Sigma_{[\mathbb{D}_4]}/\mathrm{SO}(3)) = 6, \quad \dim(\Sigma_{[\mathbb{D}_3]}/\mathrm{SO}(3)) = 6,$$
$$\dim(\Sigma_{[\mathrm{O}(2)]}/\mathrm{SO}(3)) = 5, \quad \dim(\Sigma_{[\mathbb{O}]}/\mathrm{SO}(3)) = 3, \quad \dim(\Sigma_{[\mathrm{SO}(3)]}/\mathrm{SO}(3)) = 2.$$

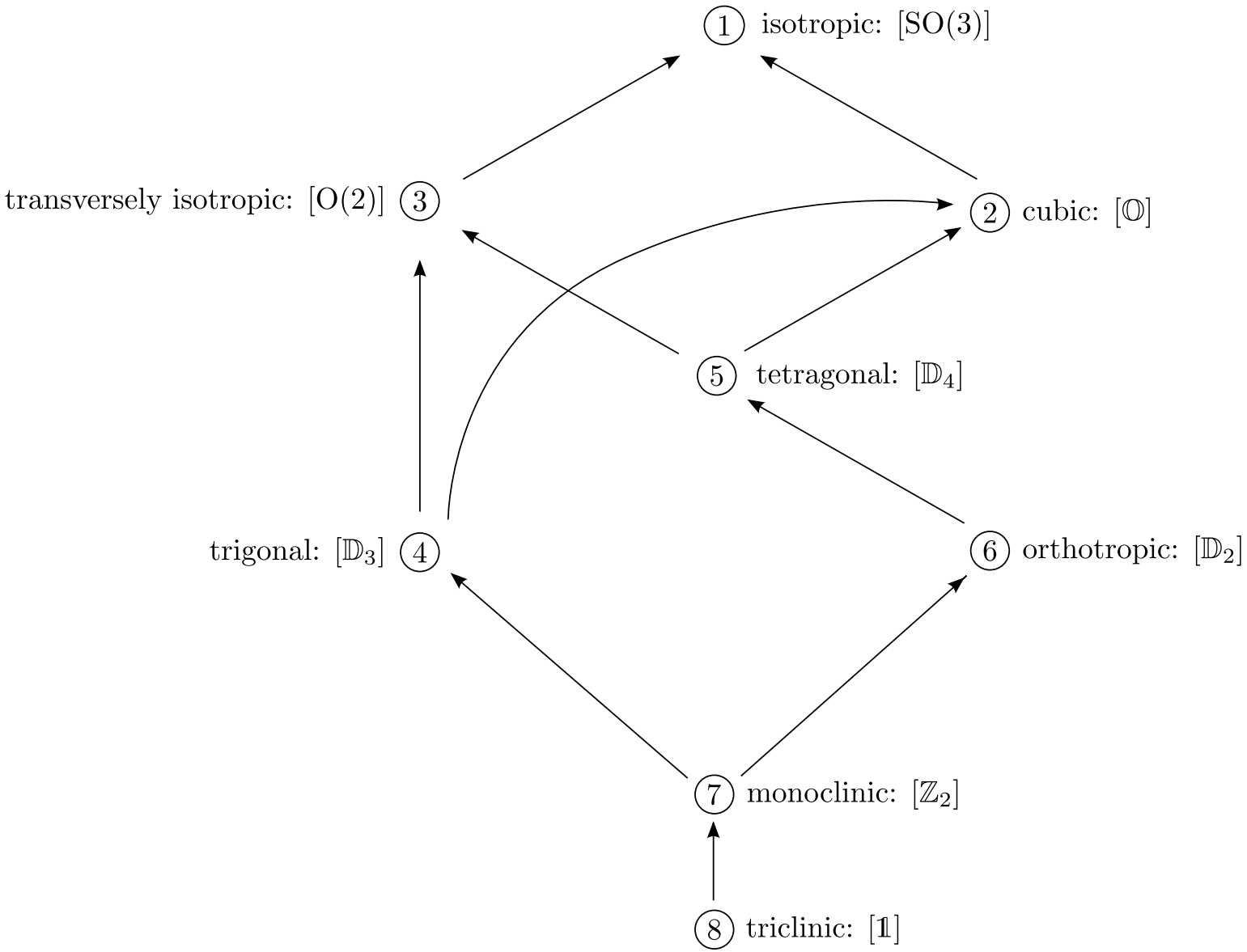


Figure: Partial ordering set for symmetry classes of $\mathbb{E}$la (Forte–Vianello, 1996).

Integrity basis

Let $\mathbb{R}[\mathsf{V}]$ be the algebra of polynomial functions on V. The invariant algebra of V is the subalgebra

$$\mathbb{R}[\mathsf{V}]^G := \{\mathrm{p} \in \mathbb{R}[\mathsf{V}];\ \mathrm{p}(g \star \boldsymbol{v}) = \mathrm{p}(\boldsymbol{v}), \quad \forall g \in G, \forall \boldsymbol{v} \in \mathsf{V}\}\,.$$

of polynomial invariants.

Definition (Integrity basis)

A finite set of invariant polynomials $\{I_1, \ldots, I_k\}$ is a generating set (also called an integrity basis) of the invariant algebra $\mathbb{R}[\mathsf{V}]^G$ if each polynomial invariant J can be written as

$$J(\boldsymbol{v}) = P(I_1(\boldsymbol{v}), \ldots, I_k(\boldsymbol{v})), \qquad \boldsymbol{v} \in \mathsf{V},$$

where P is a polynomial function in k variables.
An integrity basis is minimal if no proper subset of it is an integrity basis.

A minimal integrity basis for $\mathbb{R}[\mathbb{E}\mathrm{la}]^{\mathrm{SO}(3)}$

(Olive et al, 2021, cardinal 294)

(1) 15 simple invariants:
- $\operatorname{tr}\mathbf{d}$, $\operatorname{tr}\mathbf{v}$;
- the simple invariants of $\mathbf{d}$ and $\mathbf{v}$: $\operatorname{tr}\mathbf{d}^2$, $\operatorname{tr}\mathbf{d}^3$, $\operatorname{tr}\mathbf{v}^2$, $\operatorname{tr}\mathbf{v}^3$:
- and the nine simple invariants of $\mathbf{H}$, computed first in [15]:

$$\operatorname{tr}\mathbf{d}_2,\quad \operatorname{tr}\mathbf{d}_3,\quad \operatorname{tr}\mathbf{d}_2^2,\quad \operatorname{tr}\left(\mathbf{d}_2\mathbf{d}_3\right),\quad \operatorname{tr}\mathbf{d}_2^3,$$
$$\operatorname{tr}\left(\mathbf{d}_2^2\mathbf{d}_3\right),\quad \operatorname{tr}\left(\mathbf{d}_2\mathbf{d}_3^2\right),\quad \operatorname{tr}\mathbf{d}_3^3,\quad \operatorname{tr}\left(\mathbf{d}_2^2\mathbf{d}_3^2\right);$$

(2) 4 joint invariants of $(\mathbf{d},\mathbf{v})$:

$$\operatorname{tr}\left(\mathbf{dv}\right),\quad \operatorname{tr}\left(\mathbf{d}^2\mathbf{v}\right),\quad \operatorname{tr}\left(\mathbf{dv}^2\right),\quad \operatorname{tr}\left(\mathbf{d}^2\mathbf{v}^2\right)$$

(3) 52 joint invariants of $(\mathbf{H},\mathbf{d})$, and similarly 52 joint invariants of $(\mathbf{H},\mathbf{v})$ given in Table 3, where

$$^{3,3}\boldsymbol{C} = \operatorname{tr}\left(\mathbf{H}\times\mathbf{d}_2\right),\quad ^{4\mathrm{b},5}\boldsymbol{C} = \left(\mathbf{H}^2\right)^s\times\mathbf{d}_2,\quad \boldsymbol{v}_5 = \boldsymbol{\varepsilon}:\left(\mathbf{d}_2\mathbf{c}_3\right).$$

(4) 174 joint invariant of $(\mathbf{H},\mathbf{d},\mathbf{v})$ given in Table 4, Table 5 and Table 6, where

$$^{3,7}\boldsymbol{C} = \mathbf{H}\times\left(\mathbf{H}^2\right)^s,\quad ^{3,9}\boldsymbol{C} = \left(\left(\mathbf{H}\cdot\mathbf{H}\right)^s\times\mathbf{H}\right),$$
$$^{3,5}\boldsymbol{C} = \mathbf{H}\times\mathbf{d}_2,\quad ^{4,7}\boldsymbol{C} = \left(\mathbf{H}\times\left(\mathbf{H}^3\right)^s\right).$$

#	Deg.	M. deg.	Inv.
ja_1	3	(2,1,0)	$\operatorname{tr}\left(\mathbf{ad_2}\right)$
ja_2	3	(1,2,0)	$\mathbf{a}:\mathbf{H}:\mathbf{a}$
ja_3	4	(2,2,0)	$\operatorname{tr}\left(\mathbf{a}^2\mathbf{d_2}\right)$
ja_4	4	(3,1,0)	$\operatorname{tr}\left(\mathbf{ad_3}\right)$
ja_5	4	(1,3,0)	$\mathbf{a}:\mathbf{H}:\mathbf{a}^2$
ja_6	4	(2,2,0)	$\mathbf{a}:\left(\mathbf{H}^2\right)^s:\mathbf{a}$
ja_7	5	(1,4,0)	$\mathbf{a}^2:\mathbf{H}:\mathbf{a}^2$
ja_8	5	(2,3,0)	$\mathbf{a}:\left(\mathbf{H}^2\right)^s:\mathbf{a}^2$
ja_9	5	(2,3,0)	$\mathbf{a}:\left(\mathbf{a}:\left(\mathbf{H}\cdot\mathbf{H}\right)^s:\mathbf{a}\right)$
ja_{10}	5	(3,2,0)	$\mathbf{a}:\left(\mathbf{H}^3\right)^s:\mathbf{a}$
ja_{11}	5	(3,2,0)	$\operatorname{tr}\left(\mathbf{a}^2\mathbf{d}_3\right)$
ja_{12}	5	(4,1,0)	$\operatorname{tr}\left(\mathbf{ad}_2^2\right)$
ja_{13}	5	(4,1,0)	$\mathbf{a}:\left(\mathbf{H}^2\right)^s:\mathbf{d}_2$
ja_{14}	6	(2,4,0)	$\mathbf{a}^2:\left(\mathbf{H}^2\right)^s:\mathbf{a}^2$
ja_{15}	6	(2,4,0)	$\mathbf{a}^2:\left(\mathbf{a}:\left(\mathbf{H}\cdot\mathbf{H}\right)^s:\mathbf{a}\right)$
ja_{16}	6	(3,3,0)	$\mathbf{a}:\left(\mathbf{a}:\left(\mathbf{H}^2\cdot\mathbf{H}\right)^s:\mathbf{a}\right)$
ja_{17}	6	(3,3,0)	$\mathbf{a}:\left(\mathbf{H}^3\right)^s:\mathbf{a}^2$
ja_{18}	6	(3,3,0)	$\operatorname{tr}\left(\mathbf{H}\times\mathbf{d}_2\right):\left(\mathbf{a}^2\times\mathbf{a}\right)$
ja_{19}	6	(4,2,0)	$\mathbf{d}_2^2:\mathbf{a}^2$
ja_{20}	6	(4,2,0)	$\mathbf{a}:\left(\mathbf{H}^4\right)^s:\mathbf{a}$
ja_{21}	6	(4,2,0)	$\mathbf{a}^2:\mathbf{c}_4$
ja_{22}	6	(5,1,0)	$\mathbf{a}:(\mathbf{d}_2\mathbf{d}_3)$
ja_{23}	6	(5,1,0)	$\mathbf{a}:\left(\mathbf{H}^3\right)^s:\mathbf{d}_2$
ja_{24}	7	(2,5,0)	$\mathbf{a}:\left(\mathbf{a}^2:\left(\mathbf{H}\cdot\mathbf{H}\right)^s:\mathbf{a}^2\right)$
ja_{25}	7	(3,4,0)	$\mathbf{a}:\left(\left(\mathbf{H}\times\mathbf{d}_2\right):\left(\mathbf{a}^2\times\mathbf{a}\right)\right)$
ja_{26}	7	(3,4,0)	$\mathbf{a}:\left(\mathbf{a}:\left(\mathbf{H}^2\cdot\mathbf{H}\right)^s:\mathbf{a}^2\right)$
ja_{27}	7	(4,3,0)	$\mathbf{a}:\left(\mathbf{H}^4\right)^s:\mathbf{a}^2$

#	Deg.	M. deg.	Inv.
ja_{28}	7	(4,3,0)	$\operatorname{tr}\left(\mathbf{H}\times\mathbf{c}_3\right):\left(\mathbf{a}^2\times\mathbf{a}\right)$
ja_{29}	7	(4,3,0)	$\mathbf{a}:\left(\mathbf{a}:\left(\mathbf{H}^2\cdot\mathbf{H}^2\right)^s:\mathbf{a}\right)$
ja_{30}	7	(5,2,0)	$\mathbf{a}:\left(\mathbf{H}\cdot\mathbf{d}_2^2\right)^s:\mathbf{a}$
ja_{31}	7	(5,2,0)	$\mathbf{c}_5:\mathbf{a}^2$
ja_{32}	7	(5,2,0)	$\left(\mathbf{d}_2\mathbf{c}_3\right):\mathbf{a}^2$
ja_{33}	7	(6,1,0)	$\left(\mathbf{d}_2\mathbf{c}_4\right):\mathbf{a}$
ja_{34}	7	(6,1,0)	$\mathbf{c}_3^2:\mathbf{a}$
ja_{35}	8	(7,1,0)	$\left(\mathbf{d}_2^2\mathbf{c}_3\right):\mathbf{a}$
ja_{36}	8	(7,1,0)	$\left(\mathbf{c}_4\mathbf{c}_3\right):\mathbf{a}$
ja_{37}	8	(6,2,0)	$\left(\mathbf{d}_2\mathbf{c}_4\right):\mathbf{a}^2$
ja_{38}	8	(6,2,0)	$\mathbf{c}_3^2:\mathbf{a}^2$
ja_{39}	8	(6,2,0)	$\mathbf{a}:\left(\mathbf{H}^2\cdot\mathbf{d}_2^2\right)^s:\mathbf{a}$
ja_{40}	8	(5,3,0)	$\operatorname{tr}\left(\mathbf{H}\times\mathbf{d}_2^2\right):\left(\mathbf{a}^2\times\mathbf{a}\right)$
ja_{41}	8	(5,3,0)	$\mathbf{a}:\left(\mathbf{H}\cdot\mathbf{d}_2^2\right)^s:\mathbf{a}^2$
ja_{42}	8	(4,4,0)	$\mathbf{a}:\left(^{4\mathrm{b},5}\boldsymbol{C}:\left(\mathbf{a}^2\times\mathbf{a}\right)\right)$
ja_{43}	8	(4,4,0)	$\mathbf{a}:\left(\mathbf{a}:\left(\mathbf{H}^2\cdot\mathbf{H}^2\right)^s:\mathbf{a}^2\right)$
ja_{44}	8	(3,5,0)	$\mathbf{a}^2:\left(\mathbf{a}^2:\left(\mathbf{H}^2\cdot\mathbf{H}\right)^s:\mathbf{a}\right)$
ja_{45}	9	(6,3,0)	$\left(^{3,3}\boldsymbol{C}:\mathbf{a}\right)\cdot\mathbf{a}\cdot\left(^{3,3}\boldsymbol{C}:\mathbf{a}\right)$
ja_{46}	9	(7,2,0)	$\left(\mathbf{c}_4\mathbf{c}_3\right):\mathbf{a}^2$
ja_{47}	9	(7,2,0)	$\left(\mathbf{d}_2^2\mathbf{c}_3\right):\mathbf{a}^2$
ja_{48}	9	(8,1,0)	$\left(\mathbf{d}_2\mathbf{c}_3^2\right):\mathbf{a}$
ja_{49}	9	(8,1,0)	$\mathbf{c}_4^2:\mathbf{a}$
ja_{50}	10	(8,2,0)	$\mathbf{c}_4^2:\mathbf{a}^2$
ja_{51}	10	(9,1,0)	$\left(\mathbf{d}_2^2\mathbf{c}_5\right):\mathbf{a}$
ja_{52}	11	(10,1,0)	$\boldsymbol{v}_5\cdot\mathbf{a}\cdot\boldsymbol{v}_5$

$\mathbf{a} = \mathbf{d}$ or $\mathbf{v}$.

Integrity bases are separating

Theorem (Abud–Sartori, 1983)

Let G be a compact group acting linearly on a real vector space V. Then, each integrity basis of $\mathbb{R}[\mathsf{V}]^G$ is a separating set.

Reducing the cardinal of a polynomial separating set

	Cardinal	$n=3$	$n=4$	$n=5$	$n=6$
Mininimal integrity basis	$3n+4\binom{n}{2}+7\binom{n}{3}+20\binom{n}{4}+26\binom{n}{5}+10\binom{n}{6}$	28	84	261	684
Mininimal separating set	$3n+4\binom{n}{2}+\binom{n}{3}=\frac{n(n^2+17)}{6}$	22	40	65	98

Figure: Cardinals for n symmetric 2nd-order tensors (Pennisi–Trovato, 1987).

Remark

The (cardinal 294) minimal integrity basis of $\mathbb{R}[\mathbb{E}\mathrm{la}]^{\mathrm{SO}(3)}$ has been reduced to a set of 251 separating polynomial invariants by Ming *et al* (2019).

Other reductions

There are in the literature different strategies to reduce the number of elements of a functional basis.

- change the class of its elements: usually polynomial invariants are considered but this is not mandatory;
- look for local separating sets instead of global ones: the separating property is then defined, not on the whole vector space, but only on a neighbourhood of a given tensor.
  - Bona et al (2008): 18 *local algebraic invariants*.
  - DADKO-2019: 18 *local polynomial invariants*.
- restrict the separating property to a subset of generic tensors $\rightarrow$ *weak functional bases*.

Other reductions for elasticity tensors

For three-dimensional elasticity tensors,

- a weak separating set of 39 global polynomial invariants has been provided by Boehler–Kirillov–Onat (1994),
- and a weak separating set of 18 global rational invariants has been obtained by us (2019).

To reduce the set from 294 elements to 39 and to only 18, a price (genericity) has to be paid.

Outline

Linear elasticity

Harmonic decomposition – Covariant characterization of elasticity

Functional bases and separating sets

Functional bases on symmetry strata of elasticity tensors

Conclusion

Low dimensional elasticity symmetry strata

Pertinent invariants? Continuous. On an elasticity symmetry stratum.

Isotropic stratum $\Sigma_{[\mathrm{SO}(3)]}$:	$d = 2$,
Cubic stratum $\Sigma_{[\mathbb{O}]}$:	$d = 3$,
Transversely isotropic stratum $\Sigma_{[\mathrm{O}(2)]}$:	$d = 5$,
Tetragonal stratum $\Sigma_{[\mathbb{D}_4]}$:	$d = 6$,
Trigonal stratum $\Sigma_{[\mathbb{D}_3]}$:	$d = 6$.

Reminder

$$\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H})$$

and

$$\mathbf{d} = \operatorname{tr}_{12}\mathbf{E}, \qquad \mathbf{v} = \operatorname{tr}_{13}\mathbf{E},$$
$$\mathbf{d}_2 = \operatorname{tr}_{13}\mathbf{H}^2, \qquad \mathbf{d}_3 = \operatorname{tr}_{13}\mathbf{H}^3,$$
$$I_2 = \operatorname{tr}\mathbf{d}_2 = \|\mathbf{H}\|^2, \qquad I_3 = \operatorname{tr}\mathbf{d}_3.$$

Functional basis on cubic stratum

An elasticity tensor $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H}) \in \mathbb{E}\mathrm{la}$ is cubic if and only if

$$\mathbf{d}' = \mathbf{v}' = 0, \quad \mathbf{d}_2' = 0 \quad \text{and} \quad I_2 = \operatorname{tr}\mathbf{d}_2 = \|\mathbf{H}\|^2 \neq 0.$$

Let $\mathbf{d}_3 = \operatorname{tr}_{13}\mathbf{H}^3$ and $I_3 = \operatorname{tr}\mathbf{d}_3$.

Theorem (DADOK-2021)

Let $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, 0, 0, \mathbf{H})$ *be a cubic elasticity tensor. A minimal functional basis for* $\Sigma_{[\mathbb{O}]}$ *consists in the 3 rational invariants*

$$\kappa_1 := \operatorname{tr}\mathbf{d}, \quad \kappa_2 := \operatorname{tr}\mathbf{v}, \quad \kappa_3 := \frac{I_3}{I_2}.$$

A transversely isotropic second-order covariant

We define the deviatoric second-order rational covariant

$$\mathbf{t} := 2\frac{(\|\mathbf{d}'\|^2\,\mathbf{d}'^2 + \|\mathbf{v}'\|^2\,\mathbf{v}'^2 + \mathbf{d}_2'^2)'}{\|\mathbf{d}'\|^4 + \|\mathbf{v}'\|^4 + \|\mathbf{d}_2'\|^2} \neq 0$$

for every elasticity tensor which is either trigonal, tetragonal or transversely isotropic, *i.e.* for any

$$\mathbf{E} \in \Sigma_{[\mathbb{D}_3]} \cup \Sigma_{[\mathbb{D}_4]} \cup \Sigma_{[\mathrm{O}(2)]}.$$

Functional basis on transversely isotropic stratum

If $\mathbf{E}$ is transversely isotropic, then, the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ is transversely isotropic, and thus

$$\|\mathbf{d}'\|^4 + \|\mathbf{v}'\|^4 + \|\mathbf{d}_2'\|^2 \neq 0.$$

Theorem (DADOK-2021)

Let $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H}) \in \Sigma_{[\mathrm{O}(2)]}$ *be a transversely isotropic elasticity tensor and*

$$\mathbf{t} = 2\frac{(\|\mathbf{d}'\|^2\,\mathbf{d}'^2 + \|\mathbf{v}'\|^2\,\mathbf{v}'^2 + \mathbf{d}_2'^2)'}{\|\mathbf{d}'\|^4 + \|\mathbf{v}'\|^4 + \|\mathbf{d}_2'\|^2} \in \mathbb{H}^2(\mathbb{R}^3).$$

A minimal functional basis for $\Sigma_{[\mathrm{O}(2)]}$ *consists in the 5 rational invariants*

$$\kappa_1 := \operatorname{tr}\mathbf{d}, \quad \kappa_2 := \operatorname{tr}\mathbf{v}, \quad \kappa_3 := \mathbf{d} : \mathbf{t}, \quad \kappa_4 := \mathbf{v} : \mathbf{t}, \quad \kappa_5 := \mathbf{t} : \mathbf{H} : \mathbf{t}.$$

Functional basis on tetragonal stratum

Given a tetragonal elasticity tensor $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H}) \in \Sigma_{[\mathbb{D}_4]}$, the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ is transversely isotropic and $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ is either cubic or tetragonal.

Theorem (DADOK-2021)

Let $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H})$ *be a tetragonal elasticity tensor and*

$$\mathbf{t} = 2\frac{(\|\mathbf{d}'\|^2\,\mathbf{d}'^2 + \|\mathbf{v}'\|^2\,\mathbf{v}'^2 + \mathbf{d}_2'^2)'}{\|\mathbf{d}'\|^4 + \|\mathbf{v}'\|^4 + \|\mathbf{d}_2'\|^2}.$$

A minimal functional basis for $\Sigma_{[\mathbb{D}_4]}$ *consists in the 6 rational invariants*

$$\kappa_1 := \operatorname{tr}\mathbf{d}, \qquad \kappa_2 := \operatorname{tr}\mathbf{v}, \qquad \kappa_3 := \mathbf{d} : \mathbf{t},$$
$$\kappa_4 := \mathbf{v} : \mathbf{t}, \qquad \kappa_5 := \mathbf{t} : \mathbf{H} : \mathbf{t}, \qquad \kappa_6 := I_2.$$

Elasticity trigonal stratum

Given a trigonal elasticity tensor $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H}) \in \Sigma_{[\mathbb{D}_3]}$, the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}_2')$ is transversely isotropic and $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ is either cubic or trigonal.

Theorem (DADOK-2021)

Let $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H})$ *be a trigonal elasticity tensor and*

$$\mathbf{t} = 2\frac{(\|\mathbf{d}'\|^2\,\mathbf{d}'^2 + \|\mathbf{v}'\|^2\,\mathbf{v}'^2 + \mathbf{d}_2'^2)'}{\|\mathbf{d}'\|^4 + \|\mathbf{v}'\|^4 + \|\mathbf{d}_2'\|^2}.$$

A minimal functional basis for $\Sigma_{[\mathbb{D}_3]}$ *consists in the 6 rational invariants*

$$\kappa_1 := \operatorname{tr}\mathbf{d}, \qquad \kappa_2 := \operatorname{tr}\mathbf{v}, \qquad \kappa_3 := \mathbf{d} : \mathbf{t},$$
$$\kappa_4 := \mathbf{v} : \mathbf{t}, \qquad \kappa_5 := \mathbf{t} : \mathbf{H} : \mathbf{t}, \qquad \kappa_6 = I_2.$$

A polynomial functional basis for elasticity tensors at least tetragonal or trigonal

Theorem (DADOK-2021)

Let $\mathbf{E} = (\operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{v}, \mathbf{d}', \mathbf{v}', \mathbf{H})$ *be an elasticity tensor and*

$$K_1 := \operatorname{tr}\mathbf{d}, \quad L_1 := \operatorname{tr}\mathbf{v}, \quad I_3 := \operatorname{tr}\mathbf{d}_3, \quad K_4 := \|\mathbf{d}'\|^4 + \|\mathbf{v}'\|^4 + \|\mathbf{d}_2'\|^2,$$
$$K_5 := \mathbf{d} : \mathbf{k}_4, \quad L_5 := \mathbf{v} : \mathbf{k}_4, \quad K_9 := \mathbf{k}_4 : \mathbf{H} : \mathbf{k}_4, \quad K_{10} := \|\operatorname{tr}(\mathbf{H} \times \mathbf{k}_4)\|^2.$$

where $\mathbf{k}_4 := (\|\mathbf{d}'\|^2\, \mathbf{d}'^2 + \|\mathbf{v}'\|^2\, \mathbf{v}'^2 + \mathbf{d}_2'^2)'$.

1. *A minimal functional basis for* $\Sigma_{[\mathrm{SO}(3)]} \cup \Sigma_{[\mathbb{O}]} \cup \Sigma_{[\mathrm{O}(2)]} \cup \Sigma_{[\mathbb{D}_4]}$ *(i.e. at least tetragonal elasticity tensors) consists in the 7 polynomial invariants* $K_1, L_1, I_3, K_4, K_5, L_5$ *and* K_9.
2. *A minimal functional basis for* $\Sigma_{[\mathrm{SO}(3)]} \cup \Sigma_{[\mathbb{O}]} \cup \Sigma_{[\mathrm{O}(2)]} \cup \Sigma_{[\mathbb{D}_3]}$ *(i.e. at least trigonal elasticity tensors) consists in the 7 polynomial invariants* $K_1, L_1, I_3, K_4, K_5, L_5$ *and* K_9.
3. *A minimal functional basis for* $\Sigma_{[\mathrm{SO}(3)]} \cup \Sigma_{[\mathbb{O}]} \cup \Sigma_{[\mathrm{O}(2)]} \cup \Sigma_{[\mathbb{D}_4]} \cup \Sigma_{[\mathbb{D}_3]}$ *(i.e. at least tetragonal or trigonal elasticity tensors) consists in the 8 polynomial invariants* $K_1, L_1, I_3, K_4, K_5, L_5, K_9$ *and* K_{10}.

Quelques questions

- Quand est ce que 2 tenseurs $\mathbf{E}_i$ représentent le même matériau ? (Boehler-Kirillov–Onat 1994, ...)
- Quelles sont les classes de symétrie de l'élasticité ? (Forte–Vianello, 1996)
- Comment les caractériser ? (tentative par Bona–Bucataru–Slawinski, 2004, ...)
- Quelle est la distance d'un tenseur expérimental à une classe de symétrie donnée ? (Gazis, 1963, M. François, 1995, ...)
- Quels sont les invariants pertinents du tenseur d'élasticité ? (base d'intégrité par Olive, 2014, ...)
- Extension aux lois de comportement nonlinéaires ? Aux grandes transformations ?

Travaux connexes

- Invariants des formes binaires, en lien avec la cryptographie (R. Lercier ...)
- Travaux sur les décompositions harmoniques (N. Auffray–B. Desmorat), en lien avec les milieux à gradient et les milieux architecturés ...
- Bases d'intégrités (minimales) de n tenseurs d'ordre quelconque en 2D ($G = \mathrm{SO}(2)$ ou $\mathrm{O}(2)$)
- Bases fonctionnelles (minimales) de n tenseurs d'ordre 2 en 3D (avec M. Olive)
- Travaux sur la reconstruction d'un tenseur à l'aide de ses covariants (élasticité – endommagement anisotrope, avec C. Oliver-Leblond et B. Kolev)
- Travaux sur les couplages multiphysiques (Thèse J. Taurines)
- Travaux sur les distances aux classes de symétrie (François–Geymonat–Berthaud (1995-1998), Thèse P. Azzi)
- Géométrisation (de l'anisotropie ?) des grandes transformations (avec B. Kolev)

Application exponentielle (1/2)

- Flot géodésique = solution $\phi(t, x, v_0)$ de l'équation des géodésiques

$$\begin{cases} \partial_t x = v \\ \partial_t v = -\Gamma(v, v) \\ x(t=0) = x_0 \\ v(t=0) = v_0 \end{cases}$$

- Pour un espace affine euclidien $(\mathcal{E}, \mathbf{q})$, $\Gamma(v, v) = 0$ et

$$x = x_0 + t v_0$$

- L'application exponentielle (difféomorphisme local) est le flot géodésique au temps 1, soit

$$\exp_{x_0}(v_0) := \phi(t = 1, x_0, v_0)$$

- Pour un espace affine euclidien,

$$\exp_{x_0}(v_0) = x_0 + v_0$$

Application exponentielle (2/2)

- Sur une variété Riemannienne M,

$$TM \to M \times M$$
$$(x, \delta x) \mapsto (x, \exp_x \delta x)$$

est un difféomorphisme d'un voisinage de la section nulle (0_x) sur un voisinage de la diagonale (x, x).

- Pour un espace affine euclidien $(\mathcal{E}, \mathbf{q})$,

$$T\mathcal{E} = \mathcal{E} \times E \to \mathcal{E} \times \mathcal{E}$$
$$(x, \delta x) \mapsto (x, \exp_x \delta x = x + \delta x)$$

est un difféomorphisme global.

Métriques – Tenseur des déformations

Métrique sur Ω_0 (tenseur de Cauchy Green-droit)

$$\mathbf{C} = \varphi^* \mathbf{q} = \mathbf{F}_\varphi^\star \mathbf{q}\, \mathbf{F}_\varphi = \mathbf{q}\, \mathbf{F}_\varphi^t \mathbf{F}_\varphi, \qquad \mathbf{F}_\varphi = T\varphi$$

Déplacement infinitésimal: $\boldsymbol{u}$

Soit $\varphi(s)$ un chemin de transformation tel que $\varphi(0) = \mathrm{id}$.
$\boldsymbol{u}$, champ de vecteur sur Ω_0, est défini par (Germain–Muller, 1980)

$$\boldsymbol{u}(\mathbf{x}_0) := s\, \frac{\partial}{\partial s}\bigg|_{s=0} \varphi(s, \mathbf{x}_0), \qquad \varphi(s) = \mathrm{id} + \boldsymbol{u} + o(s)$$

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Tenseur des déformations infinitésimales: $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} := s\, \frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2} \left(\varphi(s)^* \mathbf{q} - \mathbf{q}\right) = \frac{1}{2}\, \mathrm{L}_{\boldsymbol{u}}\, \mathbf{q},$$

$$\frac{1}{2} \left(\mathbf{C}(s) - \mathbf{q}\right) = \frac{1}{2} \left(\varphi(s)^* \mathbf{q} - \mathbf{q}\right) = \boldsymbol{\epsilon} + o(s)$$