La voie symplectique à l'analyse modale : application à la réduction de modèle

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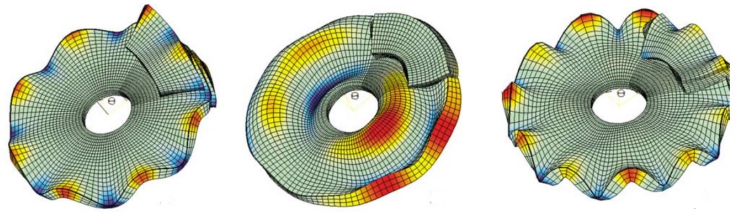
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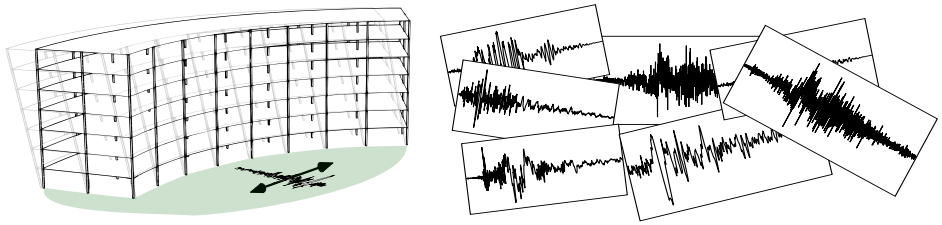


Model order Reduction (MOR)

- The modal analysis coupled with the FEM is widely used by engineers



- Working in high dimension spaces is time-consuming $\longrightarrow$ need to simplify the models



The Proper Orthogonal Decomposition (POD) is one of the most successfully used MOR, based on the metric structure space

- The Proper Symplectic Decomposition (PSD) :
  [Peng & Mohseni 2009] non linear optimization for linear systems
  Our aim is to extract from a data cloud (one or many system evolutions)
  - the leading eigenmodes
  - the principal component of the Hamiltonian

The modal analysis as an equivalence problem

- **The symplectic mechanics in a nutshell**
  (V, ω) symplectic, $(V^*, \{\cdot,\cdot\})$ Poisson
  DOF $\boldsymbol{q}$, momenta $\boldsymbol{p}$, $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} \in V = \mathbb{R}^{2N}$
  $\omega(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{x}^T \boldsymbol{J}\, \boldsymbol{x}' = \boldsymbol{q}^T \boldsymbol{p}' - \boldsymbol{p}^T \boldsymbol{q}'$
  flat isomorphism $\boldsymbol{J} : V \to V^*$
  sharp isomorphism $\boldsymbol{J}^{-1} : V^* \to V$
  Symplectic gradient $\dot{\boldsymbol{x}} = \nabla^\omega h = \boldsymbol{J} \nabla h$
  Hamiltonian $h(\boldsymbol{x})$, Hessian matrix $\boldsymbol{H} \in V^* \otimes V^*$
- **Rewriting the modal analysis**

Equivalence problem

$\exists$ a symplectic matrix $\boldsymbol{S} \in Sp(V)$ mapping $h(\boldsymbol{x})$ and $\boldsymbol{H}$ onto the Hamiltonian $h'(\boldsymbol{x}')$ of a *reduced system of independent harmonic oscillators* and its *diagonal* Hessian matrix $\boldsymbol{H}'$

$$h' = h \circ \boldsymbol{S}, \quad \text{then} \quad \boldsymbol{S}^T \boldsymbol{H}\, \boldsymbol{S} = \boldsymbol{H}' = \begin{bmatrix} diag(k_1, \ldots, k_N) & 0 \\ 0 & diag(g_1, \ldots, g_N) \end{bmatrix}$$

Intertwined eigenproblems

- Hamiltonian matrix $\boldsymbol{H}_\omega = \boldsymbol{J}^{-1}\boldsymbol{H} \in V \otimes V^*$
  The decomposition into columns $\boldsymbol{S} = [\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_N]$
  leads to two **intertwined eigenproblems**

$$\boldsymbol{H}_\omega \boldsymbol{u}_i = k_i \boldsymbol{v}_i, \quad \boldsymbol{H}_\omega \boldsymbol{v}_i = -g_i \boldsymbol{u}_i$$

  that can be transformed into a classical eigenproblem $\boldsymbol{H}_\omega^2 \boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$ where $\lambda_i = -k_i g_i$

Theorem 1

If the Hessian matrix $\boldsymbol{H}$ is positive definite:

- the eigenvalues λ of $\boldsymbol{H}_\omega^2$ are negative.
- the twin eigenvectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ of the eigenproblem are not orthogonal: $\omega(\boldsymbol{u}_i, \boldsymbol{v}_i) \neq 0$.

- for the particular case of the standard modal analysis with a decoupled Hamiltonian

$$h(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1} \boldsymbol{p} + \frac{1}{2}\boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q}, \quad \boldsymbol{u}_i^T = \left[\boldsymbol{a}_i^T, 0^T\right], \quad \boldsymbol{v}_i^T = \left[0^T, (\boldsymbol{M}\boldsymbol{a}_i)^T\right]$$

  we recover the eigenproblem $\boldsymbol{K}\,\boldsymbol{a}_i = \omega_i^2 \boldsymbol{M}\,\boldsymbol{a}_i$ with $\lambda_i = -\omega_i^2$
  and the normalization condition $\omega(\boldsymbol{u}_i, \boldsymbol{v}_i) = \boldsymbol{a}_i^T \boldsymbol{M}\,\boldsymbol{a}_i = 1$
- Our method is more general and allows to treat also cases with terms coupling $\boldsymbol{q}$ and $\boldsymbol{p}$, for instance in problems with Coriolis' force or electromagnetic fields

Symplectic spectral decomposition

From now on, the eigenvectors are scaled to fulfill the normalization condition $\omega(\boldsymbol{u}_i, \boldsymbol{v}_i) = 1$ so as to form a canonical basis

Theorem 2

If $(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_N)$ is a canonical basis,
$\boldsymbol{u}_i^* = -\boldsymbol{J}\,\boldsymbol{u}_i$ and $\boldsymbol{v}_i^* = \boldsymbol{J}\,\boldsymbol{v}_i$,

then $(\boldsymbol{v}_1^*, \cdots, \boldsymbol{v}_N^*, \boldsymbol{u}_1^*, \cdots, \boldsymbol{u}_N^*)$ is its dual basis.

It leads to the **spectral decomposition** of the identity of $\mathbb{R}^{2N}$,
next of the Hamiltonian matrix $\boldsymbol{H}_\omega$ and of the Hessian matrix $\boldsymbol{H}$:

$$\boldsymbol{I}_{2N} = \sum_{i=1}^{N} (\boldsymbol{v}_i \otimes \boldsymbol{u}_i^* + \boldsymbol{u}_i \otimes \boldsymbol{v}_i^*)$$

$$\boldsymbol{H}_\omega = \sum_{i=1}^{N} (g_i \boldsymbol{v}_i \otimes \boldsymbol{u}_i^* - k_i \boldsymbol{u}_i \otimes \boldsymbol{v}_i^*)$$

$$\boldsymbol{H} = \sum_{i=1}^{N} (g_i \boldsymbol{u}_i^* \otimes \boldsymbol{u}_i^* + k_i \boldsymbol{v}_i^* \otimes \boldsymbol{v}_i^*)$$

Proper symplectic Decomposition (PSD)

We hope to address the following problem:
data being given in the form of a structural evolution $[0, T] \to V : t \mapsto \boldsymbol{x}(t)$ (or a concatenation of structural evolutions), we want to construct an approximation of the Hamiltonian

Proposed method:

- $(W_\infty, (. \mid .))$ functional space of functions on $[0, T]$, endowed with the metrics $(f_1 \mid f_2) = \frac{1}{T} \int_0^T f_1(t)\, f_2(t)\, dt$
- $(W_m, (.,.))$ approximation space of dimension m, endowed with the metrics $(\boldsymbol{f}_1, \boldsymbol{f}_2) = \frac{1}{T} \sum_{k=1}^m f_1(t_k)\, f_2(t_k) \Delta t_k, = \frac{1}{2} \boldsymbol{f}_1^T \boldsymbol{G}_t \boldsymbol{f}_2, \quad \boldsymbol{G}_t = \frac{1}{T}\, diag(\Delta t_1, \cdots, \Delta t_m)$
- **covariant snapshot matrix**, built through the isomorphism $\boldsymbol{J} : V \to V^*$

$$\boldsymbol{X}^* = [\boldsymbol{J}\boldsymbol{x}(t_1), \cdots, \boldsymbol{J}\boldsymbol{x}(t_m)] : W_m \to V^*$$

- Inspiring from the SVD, we introduce the positive semi-definite symmetric matrix

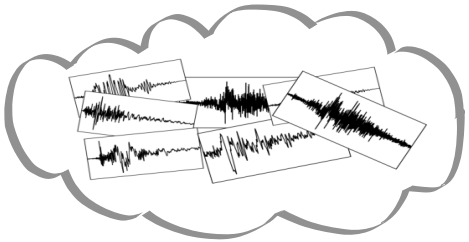
$$\boldsymbol{H} = 2\, \boldsymbol{X}^*\, \boldsymbol{G}_t\, (\boldsymbol{X}^*)^T \in V^* \otimes V^*$$

Remark: in the SVD, the domain and codomain of the snapshot matrix are Euclidean spaces, the codomain is not necessarily of even dimension and Gram's matrix is the identity.
The factor 2 is required because of the factor $1/2$ in the second order term of the Taylor expansion of the Hamiltonian around $\boldsymbol{x} = 0$

Algorithm and convergence properties of the method

Algorithm

data cloud



covariant snapshot matrix

$$X^* = [Jx(t_1), \cdots, Jx(t_m)]$$

reduced Hamiltonian

$$H = 2\, X^*\, G_t\, (X^*)^T$$

apply sharp operator

$$H_\omega = J^{-1} H$$

solve

$$H_\omega^2 u_i = \lambda_i u_i$$

eigenmodes

$$u_i, v_i$$

eigenvalues

$$\lambda_i$$

Convergence properties

$$H = 2\, X^*\, G_t\, (X^*)^T$$

converges to a matrix which allows to find the exact eigenvectors and eigenvalues when

- the time interval of the data increases
- the time discretization is refined
- and the mode combination is balanced

- If some eigenmodes are poorly represented, we find only a reduced Hamiltonian
- If we enrich the data cloud, we converge to the exact Hamiltonian

Convergence properties of the method

Sketch of the proof

- Vibrations of a discrete system with non null initial velocity
  $\boldsymbol{q}(t) = \sum_{i=1}^{N} \alpha_i\, f_i(t)\, \boldsymbol{a}_i, \quad \boldsymbol{p}(t) = \boldsymbol{M}\,\dot{\boldsymbol{q}}(t) = \sum_{i=1}^{N} \alpha_i\, \dot{f}_i(t)\, \boldsymbol{M}\,\boldsymbol{a}_i$
  with $f_i(t) = \sin(\omega_i t)$ and $\dot{f}_i(t) = \omega_i \cos(\omega_i t)$

- Covariant vector $\boldsymbol{x}^*(t) = \sum_{i=1}^{N} \alpha_i \left[\dot{f}_i(t)\, \boldsymbol{v}_i^* - f_i(t)\, \boldsymbol{u}_i^*\right]$
  where $(\boldsymbol{u}_i^*)^T = \left[0^T, \boldsymbol{a}_i^T\right],\ (\boldsymbol{v}_i^*)^T = \left[(\boldsymbol{M}\boldsymbol{a}_i)^T, 0^T\right]$

- Refining the time discretization:
  $\lim\limits_{m\to\infty} \boldsymbol{H} = \lim\limits_{m\to\infty} \frac{2}{T} \sum_{k=1}^{m} \Delta t_k \boldsymbol{x}^*(t_k) \otimes \boldsymbol{x}^*(t_k) = \frac{2}{T} \int_0^T \boldsymbol{x}^*(t) \otimes \boldsymbol{x}^*(t)\, dt$

- It remains: $\lim\limits_{m\to\infty} \boldsymbol{H} = 2 \sum_{i=1}^{N} \alpha_i^2 \left[(f_i \mid f_i)\, \boldsymbol{u}_i^* \otimes \boldsymbol{u}_i^* + (\dot{f}_i \mid \dot{f}_i)\, \boldsymbol{v}_i^* \otimes \boldsymbol{v}_i^*\right]$
  where $(f_i \mid f_i) = \frac{1}{2}\left[1 - \frac{\sin(2\,\omega_i T)}{2\,\omega_i T}\right], \qquad (\dot{f}_i \mid \dot{f}_i) = \frac{\omega_i^2}{2}\left[1 + \frac{\sin(2\,\omega_i T)}{2\,\omega_i T}\right]$

- Finally $\lim\limits_{T\to\infty} \lim\limits_{m\to\infty} \boldsymbol{H} = \sum_{i=1}^{N} \alpha_i^2 \left[\boldsymbol{u}_i^* \otimes \boldsymbol{u}_i^* + \omega_i^2 \boldsymbol{v}_i^* \otimes \boldsymbol{v}_i^*\right]$
  Comparing to the spectral decomposition, we obtain by identification:
  $g_i = \alpha_i^2, \qquad k_i = \alpha_i^2 \omega_i^2, \qquad \lambda_i = -k_i g_i = -\alpha_i^4 \omega_i^2$
  If the mode combination is balanced ($\alpha_i = 1$), we find the exact eigenvalues

Numerical application

Clamped-free elastic bar in traction-compression of length L, cross-section area S, elastic modulus E and mass μ per length unit and displacement field

$$u(x) = \frac{x}{L}\, q_1 + \left(\frac{x}{L}\right)^2 q_2$$

$$\boldsymbol{K} = \frac{E\,S}{L} \begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix}, \qquad \boldsymbol{M} = \mu\, L \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

eigenvalues: $\omega_1^2 = 32.18070, \quad \omega_2^2 = 2.48596$

- **Sample 1: balanced combination of the two modes** $(\alpha_1 = \alpha_2 = 1)$ as data:

$$\boldsymbol{q}(t) = \begin{bmatrix} -6.4220 \\ 8.8665 \end{bmatrix} \sin(5.672\, t) + \begin{bmatrix} -19.586 \\ 8.8665 \end{bmatrix} \sin(1.556\, t)$$

T	m	value of λ_1'	value of λ_2'	error on λ_1'	error on λ_2'
10	10	-31.83290	-2.21129	$1.08\,10^{-2}$	0.11
10	40	-32.23037	-2.46422	$1.54\,10^{-3}$	$8.74\,10^{-3}$
20	100	-32.18026	-2.48581	$1.35\,10^{-5}$	$5.90\,10^{-5}$
$+\infty$	$+\infty$	-32.18070	-2.48596	-	-

Numerical application

- **Sample 2: non balanced combination of the two modes** $(\alpha_1 = \frac{1}{2},\ \alpha_2 = 1)$

$$\boldsymbol{q}(t) = 0.5 \begin{bmatrix} -6.4220 \\ 8.8665 \end{bmatrix} \sin(5.672\, t) + \begin{bmatrix} -19.586 \\ 8.8665 \end{bmatrix} \sin(1.556\, t)$$

T	m	value of λ'_1	value of λ'_2	error on λ'_1	error on λ'_2
20	100	-2.01087	-2.48629	$2.07\,10^{-4}$	$1.34\,10^{-4}$
$+\infty$	$+\infty$	-2.01129	-2.48596	-	-

$\lambda_1 = -\alpha_1^4\omega_1^2 = -\left(\frac{1}{2}\right)^4 \times 32.18070 = -2.01129$

- **Sample 3: only the second eigenmode** $(\alpha_1 = 0, \alpha_2 = 1)$

$$\boldsymbol{q}(t) = \begin{bmatrix} -19.586 \\ 8.8665 \end{bmatrix} \sin(1.556\, t)$$

With $T = 20$ and $m = 100$, we obtain two eigenvalues of multiplicity 2:

$$\lambda'_1 = -5.5511 \cdot 10^{-17}, \qquad \lambda'_2 = -2.48592$$

Merci !

