Relative Gravitation and Continuum Plasticity : (Un)-expected link

L Rakotomanana

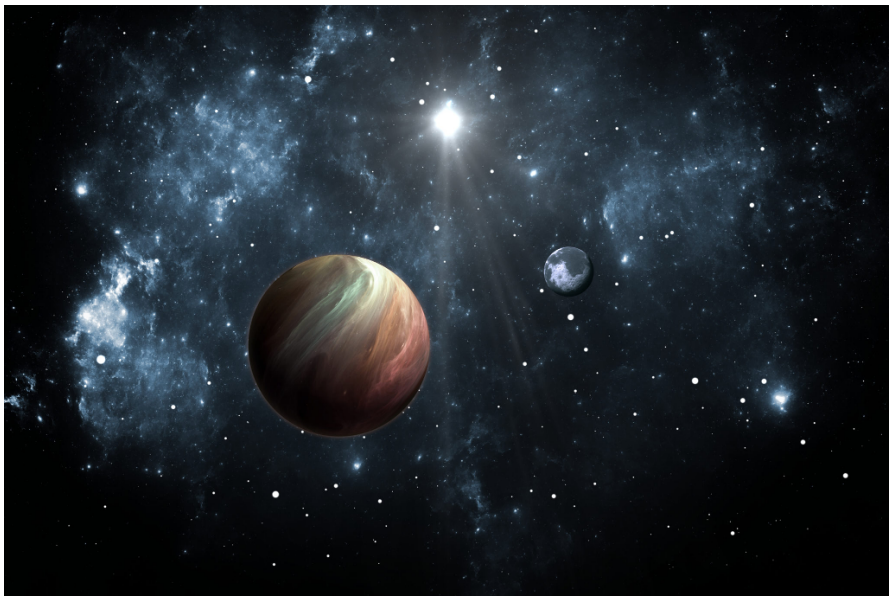
IRMAR, UMR 6625, Université de RENNES 1, France

Mécanique des Milieux Continus et Relativité
La Rochelle, July 06^{th}, 2021

Planning

1. Overview of Einstein Relative Gravitation
2. Historical of plastic deformation
3. Physics of plastic deformation
4. Geometric Foundations of Plastic Deformation
5. Covariant models of Continuum Plasticity
6. Continuum Plasticity as theory of ISV
7. Continuum Plasticity & Relative Gravitation
8. Concluding remarks

I. Overview of Einstein Relative Gravitation



- Spacetime
- Relative Gravitation

Spacetime historical

Space and **time** are the most fundamental concepts of mechanics / physics. Matter would be added.

1. **Newton space and time (1686):** There was absolute 1-continuum time (same for all), absolute **Euclidean** 3-spatial continuum, **but no** absolute rest frame;

2. **Minkowski spacetime (1908):** Space and time fused to form **Minkowskian** flat spacetime to render compatible classical mechanics and electromagnetism;

3. **Einstein spacetime (1915) :** Space and time fused to form a **Riemanian** 4-continuum spacetime, **not an inert entity**,; **Gravitation is a manifestation of spacetime geometry**

Spacetime continuum

1. Spacetime is a **continuum** (i.e. compact and connected) with :
   - a metric tensor $\mathbf{g}$ with components $g_{\alpha\beta}(\mathbf{x})$
   - an affine connection ∇ with coefficients $\Gamma^{\gamma}_{\alpha\beta}(\mathbf{x})$
2. Riemann and Ricci **curvatures** are:

$$\Re^{\lambda}_{\alpha\beta\gamma} := \frac{\partial \Gamma^{\lambda}_{\beta\gamma}}{\partial X^{\alpha}} - \frac{\partial \Gamma^{\lambda}_{\alpha\gamma}}{\partial X^{\beta}} + \Gamma^{\mu}_{\beta\gamma}\Gamma^{\lambda}_{\alpha\mu} - \Gamma^{\mu}_{\alpha\gamma}\Gamma^{\lambda}_{\beta\mu}, \quad \Re_{\alpha\beta} := \Re^{\lambda}_{\lambda\alpha\beta}$$

3. Metric compatible and torsionless (particular case)

$$\overline{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2}\, g^{\gamma\lambda}\left(\frac{\partial g_{\alpha\lambda}}{\partial X^{\beta}} + \frac{\partial g_{\lambda\beta}}{\partial X^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial X^{\lambda}}\right)$$

Newtonian spacetime

1. **Classical spacetime** of mechanics is described by metric $h^{\alpha\beta}$, 1-form τ_α, and a torsionless connection $\Gamma^\gamma_{\alpha\beta}$ satisfying (e.g. Bain 2004) [1]:

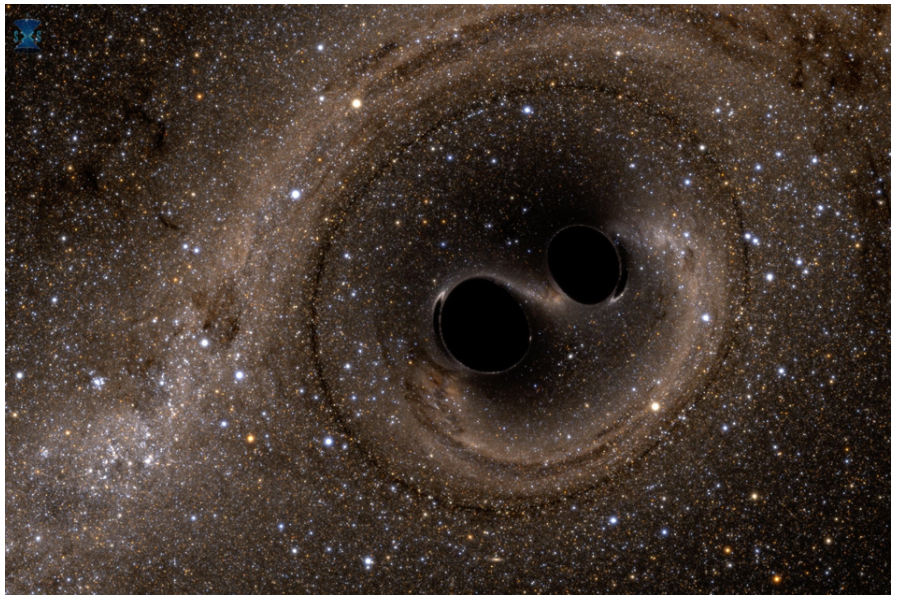
$$\left\{ \begin{array}{rcl} h^{\alpha\beta}\,\tau_\beta & = & 0 \\ \nabla_\gamma h^{\alpha\beta} & = & 0 \\ \Re^\gamma_{\alpha\beta\lambda} & = & 0 \end{array} \right. \quad \text{and} \quad \nabla_\alpha \tau_\beta = 0$$

2. **Elastic continuum** has exactly the same characteristics, the curvature $\Re^\gamma_{\alpha\beta\lambda} \equiv 0$ remains equal to zero regardless of the elastic defomations (e.g. Marsden & Hughes, 1983).

[1] We define the 3-dimensional metric $h_{\alpha\beta}$ and 1-form τ_α instead of $g_{\alpha\beta}$.

Einstein spacetime (Relative Gravitation)





1. The **Hilbert-Einstein** action governs the gravitation:

$$\mathscr{S} := \frac{1}{2\chi} \int_{\mathscr{M}} \overline{\Re}\, \omega_n, \quad \overline{\Re} := g_{\alpha\beta}\, \overline{\Re}^{\alpha\beta}$$

Gravitation in vacuum

1. The variation induces the Einstein's field equation:

$$\Delta\mathscr{S} = \frac{1}{2\chi}\int_{\mathscr{M}}\left(\overline{\Re}^{\alpha\beta} - \frac{\overline{\Re}}{2}g^{\alpha\beta}\right)\delta g_{\alpha\beta}\ \omega_n + \mathrm{BT} = 0$$

2. Einstein field equation (not a conservation laws) holds after **Eulerian variation**:

$$\overline{G}^{\alpha\beta} := \overline{\Re}^{\alpha\beta} - \frac{\overline{\Re}}{2}g^{\alpha\beta} = 0$$

3. **Poincaré's Invariance**. By considering variation as **Lie derivative** along the vector ξ, $\delta g_{\alpha\beta} = \mathcal{L}_\xi g_{\alpha\beta} = \overline{\nabla}_\alpha\xi_\beta + \overline{\nabla}_\beta\xi_\alpha$, we deduce the conservation laws:

$$\overline{\nabla}_\beta\overline{G}^{\alpha\beta} = 0$$

Gravitation in vacuum

1. Einstein spacetime continuum:

$$\left\{ \begin{array}{ccc} g_{\alpha\beta} & \text{such that} & \text{Hilbert's causality satisfied} \\ \overline{\nabla}_\gamma g_{\alpha\beta} & = & 0 \\ \overline{\mathfrak{R}}^\gamma_{\alpha\beta\lambda} & \neq & 0 \end{array} \right.$$

   Torsion $\aleph^\gamma_{\alpha\beta} \neq 0$ may be eventually added for Riemann-Cartan gravitation.

2. The conservation laws equation conforms to the **Bianchi's identity**.

Some remarks

1. **Covariant Lagrangian** for RG (Lovelock 1995) :

$$\mathscr{L}\left(g_{\alpha\beta}, \overline{\Gamma}^{\gamma}_{\alpha\beta}, \partial_\lambda \overline{\Gamma}^{\gamma}_{\alpha\beta}\right) = \mathscr{L}\left(g_{\alpha\beta}, \overline{\Re}^{\gamma}_{\alpha\beta\lambda}\right) \quad \rightarrow \quad \mathscr{L} = \frac{1}{2\chi}\,\Re$$

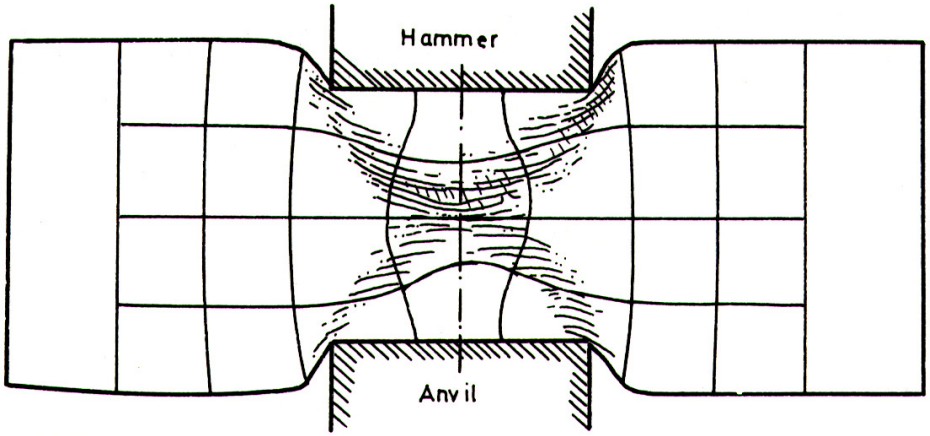
2. Expression of curvature in Einstein Relative Gravitation :

$$\begin{aligned}
\overline{\Re}^{\lambda}_{\alpha\beta\mu} &= (1/2)g^{\lambda\sigma}\left(\partial_\mu\partial_\alpha g_{\sigma\beta} - \partial_\mu\partial_\beta g_{\sigma\alpha} + \partial_\sigma\partial_\beta g_{\mu\alpha} - \partial_\sigma\partial_\alpha g_{\mu\beta}\right) \\
&+ (1/4)g^{\lambda\sigma}\left(\partial_\alpha g_{\sigma\gamma} + \partial_\gamma g_{\alpha\sigma} - \partial_\sigma g_{\alpha\gamma}\right) g^{\gamma\kappa}\left(\partial_\beta g_{\kappa\mu} + \partial_\mu g_{\beta\kappa} - \partial_\kappa g_{\beta\mu}\right) \\
&- (1/4)g^{\lambda\sigma}\left(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\beta\sigma} - \partial_\sigma g_{\beta\gamma}\right) g^{\gamma\kappa}\left(\partial_\alpha g_{\kappa\mu} + \partial_\mu g_{\alpha\kappa} - \partial_\kappa g_{\alpha\mu}\right)
\end{aligned}$$

Spacetime in Relative Gravitation is a **gradient continuum** :

$$\mathscr{L}\left(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}, \partial_\lambda \partial\mu g_{\alpha\beta}\right)$$

II. Historical of Continuum Plasticity



Exemple historique pour déterminer le critère de résistance des métaux (d'après Tresca).

Figure: Shear bands of Tresca (1864).

- Phenomenological Approach
- Multiplicative decomposition
- Geometric approaches
- Plastic deformation : Physical basis

Continuum Plasticity

1. **Plasticity as a science:** Tresca (1864), St-Venant (1870) and Levy (1870), Von Mises (1913)

2. **Plasticity as continuum mechanics:** Prager (1945), Drucker (1947), Hill (1950) ...

3. **Continuum plasticity / small strain theory:** Prager & Hodge (1951), Drucker (1960), Lemaître & Chaboche (1990), ... (Review: Nagdhi 1990)

4. **Internal State Variables:** Moreau (1970), Halphen & Nguyen (1975), Germain, Nguyen & Suquet (1983), ... (Review : Horstmeyer & Bammann 2019)

Phenomenological approach of plastic deformation

1. Green & Nagdhi 1965 : $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p$
   - $\mathbf{E}^p$: primal variable
   - $\phi(\mathbf{E}, \mathbf{E}^p)$: Helmholtz free energy
   - $\mathbf{E} - \mathbf{E}^p$: not necessarily an elastic strain (1966)

   Application : Mostly only used for small deformations (despite Nagdhi 1990)

2. Lee & Liu 1967 : $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$
   - Dislocations & crystallin backgrounds : Kondo (1949), Bilby (1957), Kröner (1959)
   - Incompatible intermediate deformation (non Euclidean configuration)
   - Existence of numerous partitions of rate of deformations : $\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p$, $\mathbf{L} = \mathbf{L}^e + \mathbf{L}^p, \cdots$, and problem of plastic spin.

   Problem : Determination of intermediate configuration, choice of plastic deformation and plastic spin and their objective rates

More on the multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$

1. **Recent results for multiplicative decomposition**
   - Rigorous relation with crystallin plasticity (Reina et al. 2017)
   - Existence and uniqueness of $\mathbf{F}^p$ (Reina & Conti 2017)
   - Advances on rates of deformations : Lubarda (2004), Bruhns (2009), Volokh (2013), ...

2. **Geometrically Necessary Dislocation (GND)**
   - Measure of incompatibility $\mathbf{G} := (\mathrm{Det}\mathbf{F}^p)^{-1}\mathbf{F}^p\ \overline{\mathrm{Rot}}\mathbf{F}^p$ (Cermelli & Gurtin 2001, Reina & Conti 2014)
   - $\mathscr{F}$ invariant iff $\mathscr{F}\left(\mathbf{F}^p, \overline{\nabla}\mathbf{F}^p\right) = \mathscr{F}\left(\mathbf{G}\right)$

Hypotheses : isovolume plastic strain, no plastic spin, strain gradient with micro-stress...

Geometric approach for plastic deformation

1. Continuum elastoplasticity based on defects : Kröner 1963, Le & Stumpf 1996, Acharaya & Bassani, 2000, Yvari & Goriely 2012, ...

2. Elastoplasticity models based on **incompatibility** : Kondo 1968, Steinmann 2015, Lychev 2019, ...

3. Elastoplasticity models based on Riemann-Cartan geometry : Kröner 1961, Noll 1967, Wang 1967, R 1995, Malyshev 2000, Lazar 2002, ...

Search of geometric foundations: RC geometry not sufficiently related with phenomenological approach, considering deformation incompatibilities and experimental observations may be interesting

III. Physics of Plastic Deformation

Main Interest : Conductive line for modeling

- Adiabatic Shear band
- Severe Plastic Deformation

Deformation of material : typical curves

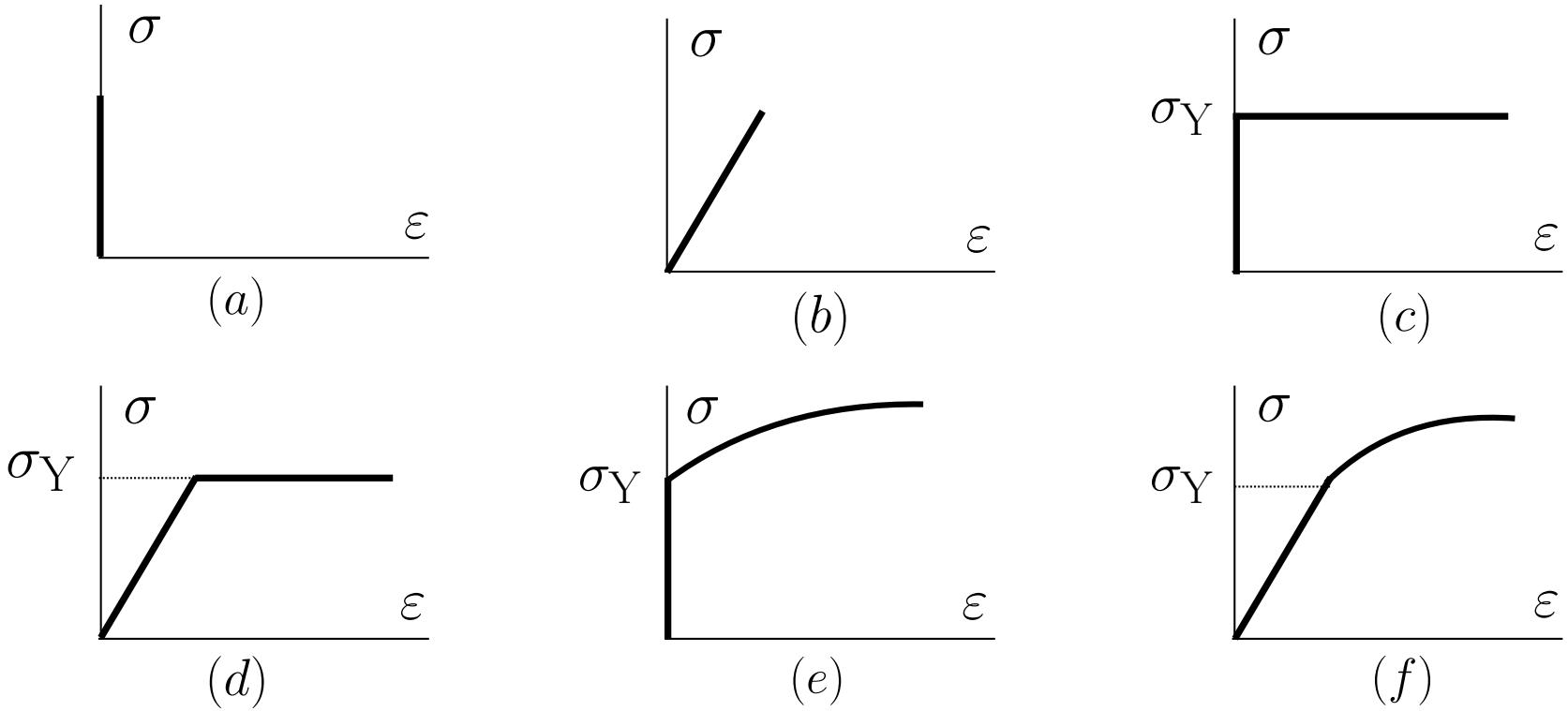


Figure: Stress σ strain ε curves for typical materials : (a) rigid ; (b) elastic; (c) rigid-plastic; (d) elastoplastic; (e) rigid plastic with hardening; and (f) elastoplastic with hardening.

Adiabatic Shear band

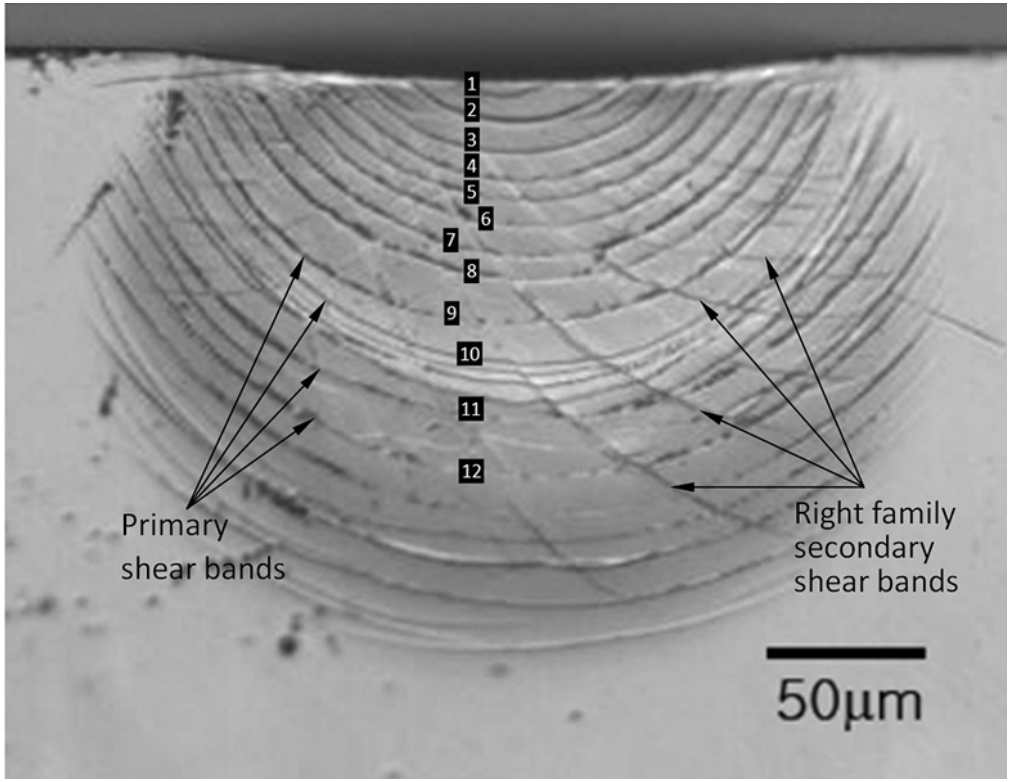


Figure: Identation : Primary Shear Band (PSB) and secondary (SSB) in metallic glass $\ell_c = 50\mu m$. (from Chen & Lin, 2010)

Adiabatic Shear Band

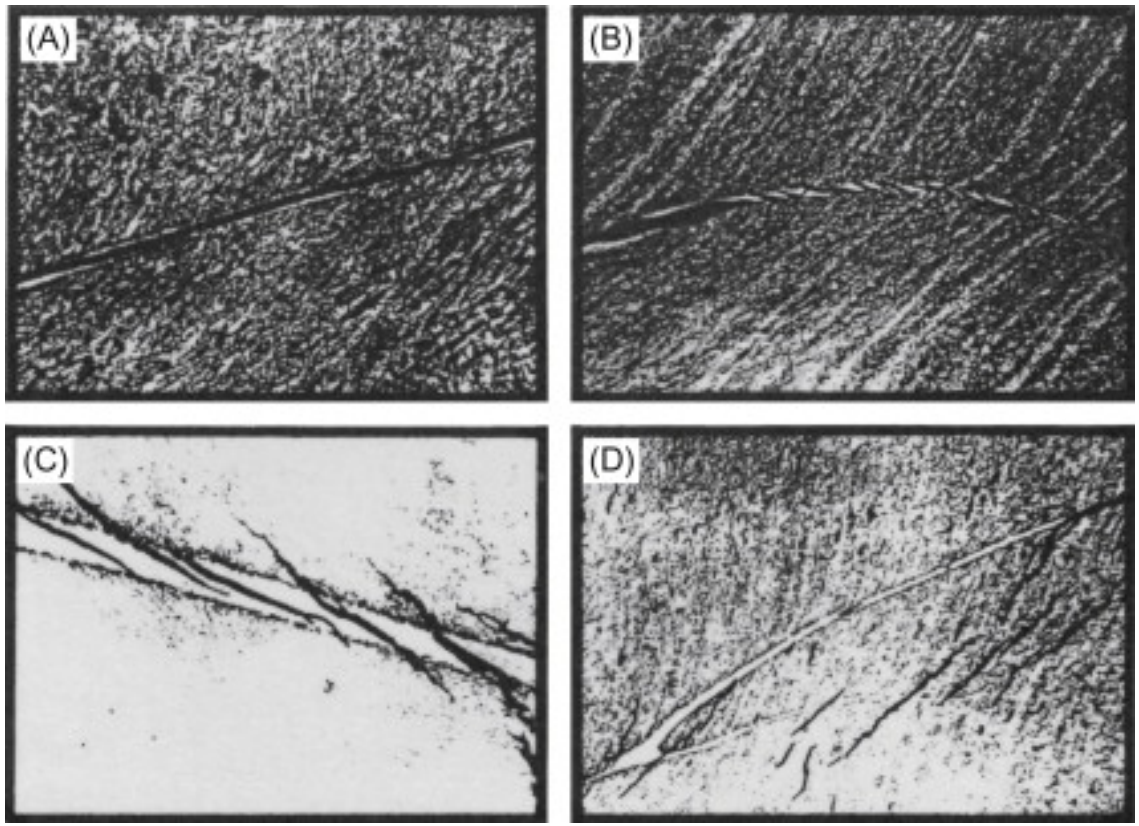


Figure: Several types of brittle fracture of Adiabatic Shear Bands.

Adiabatic Shear Band

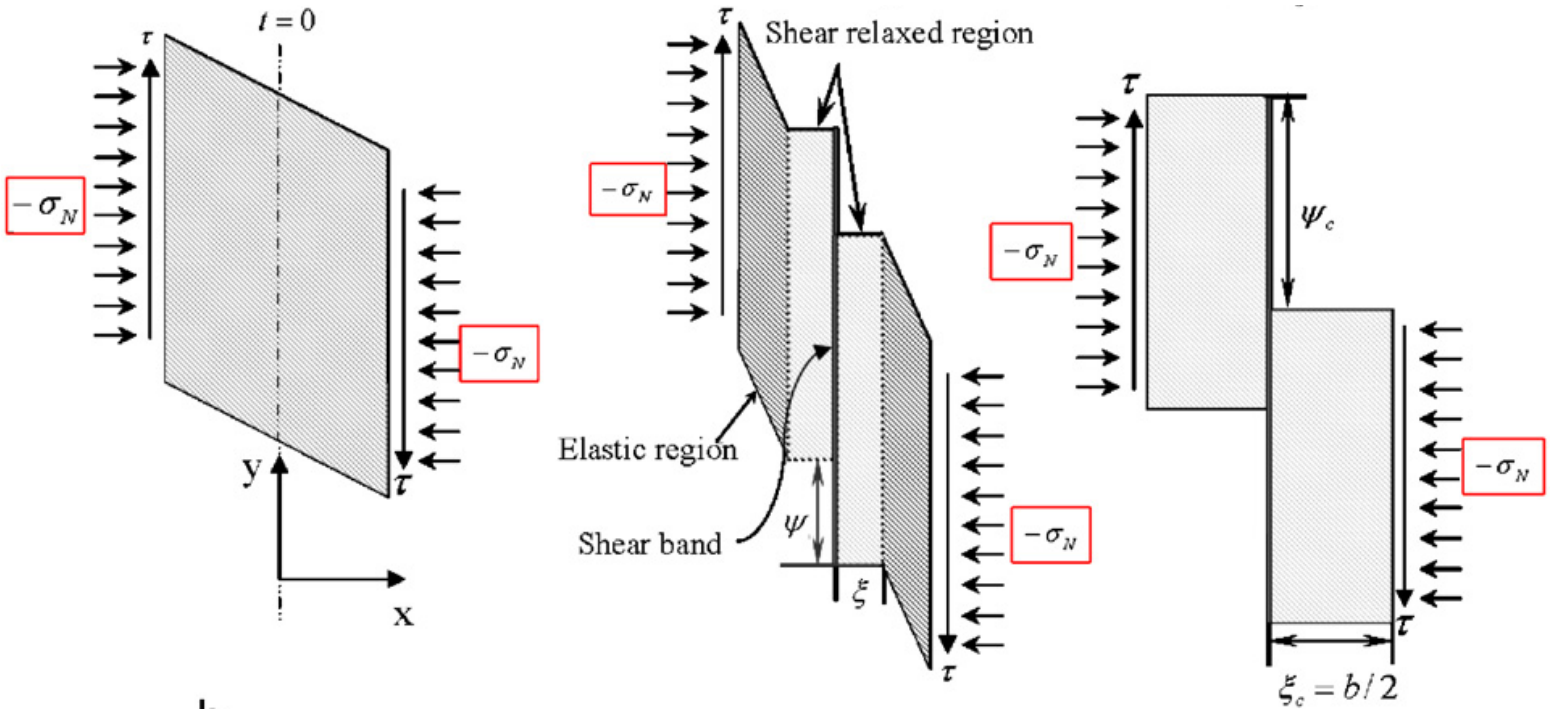


Figure: Formation of Adiabatic Shear Bands depending on the loading rate.

Severe Plastic Deformation

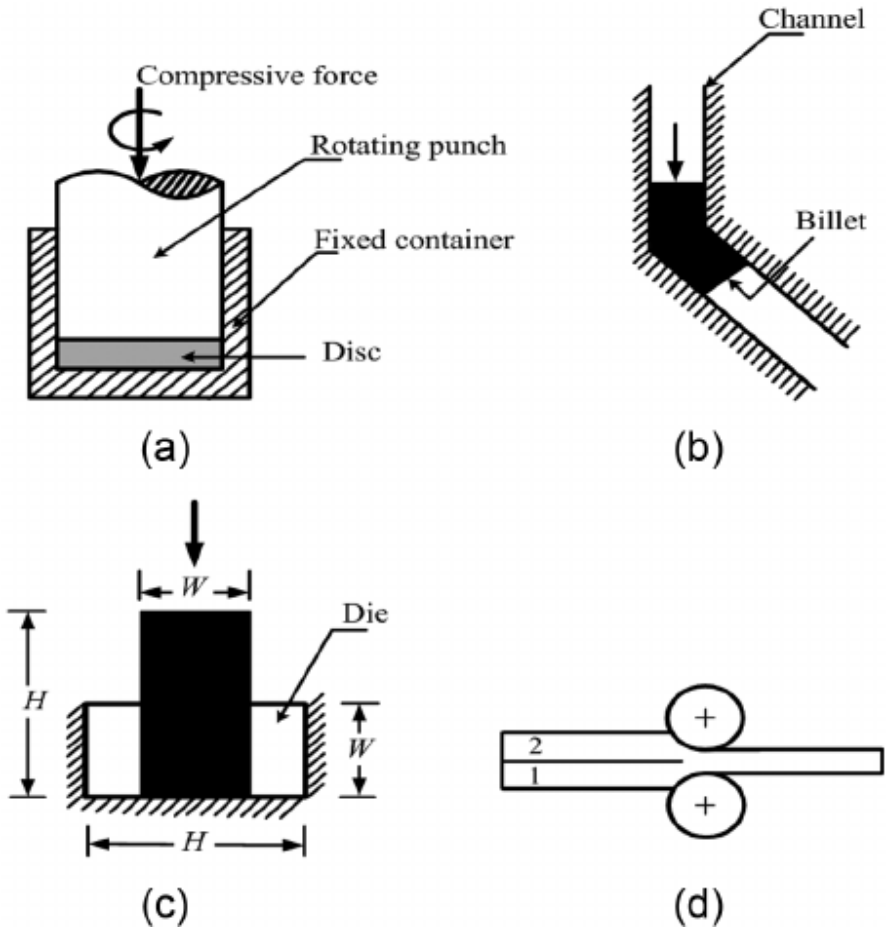


Figure: Material Processing with **Grain Refinement** by Severe Plastic Deformation : Strengthening of material.

Severe Plastic Deformation

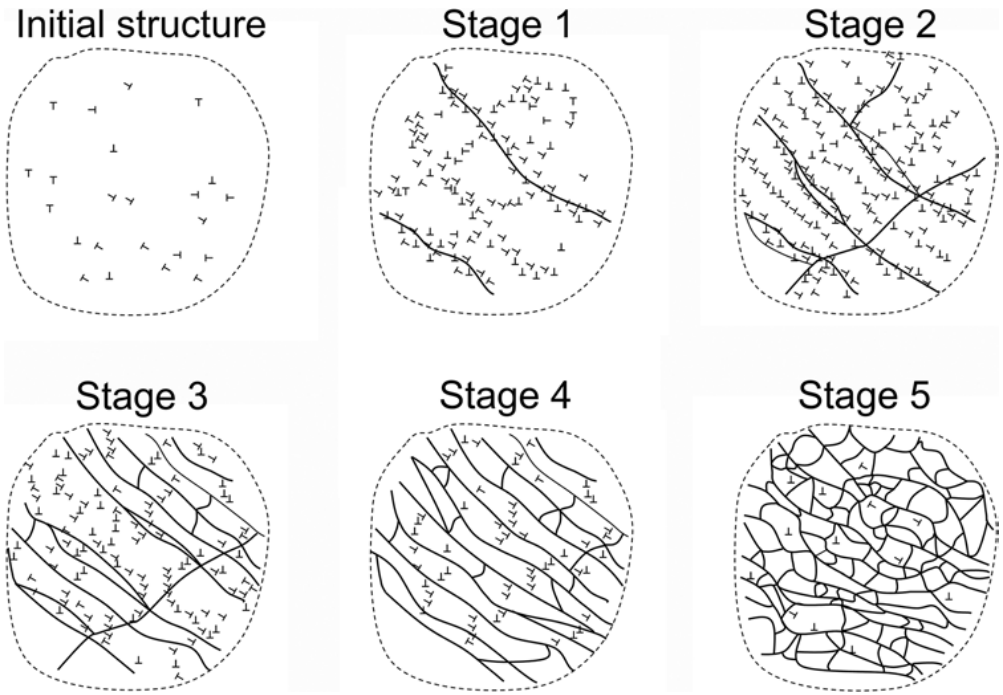


Figure: Grain size refinement under SPD (Cao et al. 2018):
(1) Initial : formation of large size dislocations cell blocks containing dislocations and dislocation cells structures;
(2) Formation of micro-bands and transformation of some nearly dislocations cells into cells blocks;
(3) Formation of lamellar sub-grains containing numerousdislocations;
(4) Formation of well-developed lamellar sub-grains and some equiaxed sub-grains;
(5) Homogeneous distribution of equiaxed ultrafine grains or nano-grains..

Severe Plastic Deformation

1. SPD appears after repetitive cyclic plastic deformation (possibly isovolumic). Significative changes occur at the microstructure level.

2. Physically, SPD in metallic systems is accompanied by **grain refinement** (e.g. Zhylaiev & Langdon 2008), and by generation and organization of **high density of crystal defects** (e.g. Cao et al. 2018).

Conclusion : **Metric** is the essential variable capturing deformation but both the SPD and the ASB involve the deformation **incompatibilities** and **relative motions** of sub-regions (grains) of the body.

IV. Geometric Foundations of Plastic Deformation

Motivation: Definition of the plastic deformation variable from physics

Plastic deformation : standard approach

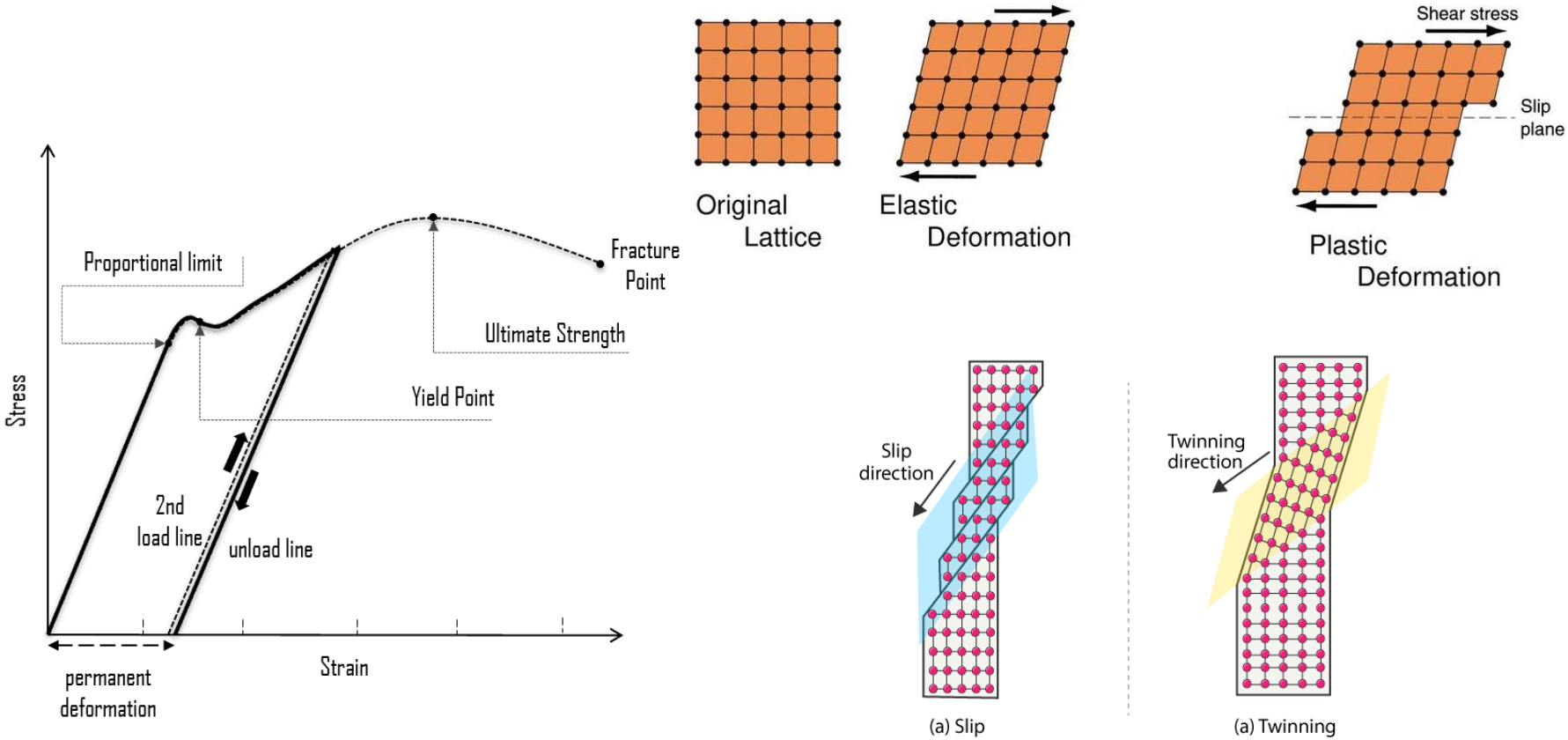


Figure: Stress-strain curve; scheme of superposition for plastic (slip) and elastic deformations, slip and twinning of sample. (Metric approach)

Severe Plastic deformation : Example of HPT

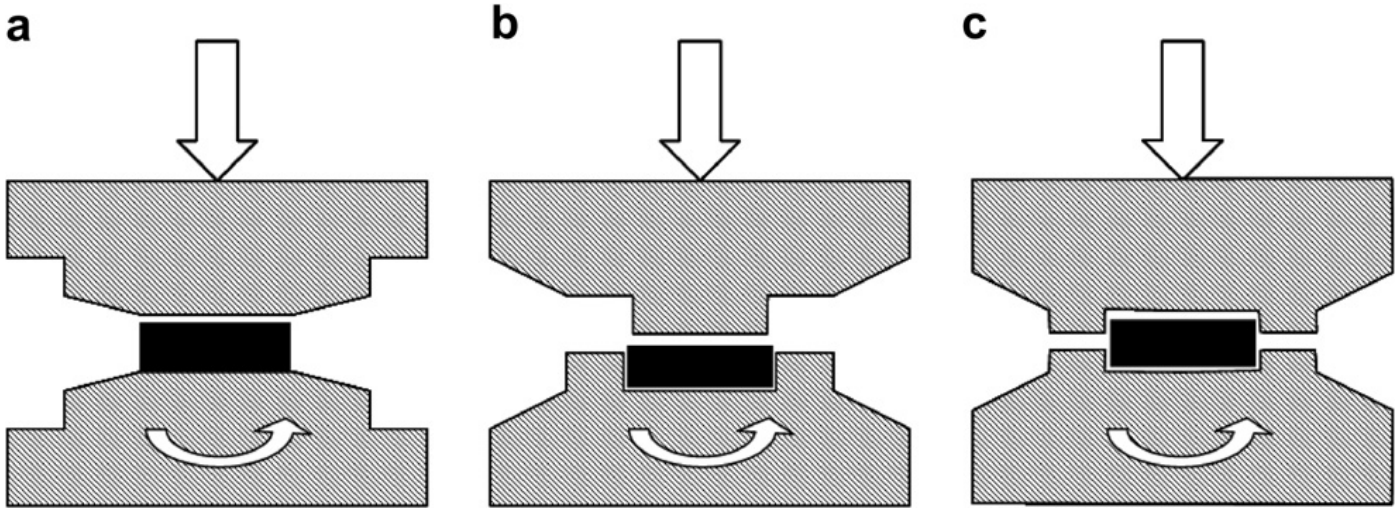


Figure: Hight Pressure Torsion Loadings : (a) Unconstrained; (b) and (c) Constrained.

Exemples of HPT plastification

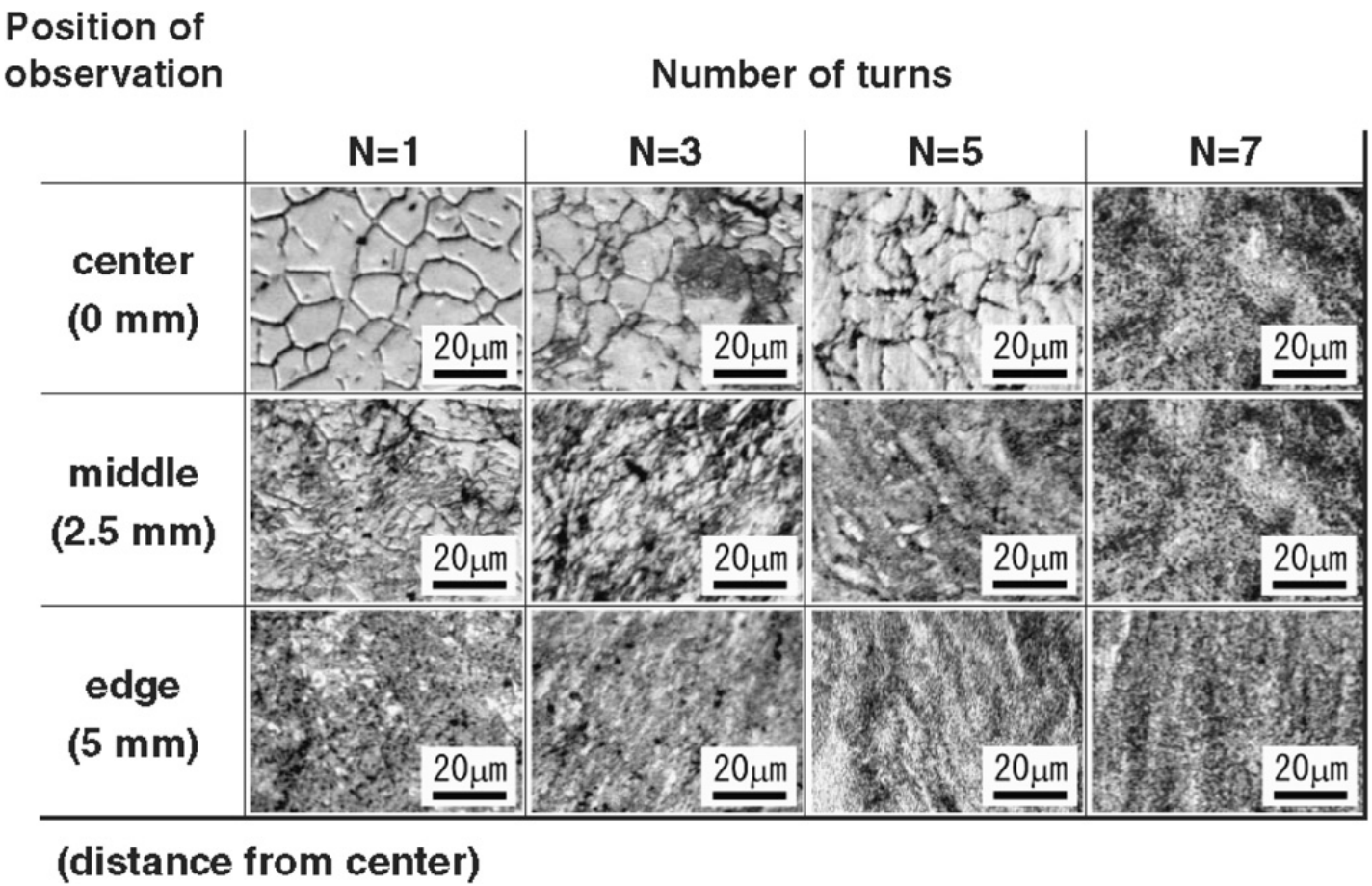


Figure: Optical Micrographs : Microstructures after HPT at the center, half-way position and edge of the disk in a magnesium AZ61 alloy after processing N turns at 423 K (Zhilayev & Langdon 2008).

Plastic deformation : Some conclusions

1. **Twinning** and **relative motions** of grain with refinement are mostly the mechanisms underlying plastic deformation.

2. In addition, **density of dislocation** augments when the plastic deformation increases.

3. **Asympotic response**. For complicated deformation operating in SPD, microstructure mostly reach a steady state at which further deformation does not change the overall microstructure.

V. Covariant models of Continuum Plasticity

Motivation : How to measure sharp gradients (defects and grain refinement) in a continuum plasticity ?

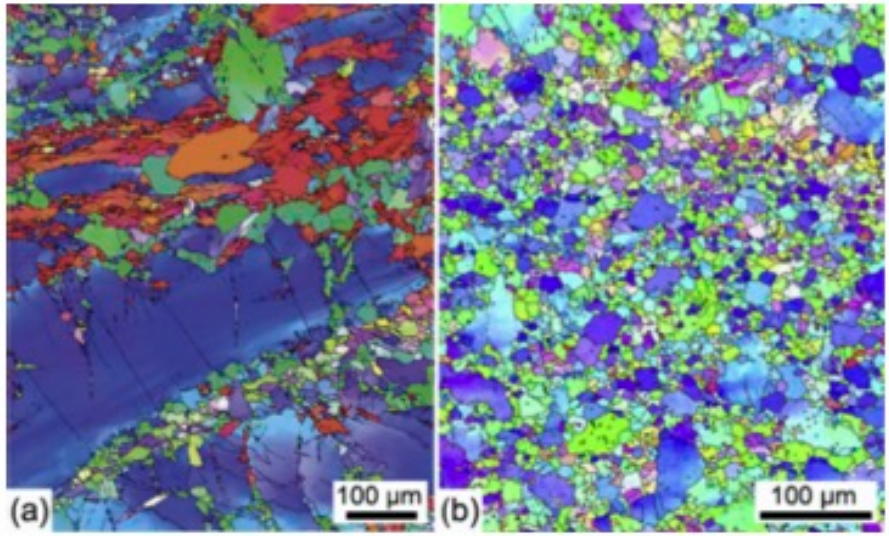
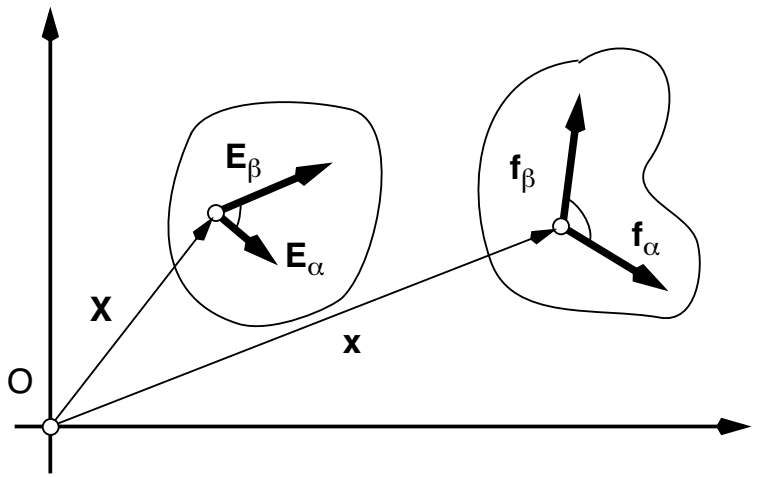


Figure: (a) One Single Crystal Mg after one Equal Canal Angular Pressing (ECAP) (Sedá et al. 2012); (b) a Polycrystalline Mg after four ECAP (Biswas et al. 2010).

Deformation : "Simple Material" continuum (Noll, 1958)



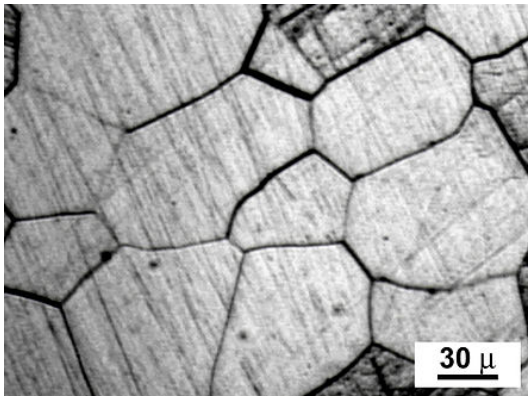
- **Metric** $\mathbf{g}$ measures local deformation : **(length & angle)** :

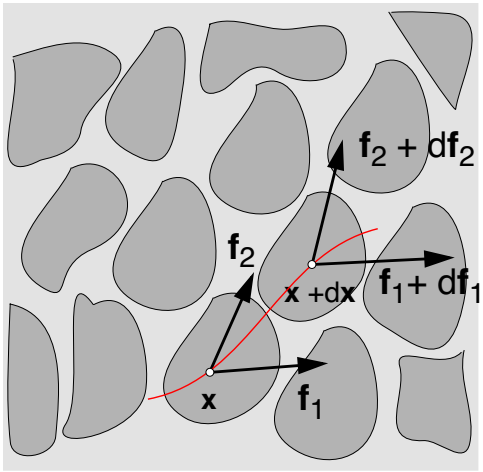
$$g_{\alpha\beta} = \mathbf{g}\left(\mathbf{f}_\alpha, \mathbf{f}_\beta\right) := \mathbf{f}_\alpha \cdot \mathbf{f}_\beta \quad \Longrightarrow \quad \left\{ \begin{array}{rcl} \|\mathbf{f}_\alpha\| & = & \sqrt{\mathbf{f}_\alpha \cdot \mathbf{f}_\alpha} \\ \cos\left(\mathbf{f}_\alpha, \mathbf{f}_\beta\right) & = & \dfrac{\mathbf{f}_\alpha \cdot \mathbf{f}_\beta}{\|\mathbf{f}_\alpha\| . \|\mathbf{f}_\beta\|} \end{array} \right.$$

- **Triads** $F_\alpha^i\left(\mathbf{X}\right)$ defines local mapping $\mathbf{f}_\alpha = F_\alpha^i\ \mathbf{E}_i$

Deformation : Nonhomogeneous Continuum (Noll 1967)

Grains move **relatively each other (discontinuity).**





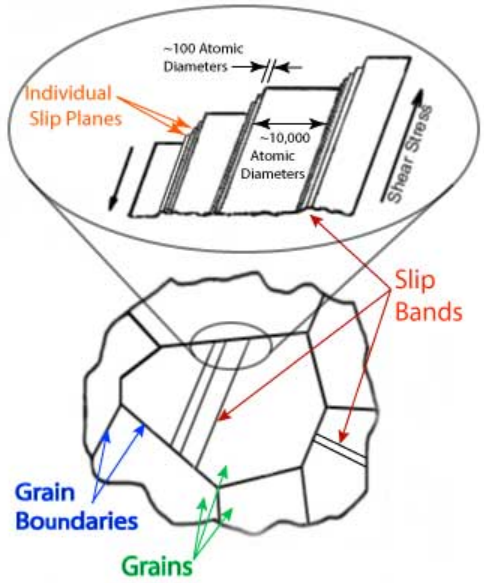


Figure: {**Grains**} (Gonseth, 1929), $d\mathbf{f}_\beta = \nabla_\alpha \mathbf{f}_\beta dx^\alpha$ and $\nabla_\alpha \mathbf{f}_\beta := \Gamma^\gamma_{\alpha\beta} \mathbf{f}_\gamma$

- **Microcosm** deformation is defined by tetrads : $\mathbf{f}_\alpha := F^i_\alpha \, \mathbf{E}_\alpha$
- **Relative motion** of microcosms is defined by **connection** ∇_α with its coefficients : $\Gamma^\gamma_{\alpha\beta} := \mathbf{f}^\gamma (\nabla_\alpha \mathbf{f}_\beta)$.

Dislocations density and relative motions

Consider a scalar field and vector field (Example : density field, velocity field)

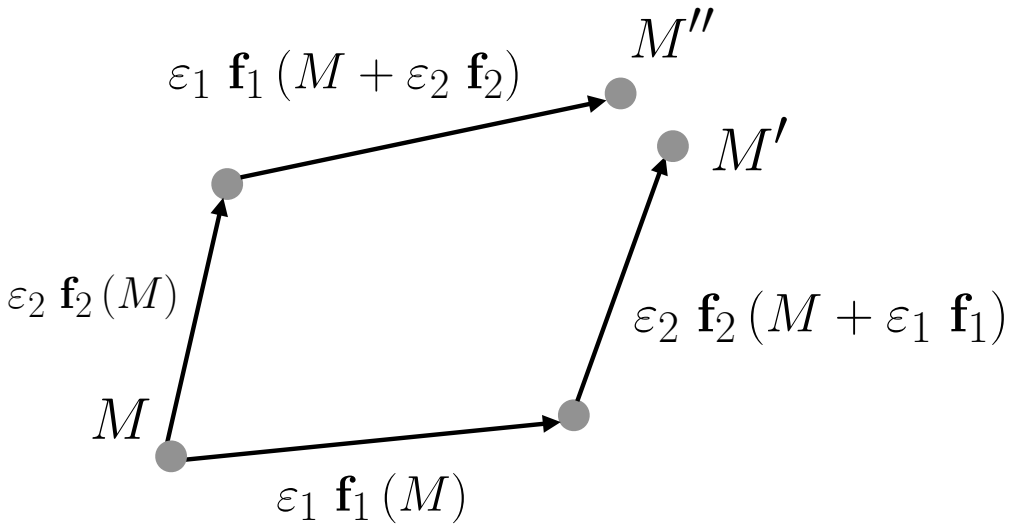


Figure: Parallelogram of Cartan. **Calculus done at point M**.

$\theta\,(\mathbf{x})$, and $\mathbf{w}\,(\mathbf{x})$ are scalar and vector fields on $\mathscr{B}$. We notice $\theta' := \theta\,(M')$, and $\theta'' := \theta\,(M'')$; and $\mathbf{w}' := \mathbf{w}\,(M')$, and $\mathbf{w}'' := \mathbf{w}\,(M'')$.

Density of dislocations and grains relative motions

Theorem

Say an affinely connected manifold $\mathscr{B}$. Let consider two arbitrary vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ at a point M, they define two paths of length ϵ_1 and ϵ_2. Then:

$$\left\{\begin{array}{lcl} \lim\limits_{(\epsilon_1,\epsilon_2)\to 0} \dfrac{(\theta' - \theta'')}{\epsilon_1\epsilon_2} & = & \aleph\left(\mathbf{f}_1, \mathbf{f}_2\right)[\theta] \\ \lim\limits_{(\epsilon_1,\epsilon_2)\to 0} \dfrac{(\mathbf{w}'' - \mathbf{w}')}{\epsilon_1\epsilon_2} & = & \Re\left(\mathbf{f}_1, \mathbf{f}_2, \mathbf{w}\right) - \nabla_{\aleph(\mathbf{f}_1,\mathbf{f}_2)}\mathbf{w} \end{array}\right.$$

Proof (Partly R 1997, complete R 2021) With

$$\begin{array}{lcl} \aleph\left(\mathbf{f}_1, \mathbf{f}_2\right)[\theta] & := & \left(\nabla_{\mathbf{f}_1}\mathbf{f}_2 - \nabla_{\mathbf{f}_2}\mathbf{f}_1\right)[\theta] - [\mathbf{f}_1, \mathbf{f}_2][\theta] \\ \Re\left(\mathbf{f}_1, \mathbf{f}_2, \mathbf{w}\right) & := & \nabla_{\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w} - \nabla_{\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w} - \nabla_{[\mathbf{f}_1,\mathbf{f}_2]}\mathbf{w} \end{array}$$

Sketch of the proof (1)

1) Consider a scalar field $\theta(M)$, we have respectively the relations:

$$\left\{ \begin{array}{lcl} \theta(M_1) & = & \theta(M + \varepsilon_1\ \mathbf{f}_1) = \theta(M) + \nabla_{\varepsilon_1 \mathbf{f}_1(M)} \theta(M) \\ \theta(M') & = & \theta(M_1 + \varepsilon_2 \mathbf{f}_2(M + \varepsilon_1 \mathbf{f}_1)) = \theta(M_1) + \nabla_{\varepsilon_2 \mathbf{f}_2(M+\varepsilon_1 \mathbf{f}_1)}\ \theta(M_1) \end{array} \right.$$

We obtain the value of scalar field at M' in terms of **its value at M**, by noticing $\theta(M') = \theta'$ and $\theta(M) = \theta$, :

$$\begin{array}{lcl} \theta' & = & \theta + \varepsilon_1 \nabla_{\mathbf{f}_1} \theta + \varepsilon_2 \nabla_{\mathbf{f}_2} \theta + \varepsilon_2 \varepsilon_1 \nabla_{\nabla_{\mathbf{f}_1} \mathbf{f}_2} \theta + \varepsilon_2 \varepsilon_1 \nabla_{\mathbf{f}_2} \nabla_{\mathbf{f}_1} \theta \\ & + & \varepsilon_2 \varepsilon_1^2 \nabla_{\nabla_{\mathbf{f}_1} \mathbf{f}_2} \nabla_{\mathbf{f}_1} \theta \end{array} \quad (1)$$

2) Similarly, we also obtain:

$$\begin{array}{lcl} \theta'' & = & \theta + \varepsilon_2 \nabla_{\mathbf{f}_2} \theta + \varepsilon_1 \nabla_{\mathbf{f}_1} \theta + \varepsilon_1 \varepsilon_2 \nabla_{\nabla_{\mathbf{f}_2} \mathbf{f}_1} \theta + \varepsilon_1 \varepsilon_2 \nabla_{\mathbf{f}_1} \nabla_{\mathbf{f}_2} \theta \\ & + & \varepsilon_1 \varepsilon_2^2 \nabla_{\nabla_{\mathbf{f}_2} \mathbf{f}_1} \nabla_{\mathbf{f}_2} \theta \end{array} \quad (2)$$

Sketch of the proof (2)

3) Following the same method, we have, with $\mathbf{w} := \mathbf{w}(M)$,:

$$\begin{aligned} \mathbf{w}' &= \mathbf{w} + \varepsilon_1 \nabla_{\mathbf{f}_1}\mathbf{w} + \varepsilon_2 \nabla_{\mathbf{f}_2}\mathbf{w} + \varepsilon_2\varepsilon_1 \nabla_{\nabla_{\mathbf{f}_1}\mathbf{f}_2}\mathbf{w} + \varepsilon_2\varepsilon_1 \nabla_{\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w} \\ &+ \varepsilon_2\varepsilon_1^2 \nabla_{\nabla_{\mathbf{f}_1}\mathbf{f}_2}\nabla_{\mathbf{f}_1}\mathbf{w} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \mathbf{w}'' &= \mathbf{w} + \varepsilon_2 \nabla_{\mathbf{f}_2}\mathbf{w} + \varepsilon_1 \nabla_{\mathbf{f}_1}\mathbf{w} + \varepsilon_1\varepsilon_2 \nabla_{\nabla_{\mathbf{f}_2}\mathbf{f}_1}\mathbf{w} + \varepsilon_1\varepsilon_2 \nabla_{\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w} \\ &+ \varepsilon_1\varepsilon_2^2 \nabla_{\nabla_{\mathbf{f}_2}\mathbf{f}_1}\nabla_{\mathbf{f}_2}\mathbf{w} \end{aligned} \tag{4}$$

4) Final step:

(1) - (2) induces the first relation on torsion.

(3) - (4) induces the second result on curvature (and torsion). □

Remarque

Calculus is done **exculsively** *at point M (then $\mathscr{T}_M\mathscr{B}$). Without curvature but with torsion, we may have vector field discontinuities.*

Plasticity variables : Some conclusions

1. Plastic deformation consists in local **discontinuities of scalar and vector fields**. Continuum Plasticity requires a **Riemann-Cartan continuum** as geometric foundations.

2. **Torsion** and **curvature** are variables measuring the density of these discontinuities (dislocations density and grain refinement, relative motions of grains) !

3. The question is now if they are :
   1. Internal variables
   2. Primal Variables (as for metric / strain)

Covariance of Lagrangian $\mathscr{L}$ (Antonio & R 2011)

Theorem

The model of continuum defined by the Lagrangian

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \Gamma^{\gamma}_{\alpha\beta},\ \partial_{\lambda}\Gamma^{\gamma}_{\alpha\beta})$$

is **covariant** *if and only if*

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \aleph^{\gamma}_{\alpha\beta},\ \mathfrak{R}^{\gamma}_{\alpha\beta\lambda})$$

Remarks :

- **Primal / internal variables** are metric $g_{\alpha\beta}$, torsion $\aleph^{\gamma}_{\alpha\beta}$, and curvature $\mathfrak{R}^{\gamma}_{\alpha\beta\lambda}$ (**Continuum physics** : elasticity, fluid mechanics, gravitation, electromagnetism, plastic deformation ...)
- This theorem extends **Cartan** (1922) and **Lovelock-Rund** (1971, 1975) theorems from Riemann to **Riemann-Cartan continuum**.

Differential geometry links RG and CP

- **Continuum elasticity**

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta})$$

where $g_{\alpha\beta} := \mathbf{g}\,(\mathbf{f}_\alpha, \mathbf{f}_\beta)$ form the Cauchy-Green tensor.

- Einstein Relative Gravitation

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \mathfrak{R}^{\gamma}_{\alpha\beta\lambda}) \qquad \text{as} \qquad \mathscr{L} = (1/2\chi)\ g^{\alpha\beta}\ \mathfrak{R}^{\lambda}_{\lambda\alpha\beta}$$

extended to **Einstein-Cartan** $\mathscr{L} = \mathscr{L}\left(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \mathfrak{R}^{\gamma}_{\alpha\beta\lambda}\right)$.

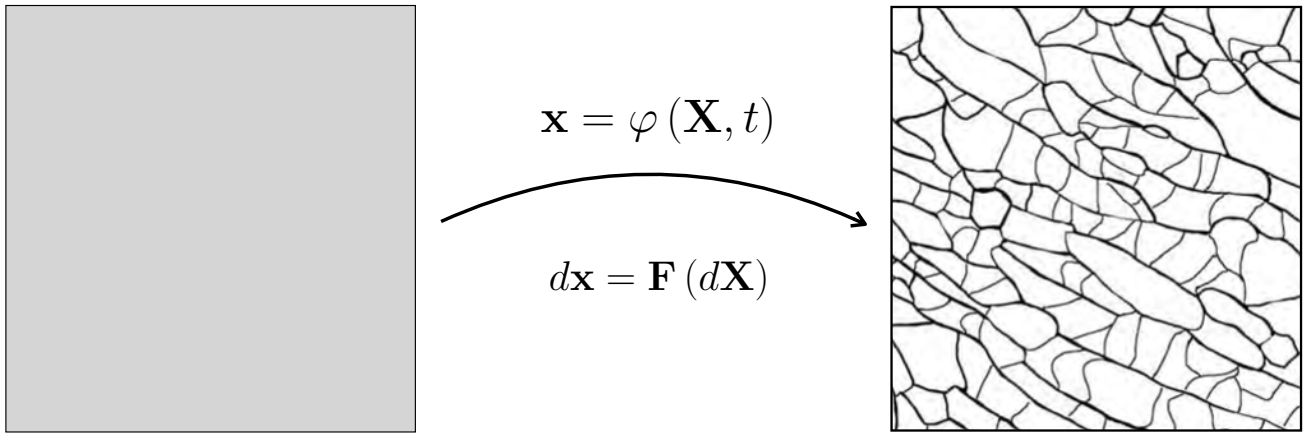
- **Continuum Elastoplasticity** $\mathscr{L} = \mathscr{L}\left(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \mathfrak{R}^{\gamma}_{\alpha\beta\lambda}\right)$

Question: Does metric include part of plastic deformation ?

VI. Continuum Plasticity as Theory of **Internal State Variables** $\mathscr{L} = \mathscr{L}(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \Re^{\lambda}_{\alpha\beta\gamma})$

$g_{\alpha\beta}(\mathbf{X}, 0)$

$g_{\alpha\beta}(\mathbf{X}, t) = F^{\mu}_{\alpha}(\mathbf{X}, t)\ g_{\mu\nu}(\mathbf{X}, 0)\ F^{\nu}_{\beta}(\mathbf{X}, t)$



$$\aleph^{\gamma}_{\alpha\beta}(\mathbf{X}, 0) \equiv 0 \quad \rightarrow \quad \aleph^{\gamma}_{\alpha\beta}(\mathbf{X}, t) \neq 0$$

$$\Re^{\gamma}_{\alpha\beta\lambda}(\mathbf{X}, 0) \equiv 0 \quad \rightarrow \quad \Re^{\gamma}_{\alpha\beta\lambda}(\mathbf{X}, t) \neq 0$$

Figure: Elastoplastic deformation includes : (1) variation of the metric components; (2) variations of the torsion and curvature.

Thermodynamic process (Coleman & Gurtin 1967)

Thermodynamical Process is defined by :

1. spatial position of each point $M \in \mathscr{B}$: $\mathbf{x} = \varphi(\mathbf{X}, t)$;
2. temperature $\theta(M, t)$;
3. stress tensor field $\sigma(M, t)$ and external body force $\rho\mathbf{b}(M, t)$;
4. entropy $s(M, t)$;
5. Helmholtz free energy $\phi(M, t)$;
6. heat flux $\mathbf{q}(M, t)$;
7. heat source $r(M, t)$

8. (Extension) set of **Internal Variables** : torsion $\aleph$ and curvature $\Re$ (Ramaniraka & R 2000, R 2003).

Notion of objective rate of tensor fields (R 1997)

Definition

($\mathscr{B}$-derivative) Let $\mathbf{A}$ be a tensor of type (p, q) on $\mathscr{B}$. The time derivative of $\mathbf{A}$ with respect to $\mathscr{B}$ is a tensor of the same type as $\mathbf{A}$, which satisfies for any p-uplet of vectors $(\mathbf{u}_1, ..., \mathbf{u}_p)$ and for any q-uplet of 1-forms $(\omega^1, ..., \omega^q)$, macroscopically embedded in $\mathscr{B}$,:

$$\frac{d^{\mathscr{B}}\mathbf{A}}{dt}\left(\mathbf{u}_1, ..., \mathbf{u}_p, \omega^1, ..., \omega^q\right) \equiv \frac{d}{dt}[\mathbf{A}(\mathbf{u}_1, \cdots, \mathbf{u}_p, \omega^1, \cdots, \omega^q)]$$

Plastic continuum: The $\mathscr{B}$-derivatives of the geometric variables are given by:

$$\mathbf{D}_g \equiv \frac{1}{2}\frac{d^{\mathscr{B}}\mathbf{g}}{dt} \qquad \mathbf{D}_{\aleph} \equiv \frac{d^{\mathscr{B}}\aleph}{dt} \qquad \mathbf{D}_{\Re} \equiv \frac{d^{\mathscr{B}}\Re}{dt}$$

Definition includes all so-called objective rate of tensor of classical continuum mechanics (and Lie derivatives on Riemann manifolds) !

Constitutive laws of rate-independent plastic Continuum

Theorem

A Plastic Continuum of the rate type $\mathscr{B}$ is defined by constitutive tensor functions : $\Im\{\sigma, \phi, s\}$:

$$\Im = \tilde{\Im}(\mathbf{g}, \aleph, \Re, \mathbf{D}_g, \mathbf{D}_\aleph, \mathbf{D}_\Re)$$

Then the free energy ϕ takes necessarily the form $\phi = \tilde{\phi}(\mathbf{g}, \aleph, \Re)$ and the entropy inequality as:

$$\mathbf{J}_g : \mathbf{D}_g + \mathbf{J}_\aleph : \mathbf{D}_\aleph + \mathbf{J}_\Re : \mathbf{D}_\Re \geq 0$$

with the dual variables defined by:

$$\mathbf{J}_g \equiv \sigma - 2\rho\frac{\partial\phi}{\partial\mathbf{g}} \qquad \mathbf{J}_\aleph \equiv -\rho\frac{\partial\phi}{\partial\aleph} \qquad \mathbf{J}_\Re \equiv -\rho\frac{\partial\phi}{\partial\Re}$$

Proof Apply Coleman and Noll theorem (R 2003)□. To ensure the entropy inequality, dissipation potential function ψ is positive, convex and zero when the rates are equal to zero (Moreau 1970, Germain 1973).

Steps of normal dissipation theory

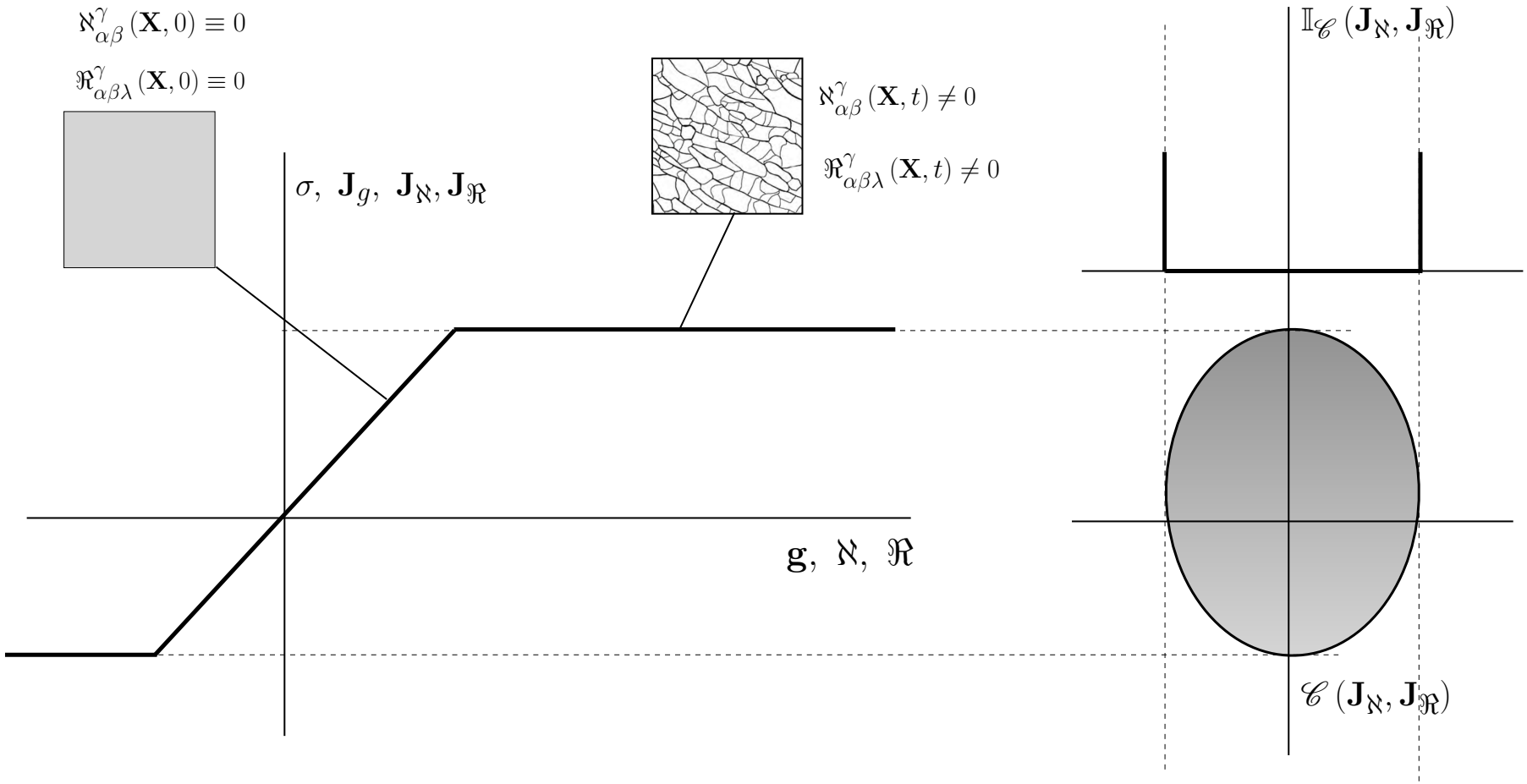


Figure: Normal dissipation : "Stress-strain curve"; Convex set where no nonholonomic variables change $\mathscr{C}$; Indicator function $\mathbb{I}_{\mathscr{C}}$.

Normal Dissipation (Ramaniraka & R 2000)

1. The **Helmholtz free energy** ϕ takes the form of:

$$\phi = \hat{\phi}\left(\mathbf{g}, \aleph, \Re\right)$$

2. The convex, homogeneous of degree one **dissipation potential** $\psi = \hat{\psi}\left(\mathbf{D}_{\aleph}, \mathbf{D}_{\Re}\right)$ vanishes when rates are zero:

$$\begin{aligned} \hat{\psi}^{*}\left(\mathbf{J}_{\aleph}, \mathbf{J}_{\Re}\right) &:= \mathrm{Sup}_{\mathbf{D}_{\aleph}, \mathbf{D}_{\Re}} \left[\mathbf{J}_{\aleph} : \mathbf{D}_{\aleph} + \mathbf{J}_{\Re} : \mathbf{D}_{\Re} - \hat{\psi}\left(\mathbf{D}_{\aleph}, \mathbf{D}_{\Re}\right)\right] \\ &= \begin{cases} 0 & \text{if} \quad \left(\mathbf{J}_{\aleph}, \mathbf{J}_{\Re}\right) \in \mathscr{C} \\ \infty & \text{if} \quad \left(\mathbf{J}_{\aleph}, \mathbf{J}_{\Re}\right) \notin \mathscr{C} \end{cases} \end{aligned}$$

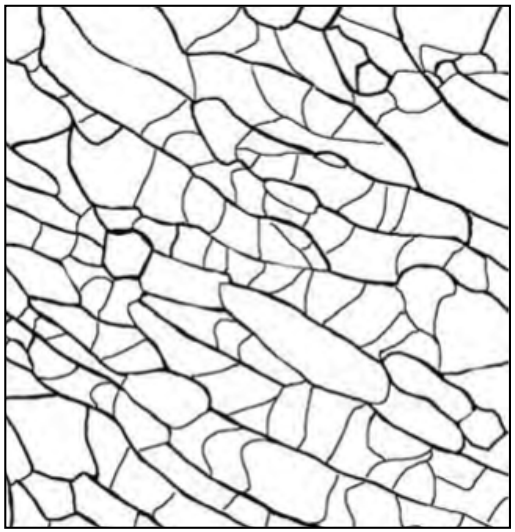
3. Rates of plastic variables are obtained as **plastic flow rule**:

$$\mathbf{D}_{\aleph} \in \partial \hat{\psi}^{*}_{\mathbf{J}_{\aleph}}\left(\mathbf{J}_{\aleph}, \mathbf{J}_{\Re}\right) \quad \text{and} \quad \mathbf{D}_{\Re} \in \partial \hat{\psi}^{*}_{\mathbf{J}_{\Re}}\left(\mathbf{J}_{\aleph}, \mathbf{J}_{\Re}\right)$$

Note: $\mathbf{D}_{\aleph}$ and $\mathbf{D}_{\Re}$ are the covariant analogous of rates $\mathbf{D}^{p} := \dot{\mathbf{F}}^{p}\ \mathbf{F}^{p-1}$ and $\overline{\nabla}\mathbf{D}^{p}$ in the framework of GND e.g. (Cermelli & Gurtin 2001).

VII. Continuum Plasticity & Relative Gravitation

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \Re^{\lambda}_{\alpha\beta\gamma})$$



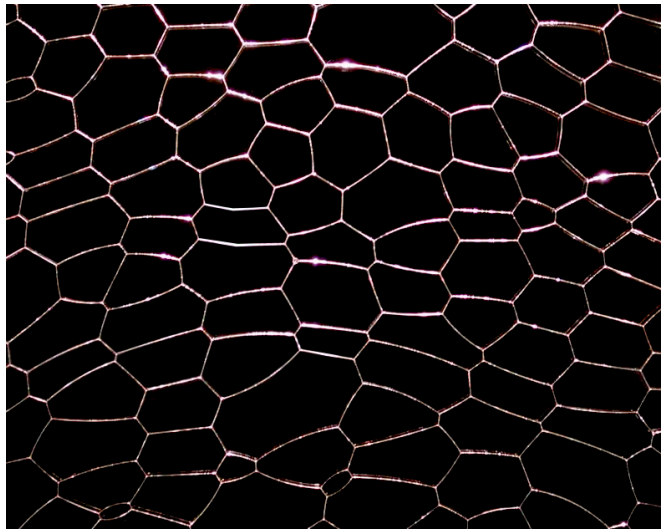


Figure: **Left**: Plastic deformation of (Metallic alloy, set of **microcosms**); **Middle**: Spacetime in Loop Quantum Gravitation (set of "**quanta**").

Note: Torsion and curvature are considered as primal variables

Poincaré's gauge invariance & Conservation laws

For the general case, we have.

1. **Principle of General Covariance.**[2] (e.g. Souriau 1975, Duval & Künzle 1978, R 2018) The basic method is to express variation of action resulting from the **Lie derivatives**, neglecting $\mathscr{O}(d\lambda)$,:

$$\begin{aligned}\Delta\mathscr{S} &= d\lambda\int_{\mathscr{B}}\left(\sigma^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta}+\Sigma^{\alpha\beta}_{\gamma}\mathcal{L}_\xi\aleph^{\gamma}_{\alpha\beta}+\Xi^{\alpha\beta\mu}_{\lambda}\mathcal{L}_\xi\Re^{\lambda}_{\alpha\beta\mu}\right)\omega_n\\ &= 0\end{aligned}$$

2. **Conservation laws** are obtained when the trajectory is shifted while the action is left unchanged. Infinitesimal active diffeomorphisms are defined by **Lie derivative** variations

$$(\mathcal{L}_\xi g_{\alpha\beta},\mathcal{L}_\xi\aleph^{\gamma}_{\alpha\beta},\mathcal{L}_\xi\Re^{\lambda}_{\alpha\beta\mu})$$

[2] This Principle can be extended to Noether-Klein method and furnishes identities in addition to conservation laws by pointing out the variations ξ, $\nabla\xi$, $\nabla\nabla\xi$, ... Lompay & Petrov 2013

Lagrangian of the form $\mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta}, \mathcal{K}^{\gamma}_{\alpha\beta\lambda})$

1. **Modified Lagrangian**. We consider a Lagrangian function with a list of arguments is: $\mathscr{L} := \mathscr{L}\left(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta}, \overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\beta}\right)$ where:

$$\left\{\begin{array}{rcl} \mathcal{T}^{\gamma}_{\alpha\beta} & := & \Gamma^{\gamma}_{\alpha\beta} - \overline{\Gamma}^{\gamma}_{\alpha\beta} \\ \mathcal{K}^{\gamma}_{\alpha\beta\lambda} & := & \overline{\nabla}_{\alpha}\mathcal{T}^{\gamma}_{\lambda\beta} - \overline{\nabla}_{\beta}\mathcal{T}^{\gamma}_{\lambda\alpha} + \left(\mathcal{T}^{\gamma}_{\beta\mu}\mathcal{T}^{\mu}_{\alpha\lambda} - \mathcal{T}^{\gamma}_{\alpha\mu}\mathcal{T}^{\mu}_{\beta\lambda}\right) \end{array}\right.$$

Note: This avoids the use of ∇.

2. **Principle of General Covariance.** For any local translation $\xi(\mathbf{x})$, we should have:

$$\delta\left(\mathscr{L}\omega_n\right) = \left[\sigma^{\alpha\beta}\ \mathcal{L}_{\xi}g_{\alpha\beta} + \Sigma^{\alpha\beta}_{\gamma}\ \mathcal{L}_{\xi}\mathcal{T}^{\gamma}_{\alpha\beta} + \Xi^{\lambda\alpha\beta}_{\gamma}\ \mathcal{L}_{\xi}\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\beta}\right]\omega_n = 0$$

Note: This principle includes the covariance and the gauge invariance conditions.

1. **Constitutive laws.** For Riemann-Cartan continuum, The dual variables are obtained from Lagrangian

$$\sigma^{\alpha\beta} := \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}} - \frac{\mathscr{L}}{2} g^{\alpha\beta}, \quad \Sigma^{\alpha\beta}_{\gamma} := \frac{\partial \mathscr{L}}{\partial T^{\gamma}_{\alpha\beta}}, \quad \Xi^{\lambda\alpha\beta}_{\gamma} := \frac{\partial \mathscr{L}}{\partial \overline{\nabla}_{\lambda} T^{\gamma}_{\alpha\beta}}$$

2. **Correspondence with classical Plasticity Theory.** The internal virtual power is given by (Gurtin & Anand 2005), $\mathbf{F} := \mathbf{F}^e \mathbf{F}^p$, :

$$\delta \mathscr{W}_i := \int_{\mathscr{B}} \mathbf{S}^e : \dot{\mathbf{F}}^e + \mathbf{T}^p : \mathbf{D}^p + \mathbb{T}^p : \nabla \mathbf{D}^p, \qquad \mathbf{D}^p := \dot{\mathbf{F}}^p \ \mathbf{F}^{p-1}$$

with stress, micro-stress and polar stress where conditions are : $\mathbf{D}^p = \mathbf{D}^{p\mathrm{T}}$, $\mathrm{Tr}\mathbf{D}^p \equiv 0$.

Warning: To be covariant $\mathbf{D}^p$ and $\nabla \mathbf{D}^p$ should be merged into the time rate of unique variable $\dot{\mathbf{G}}$ where $\mathbf{G} := (\mathrm{Det}\mathbf{F}^p)^{-1} \mathbf{F}^p \ \overline{\mathrm{Rot}} \mathbf{F}^p$ (Cermelli & Gurtin 2001).

Lagrangian of the form $\mathscr{L}(g_{\alpha\beta})$

Constitutive laws. The "stress" σ is defined as :

$$\sigma^{\alpha\beta} := \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}} - \frac{\mathscr{L}}{2}\, g^{\alpha\beta}$$

Theorem

Consider a Riemann-Cartan continuum $(\mathscr{B}, \mathbf{g}, \nabla)$ *with a Lagrangian function* $\mathscr{L}(g_{\alpha\beta})$. *Then :*

$$\nabla_\alpha \sigma^\alpha_\gamma = 0$$

Proof: 1. Local invariance. The PGC of the Lagrangian holds:

$$\begin{aligned} \Delta\mathscr{L} &:= \int \sigma^{\alpha\beta}\ \mathcal{L}_\xi g_{\alpha\beta}\ \omega_n \\ \mathcal{L}_\xi\ g_{\alpha\beta} &= \xi^\gamma\ \nabla_\gamma g_{\alpha\beta} + g_{\gamma\beta}\ \nabla_\alpha \xi^\gamma + g_{\alpha\gamma}\ \nabla_\beta \xi^\gamma \\ &+ \xi^\gamma \left(g_{\alpha\nu}\ \aleph^\nu_{\gamma\beta} + g_{\nu\beta}\ \aleph^\nu_{\gamma\alpha} \right) \end{aligned}$$

Integrate by parts and shift the boundary terms to obtain:

$$\nabla_\alpha \sigma^\alpha_\gamma + \aleph^\lambda_{\alpha\gamma}\ \sigma^\alpha_\lambda = 0$$

Proof (2)

2. Global Invariance. By considering a uniform vector field ξ such that $\nabla\xi \equiv 0$, we obtain an **identity**:

$$\sigma^{\alpha\beta}\left(g_{\alpha\nu}\ \aleph^{\nu}_{\gamma\beta} + g_{\nu\beta}\ \aleph^{\nu}_{\gamma\alpha}\right) = 2\ \sigma^{\alpha}_{\mu}\ \aleph^{\mu}_{\gamma\alpha} \equiv 0$$

We deduce the classical continuum conservation laws:

$$\nabla_{\alpha}\sigma^{\alpha}_{\gamma} = 0 \quad \square$$

- The implicit presence of the **torsion** within the covariant derivative should be accounted for (Futhazar et al. 2014).
- Other conservation laws or identities may exist according to the choice of the gauge field $\xi\,(\mathbf{x})$. (e.g. Lompay & Petrov 2013)

Lagrangian of the form $\mathscr{L} := \mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta})$

1. **Constitutive laws**. We remind the two dual variables :

$$\sigma^{\alpha\beta} := \frac{\partial \mathscr{L}}{\partial g_{\alpha\beta}} - \frac{\mathscr{L}}{2} g^{\alpha\beta}, \qquad \Sigma^{\alpha\beta}_{\gamma} := \frac{\partial \mathscr{L}}{\partial \mathcal{T}^{\gamma}_{\alpha\beta}}$$

   For RG, dual variables $\sigma^{\alpha\beta}$ and $\Sigma^{\alpha\beta}_{\gamma}$ are respectively called energy-momentum and hypermomentum (Hehl et al. 1976).

2. **Conservations laws**.

Theorem

Consider a Riemann-Cartan continuum $(\mathscr{B}, \mathbf{g}, \nabla)$ *with a Lagrangian function* $\mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta})$. *Then :*

$$\nabla_{\beta} \tilde{\sigma}^{\beta}_{\rho} = 0$$

in which we have defined the **generalized stress** $\tilde{\sigma}^{\beta}_{\rho}$ *:*

$$\tilde{\sigma}^{\beta}_{\rho} := \sigma^{\beta}_{\rho} + \frac{1}{2} \left(-\Sigma^{\alpha\gamma}_{\rho} \, \mathcal{T}^{\beta}_{\alpha\gamma} + \Sigma^{\beta\alpha}_{\gamma} \, \mathcal{T}^{\gamma}_{\rho\alpha} + \Sigma^{\alpha\beta}_{\gamma} \, \mathcal{T}^{\gamma}_{\alpha\rho} \right)$$

Proof (1)

1. Invariance of Poincaré.

1. Express the gauge invariance as Lie derivative variation:

$$\delta\left(\mathscr{L}\overline{\omega}_n\right) = \left(\sigma^{\alpha\beta}\ \mathcal{L}_\xi g_{\alpha\beta} + \Sigma^{\alpha\beta}_{\gamma}\mathcal{L}_\xi \mathcal{T}^{\gamma}_{\alpha\beta}\right)\overline{\omega}_n$$

as active diffeomorphism. The integrand should vanishes.

2. Compute the Lie derivatives of metric and contortion (tedious !)

$$\left\{\begin{array}{lcl} \mathcal{L}_\xi g_{\alpha\beta} & = & \overline{\nabla}_\alpha\xi_\beta + \overline{\nabla}_\beta\xi_\alpha + g_{\alpha\gamma}\mathcal{T}^{\gamma}_{\rho\beta}\xi^\rho + g_{\gamma\beta}\mathcal{T}^{\gamma}_{\rho\alpha}\xi^\rho \\ \mathcal{L}_\xi \mathcal{T}^{\gamma}_{\alpha\beta} & = & \xi^\rho\overline{\nabla}_\rho\mathcal{T}^{\gamma}_{\alpha\beta} - \mathcal{T}^{\rho}_{\alpha\beta}\overline{\nabla}_\rho\xi^\gamma + \mathcal{T}^{\gamma}_{\rho\beta}\overline{\nabla}_\alpha\xi^\rho + \mathcal{T}^{\gamma}_{\alpha\rho}\overline{\nabla}_\beta\xi^\rho \end{array}\right.$$

3. We write the integrand (terms in brackets):

$$\begin{array}{rcl} \delta_\xi\mathscr{L} & = & \sigma^{\alpha\beta}\left(\overline{\nabla}_\alpha\xi_\beta + \overline{\nabla}_\beta\xi_\alpha\right) + \sigma^{\alpha\beta}\left(g_{\alpha\gamma}\mathcal{T}^{\gamma}_{\rho\beta} + g_{\gamma\beta}\mathcal{T}^{\gamma}_{\rho\alpha}\right)\xi^\rho \\ & + & \left(\Sigma^{\alpha\beta}_{\gamma}\overline{\nabla}_\rho\mathcal{T}^{\gamma}_{\alpha\beta}\right)\xi^\rho \\ & + & \Sigma^{\alpha\beta}_{\gamma}\left(-\mathcal{T}^{\rho}_{\alpha\beta}\overline{\nabla}_\rho\xi^\gamma + \mathcal{T}^{\gamma}_{\rho\beta}\overline{\nabla}_\alpha\xi^\rho + \mathcal{T}^{\gamma}_{\alpha\rho}\overline{\nabla}_\beta\xi^\rho\right) = 0 \end{array}$$

It should vanish for **all possible diffeomorphisms** (as for PVP).

Proof (2)

2. Global Invariance. By considering uniform field of gauge, we easily deduce an **identity**:

$$\sigma^{\alpha\beta}\left(g_{\alpha\nu}\mathcal{T}^{\nu}_{\rho\beta}+g_{\nu\beta\nu}\mathcal{T}^{\nu}_{\rho\alpha}\right)+\Sigma^{\alpha\beta}_{\gamma}\overline{\nabla}_{\rho}\mathcal{T}^{\gamma}_{\alpha\beta}=0$$

3. Local Invariance. We consider infinitesimal (non uniform) translations $\xi(\mathbf{x})$, factorize by $\overline{\nabla}_{\beta}\xi^{\rho}$, integrate by part, and drop all terms in divergence (for the sake of the simplicity) and obtain:

$$\overline{\nabla}_{\beta}\left(2\,\sigma^{\beta}_{\rho}-\Sigma^{\alpha\gamma}_{\rho}\,\mathcal{T}^{\beta}_{\alpha\gamma}+\Sigma^{\beta\alpha}_{\gamma}\,\mathcal{T}^{\gamma}_{\rho\alpha}+\Sigma^{\alpha\beta}_{\gamma}\,\mathcal{T}^{\gamma}_{\alpha\rho}\right)=0$$

We deduce the result. □

Remark: Other forms of conservation laws and identities may exist depending on the choice of gauge field $\xi(\mathbf{x})$ (e.g. Lompay & Petrov 2013) (uniform, linear, nonlinear, ...)

Some remarks (1)

1. **Literature**. Similar equations were obtained in the past obtained for Relative Gravitation e.g. (Souriau 1975, Hehl et al. 1975, Lompay & Petrov 2013, ...), and in Newton-Cartan Gravitation in e.g. (Duval & Künzle, 1978)

2. The **generalized stress tensor** $\tilde{\sigma}_{\rho}^{\beta}$ extends the classical Cauchy symmetric stress σ_{ρ}^{β}.

3. The generalized stress includes two contributions : "macro" σ_{ρ}^{α} and "micros" $\Sigma_{\gamma}^{\alpha\beta}\mathcal{T}_{\rho\beta}^{\gamma}$ due to change of grain structure and dislocations density due to plastic deformation. Generalized stress is not necessarily symmetric.

Some remarks (on the hyperstress / hypermomentum)

The hypermomentum / hyperstress $\Sigma_\gamma^{\alpha\beta}$ consists in three terms (Mindlin 1964, Hehl et al 1976, 1977, Gordeeva et al. 2010):

$$\Sigma_\gamma^{\alpha\beta} = \Sigma_\gamma^{[\alpha,\beta]} + \frac{\delta_\gamma^\alpha}{n}\,\overline{\Sigma}^\beta + \overline{\Sigma}_\gamma^{(\alpha,\beta)}$$

- Spin-angular momentum / rotatory hyperstress

$$\Sigma_\gamma^{[\alpha,\beta]} \qquad \text{skew symmetric part}$$

- Proper hypermomentum / proper hyperstress

$$\Sigma_\gamma^{(\alpha,\beta)} \qquad \text{symmetric part} \quad \text{and} \quad \Sigma_\alpha^{(\alpha,\beta)} = \overline{\Sigma}^\beta \quad \text{dilatational}$$

- Traceless proper hypermomentum / proper hyperstress

$$\overline{\Sigma}_\gamma^{(\alpha,\beta)} := \Sigma_\gamma^{(\alpha,\beta)} - \frac{\delta_\gamma^\alpha}{n}\,\overline{\Sigma}^\beta$$

Lagrangian of the form $\mathscr{L} := \mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta}, \overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\beta})$

Theorem

Consider a Riemann-Cartan continuum $(\mathscr{B}, \mathbf{g}, \nabla)$ *with a Lagrangian function* $\mathscr{L} := \mathscr{L}(g_{\alpha\beta}, \mathcal{T}^{\gamma}_{\alpha\beta}, \overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\beta})$. *Then :*

$$\nabla_{\beta}\tilde{\sigma}^{\beta}_{\rho} = 0$$

with the **generalized stress** $\tilde{\sigma}^{\beta}_{\rho}$ *:*

$$\begin{aligned}\tilde{\sigma}^{\beta}_{\rho} &:= \sigma^{\beta}_{\rho} + \frac{1}{2}\left(-\Sigma^{\alpha\gamma}_{\rho}\,\mathcal{T}^{\beta}_{\alpha\gamma} + \Sigma^{\beta\alpha}_{\gamma}\,\mathcal{T}^{\gamma}_{\rho\alpha} + \Sigma^{\alpha\beta}_{\gamma}\,\mathcal{T}^{\gamma}_{\alpha\rho}\right)\\ &+ \frac{1}{2}\left(-\Xi^{\lambda\alpha\gamma}_{\rho}\overline{\nabla}_{\lambda}\mathcal{T}^{\beta}_{\alpha\gamma} + \Xi^{\beta\alpha\lambda}_{\gamma}\overline{\nabla}_{\rho}\mathcal{T}^{\gamma}_{\alpha\lambda} + \Xi^{\lambda\beta\alpha}_{\gamma}\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\rho\alpha} + \Xi^{\lambda\alpha\beta}_{\gamma}\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\rho}\right)\end{aligned}$$

- The **generalized stress** includes contributions : σ^{β}_{ρ}, $\Sigma^{\alpha\beta}_{\gamma}\mathcal{T}^{\gamma}_{\rho\beta}$, and $\Xi^{\lambda\alpha\beta}_{\gamma}\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\rho\beta}$ where $\Xi^{\lambda\alpha\beta}_{\gamma}$ corresponds to (mass) quadrupole moment in RG (Dixon 1970, Souriau 1974, Bayley & Israel 1975).
- The same form of **conservation laws** is obtained.

Proof (1)

- Express the active variation of Lagrangian:

$$\delta\mathscr{L} = \sigma^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} + \Sigma^{\alpha\beta}_\gamma \mathcal{L}_\xi \mathcal{T}^\gamma_{\alpha\beta} + \Xi^{\lambda\alpha\beta}_\gamma \mathcal{L}_\xi \overline{\nabla}_\lambda \mathcal{T}^\gamma_{\alpha\beta}$$

  with the Lie derivatives $\mathcal{L}_\xi g_{\alpha\beta}$ and $\mathcal{L}_\xi \mathcal{T}^\gamma_{\alpha\beta}$.

- Compute the Lie derivative of the covariant derivative:

$$\begin{aligned}\mathcal{L}_\xi \overline{\nabla}_\lambda \mathcal{T}^\gamma_{\alpha\beta} &= \xi^\rho \overline{\nabla}_\rho \overline{\nabla}_\lambda \mathcal{T}^\gamma_{\alpha\beta} - \overline{\nabla}_\lambda \mathcal{T}^\rho_{\alpha\beta} \overline{\nabla}_\rho \xi^\gamma \\ &+ \overline{\nabla}_\rho \mathcal{T}^\gamma_{\alpha\beta} \overline{\nabla}_\lambda \xi^\rho + \overline{\nabla}_\lambda \mathcal{T}^\gamma_{\rho\beta} \overline{\nabla}_\alpha \xi^\rho + \overline{\nabla}_\lambda \mathcal{T}^\gamma_{\alpha\rho} \overline{\nabla}_\beta \xi^\rho\end{aligned}$$

- Consider a uniform gauge field, then we obtain the **identity**:

$$\sigma^{\alpha\beta}\left(g_{\alpha\gamma}\mathcal{T}^\gamma_{\rho\beta} + g_{\gamma\beta}\mathcal{T}^\gamma_{\rho\alpha}\right) + \Sigma^{\alpha\beta}_\gamma \overline{\nabla}_\rho \mathcal{T}^\gamma_{\alpha\beta} + \Xi^{\lambda\alpha\beta}_\gamma \overline{\nabla}_\rho \overline{\nabla}_\lambda \mathcal{T}^\gamma_{\alpha\beta} = 0$$

Proof (2)

- usinf of same method as previously, we factorize the remaining term by $\overline{\nabla}_\beta \xi^\rho$, and obtain :

$$
\begin{aligned}
\delta_\xi \mathcal{L} &= \sigma^{\alpha\beta} \left(\overline{\nabla}_\alpha \xi_\beta + \overline{\nabla}_\beta \xi_\alpha \right) \\
&+ \Sigma_\gamma^{\alpha\beta} \left(-\mathcal{T}_{\alpha\beta}^\rho \overline{\nabla}_\rho \xi^\gamma + \mathcal{T}_{\rho\beta}^\gamma \overline{\nabla}_\alpha \xi^\rho + \mathcal{T}_{\alpha\rho}^\gamma \overline{\nabla}_\beta \xi^\rho \right) \\
&- \Xi_\rho^{\lambda\alpha\gamma} \, \overline{\nabla}_\lambda \mathcal{T}_{\alpha\gamma}^\beta \, \overline{\nabla}_\beta \xi^\rho + \Xi_\gamma^{\beta\alpha\lambda} \, \overline{\nabla}_\rho \mathcal{T}_{\alpha\lambda}^\gamma \, \overline{\nabla}_\beta \xi^\rho \\
&+ \Xi_\gamma^{\lambda\beta\alpha} \, \overline{\nabla}_\lambda \mathcal{T}_{\rho\alpha}^\gamma \, \overline{\nabla}_\beta \xi^\rho + \Xi_\gamma^{\lambda\alpha\beta} \, \overline{\nabla}_\lambda \mathcal{T}_{\alpha\rho}^\gamma \, \overline{\nabla}_\beta \xi^\rho
\end{aligned}
$$

- The proof of the theorem follows. $\square$

Note: As usual, choice of another gauge allows us to obtain other conservation laws or identities (linear,, ...) Work in progress ...

Comment on generalized stress

Roughly speaking, the **generalized stress** includes three contributions:

1. the classical **Cauchy stress** / energy-momentum $\sigma^{\alpha\beta}$ deforms the underlying continuum point.

2. the **hyperstress** / hypermomentum current $\Sigma_{\gamma}^{\alpha\beta}\mathcal{T}_{\rho\beta}^{\gamma}$ (and $\Sigma_{\gamma}^{\alpha\beta}\mathcal{T}_{\alpha\rho}^{\gamma}$) induces change of density of dislocations / spacetime defects at the continuum point (discontinuity of scalar field)

3. the **polar hyperstress** / (mass) quadrupole moment : $\Xi_{\gamma}^{\lambda\alpha\beta}\overline{\nabla}_{\lambda}\mathcal{T}_{\rho\beta}^{\gamma}$ (and $\Xi_{\gamma}^{\lambda\alpha\beta}\overline{\nabla}_{\lambda}\mathcal{T}_{\alpha\rho}^{\gamma}$) drives the relative motions of grains / spacetime microcosms (discontinuity of vector field).

Klein (1918) and Noether (1918) identities

At least three variations exist for continuum, say Lagrangian, Eulerian, and Lie derivative variations, (e.g. Manoff 1999):

$$\left(\Delta g_{\alpha\beta}, \Delta \mathcal{T}^{\gamma}_{\alpha\beta}\right), \quad \left(\delta g_{\alpha\beta}, \delta \mathcal{T}^{\gamma}_{\alpha\beta}\right), \quad \left(\mathcal{L}_{\xi} g_{\alpha\beta}, \mathcal{L}_{\xi} \mathcal{T}^{\gamma}_{\alpha\beta}\right)$$

The change of the Lagrangian with $\delta g_{\alpha\beta}$ and $\delta \mathcal{T}^{\gamma}_{\alpha\beta}$ gives:

$$\delta\left(\mathscr{L}\omega_n\right) = \left(\sigma^{\alpha\beta}\ \delta g_{\alpha\beta} + \Sigma^{\alpha\beta}_{\gamma}\ \delta \mathcal{T}^{\gamma}_{\alpha\beta}\right)\omega_n$$

The contortion tensor may be expressed as $\mathcal{T}^{\gamma}_{\alpha\beta} := D^{\gamma}_{\alpha\beta} + \Omega^{\gamma}_{\alpha\beta}$:

$$\Omega^{\gamma}_{\alpha\beta} = (1/2)(\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}), \quad D^{\gamma}_{\alpha\beta} = g^{\gamma\lambda} g_{\alpha\mu}\ \Omega^{\mu}_{\lambda\beta} + g^{\gamma\lambda} g_{\mu\beta}\ \Omega^{\mu}_{\lambda\alpha}$$

Research of other identities can be done by expressing the Lagrangian variation as (e.g. Hehl et al. 1995, Fatibene et al. 2010, Vitagliano et al. 2011, Lompay & Petrov 2013, R 2018, ...):

$$\delta\left(\mathscr{L}\omega_n\right) = \left(S^{\alpha\beta}\ \delta g_{\alpha\beta} + \mathbb{S}^{\alpha\beta}_{\gamma}\ \delta \Gamma^{\gamma}_{\alpha\beta}\right)\omega_n, \qquad \cdots$$

...

VIII. Concluding remarks

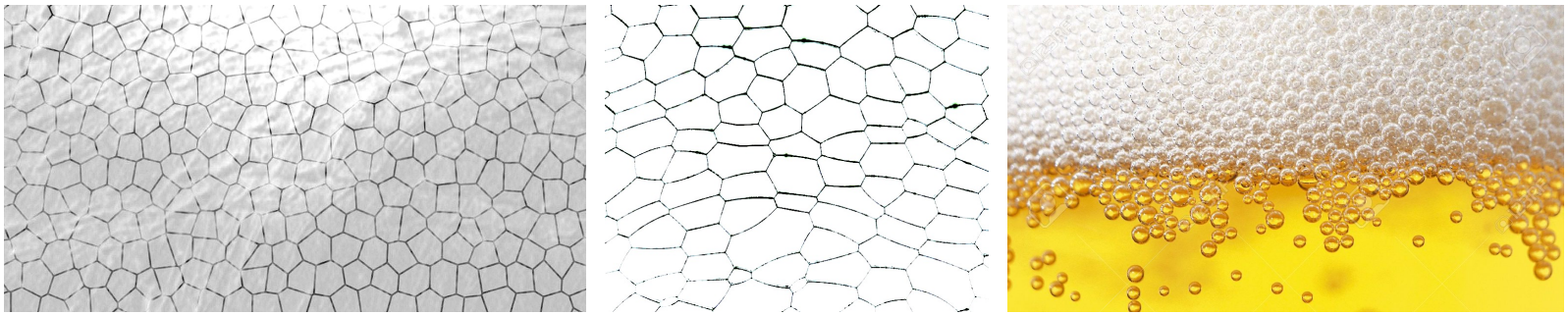


Figure: Plasticity, Gravitation and way of Life

Both the **Relative Gravitation spacetime** and the **Elastoplastic Continuum** may be judiciously modeled as Riemann-Cartan manifold with metric $g_{\alpha\beta}$, torsion $\aleph^{\gamma}_{\alpha\beta}$ (resp. contortion $\mathcal{T}^{\gamma}_{\alpha\beta}$), and curvature $\Re^{\gamma}_{\alpha\beta\lambda}$ (resp. gradient of contortion $\overline{\nabla}_{\lambda}\mathcal{T}^{\gamma}_{\alpha\beta}$).

Other concluding remarks

- **Riemann-Cartan** candidate model of (spacetime) continuum with sharp gradients and Plastic Continuum (as a mosaic of little **microcosms of spacetime / grains**).

- Plastic Continuum is *a priori* a **gradient continuum** as the Relative Gravitation spacetime . The incompatibilities of deformation are rigorously measured by torsion and curvature.

- Beyond phenomenology, **some methods of the RG** available since the seventies might be wisely and beneficially utilized for help modeling Elastoplastic Continuum.

- **Spacetime model** extended to **matter continuum physics** (acoustic, elasticity, electromagnetism, gravitation, plasticity ...)

Merci pour votre attention !

Some papers and books

- Antonio Tamarasselvame N, Rakotomanana L. On the form-invariance of Lagrangean function for higher gradient continuum, in *Mechanics of Generalized Continua*, Ed. by H Altenbach, G Maugin, V Erofeev, Springer-Verlag, 2011, pp 291-322.
- Futhazar G, Le Marrec L, Rakotomanana LR. Covariant gradient continua applied to wave propagation within defected material, *Archive for Applied Mechanics*, 2014.
- Rakotomanana RL. Contribution à la modélisation géométrique et thermodynamique d'une classe de milieux faiblement continus, *Archives for Rational Mechanics and Analysis* 141, 1997, pp 199-236.
- Rakotomanana RL. *A geometric approach to thermomechanics of dissipating continua*, in *Progress in Mathematical Physics Series*, Birkhaüser, Boston, 2003.
- Rakotomanana RL. *Covariance and Gauge Invariance in Continuum Physics*, in *Progress in Mathematical Physics Series*, Birkhaüser, Cham, 2018.
- Rakotomanana RL. More on the Geometric Approach for Gravitation Coupled with Electromagnetism within a Riemann-Cartan Vacuum Spacetime, *https://hal.archives-ouvertes.fr/hal-03216920*, 2021, pp 1-19.
- Rakotomanana RL, Curnier A, Leyvraz PF. An objective anisotropic elastic plastic model and algorithm applicable to bone mechanics, *European J. Mechanics/A* 10, 3, pp 327-342, 1991.
- Ramaniraka NA, Rakotomanana LR. Models of continuum with microcrack distribution, *Mathematics and Mechanics of Solids* 5, 2000, pp 301-336.