Coordonnées, connexions, courbures galiléennes et bargmanniennes

Géry de Saxcé

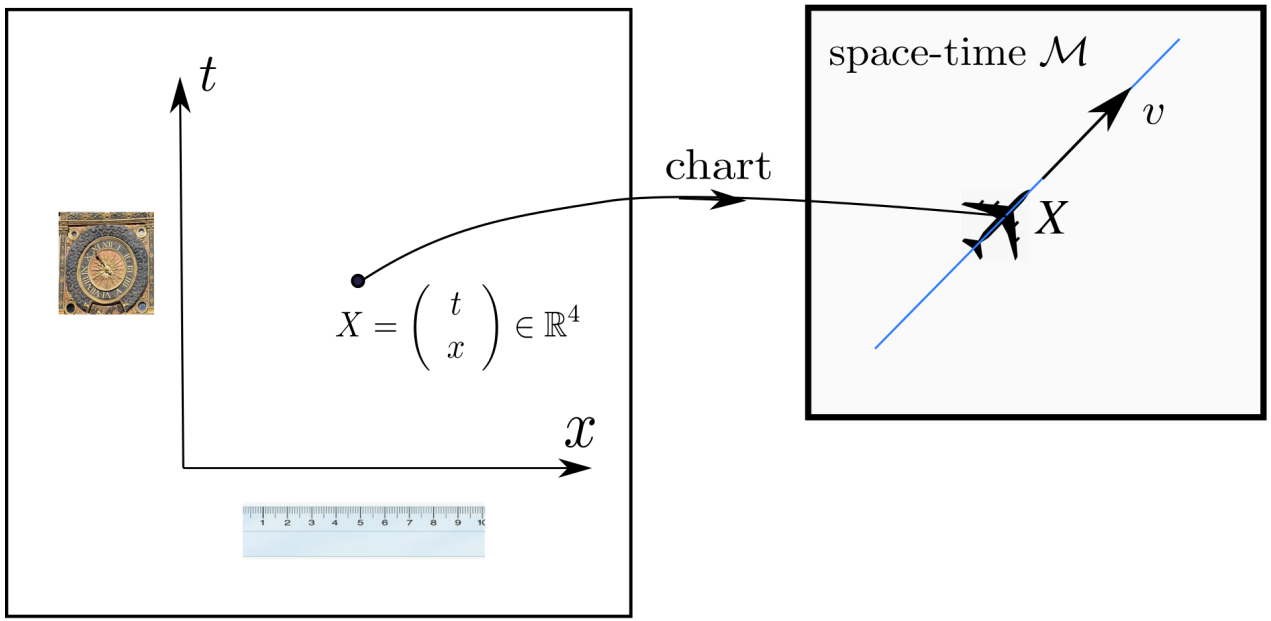
Univ. Lille, CNRS, Centrale Lille, UMR 9013 – LaMcube
Laboratoire de mécanique multiphysique multiéchelle

GDR-GDM La Rochelle 2021



Absolute space and time

The special Galilean relativity (without gravitation)



Newton-Cartan structure:

- a 1-form $\boldsymbol{\tau} = dt$ (clock-form)
- a 2-contravariant symmetric tensor $\boldsymbol{h}$ of signature $(0 + ++)$

such that $\boldsymbol{\tau} \cdot \boldsymbol{h} = 0$

To know more: Duval, Küntzle, Trautman, Toupin, Horváthy, Hartong, Bergshoeff, Van den Bleeken, ...

Galilean geometry

I use the convention $X^0 = t, X^i = x^i \, (i = 1, \ldots, 3)$

- **A geometry is a group action** `Klein Erlangen program (1872)`

  The **Galilean transformations** leave invariant:

  - the **durations** (or τ)
  - the **distances** (or h)
  - **Uniform Straight Motion**

  then affine of the form $X = P\,X' + C$ with:

$$P = \begin{pmatrix} 1 & 0 \\ v_t & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

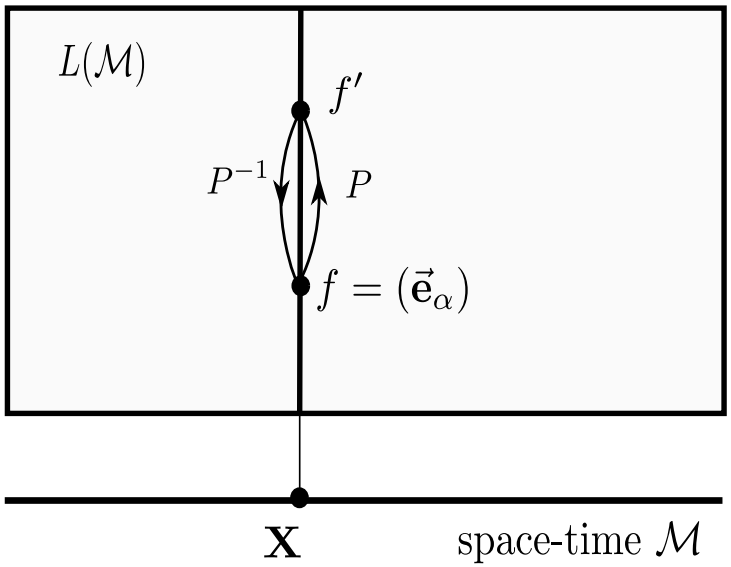
  where $v_t \in \mathbb{R}^3$ is the **velocity of transport** or **Galilean boost** and R is a rotation

- Their set is **Galileo's group** $\mathbb{GAL}$, a Lie group of dimension 10
- $\mathbb{GAL}_0$ is the subgroup of *linear* Galilean transformation P

The Galilean geometry is not Riemannian!
tensor indices may be neither lowered nor raised



G-structures



G matrix Lie group
$L(\mathcal{M})$ principal bundle of linear frames
A G-structure L_G (subbundle of $L(\mathcal{M})$) [Kobayashi 1963]
is **integrable** if $\forall \boldsymbol{X} \in \mathcal{M}, \exists$ a local chart arround $\boldsymbol{X}$
st the associated natural basis is a local cross-section of L_G
$f = \left(\frac{\partial}{\partial X^\alpha}\right), f' = \left(\frac{\partial}{\partial X'^\alpha}\right), P^{-1} = \frac{\partial X'}{\partial X} \in G$ $\quad$ (4×4 eqns. for 4 unknowns)

Galilean coordinates

- **Frobenius method** to find the integrability conditions
  $\forall dX, \delta X \in \mathbb{R}^4, \quad [\delta, d]\, X' = \delta(dX') - d(\delta X') = 0$
  $\delta(dX') = \delta(P^{-1}dX) = \delta(P^{-1})dX + P^{-1}\delta(dX)$
  $[\delta, d]\, X' = \delta(P^{-1})dX - d(P^{-1})\delta X + P^{-1}[\delta, d]\, X = 0$
  $= 0$

Theorem

$X \mapsto X'$ is such that $\frac{\partial X'}{\partial X} = P^{-1}$ is Galilean
iff $x' = (R(t))^T (x - x_0(t)), \qquad t' = t + \tau_0$
Then the **velocity of transport** is given by: $v_t = \omega(t) \times (x - x_0(t)) + \dot{x}_0(t)$
where ω is Poisson's vector such that: $\dot{R} = j(\omega) R$.

- This defines an **equivalence class** of coordinate charts
  $X \mapsto X'$ is called a **galileomorphism**,
  X and X' are called **Galilean coordinate systems**
  or **Galilean charts** or **Galilean reference frames**
- The galileomorphisms are paths in $\mathbb{R} \times \mathbb{SE}(3)$ and their set in a continuous group (but not a Lie group)
- $\boldsymbol{h}$ and $\boldsymbol{\tau}$ have constant components in Galilean charts

Galilean coordinates

[Souriau 2007 Grammaire de la Nature]

[de Saxcé Vallée 2016 Galilean Mechanics and Thermodynamics of Continua]

Theorem

The normalizer of Galileo's group in the affine group is generated by the Galilean transformations and the scale changes

$$P_u = \begin{pmatrix} T & 0 \\ 0 & L1_{\mathbb{R}^3} \end{pmatrix} .$$

Physical meaning: Each equivalence class of galileomorphisms corresponds to a particular choice of distinct units of measurements of time and length

In Relativity, there is a common unit of measurement of both time and length

Galilean connections

The general Galilean relativity: a new actor, the gravitation...
There is no more Uniform Straight Motion except within a small scale space-time window represented by the (flat) tangent space

Theorem

A **Galilean gravitation** is a symmetric connection Γ which,
in a Galilean chart,
takes its values in the Lie algebra $\mathfrak{gal}_0$
of the group of linear Galilean transformations:

$$\Gamma(dX) = \begin{pmatrix} 0 & 0 \\ \Omega \times dx - g\, dt & j(\Omega)\, dt \end{pmatrix},$$

where $j(\Omega)$ is the unique skew-symmetric matrix such that $j(\Omega)v = \Omega \times v$

- g is the classical **gravity** $(\Gamma^j_{00} = -g^j)$
- Ω is a new object called **spinning** $(\Gamma^j_{0k} = \Gamma^j_{k0} = (j(\Omega))^j_k = \Omega^j_k)$

The **inertial charts** are the ones in which the spinning Ω vanishes

Dynamics of particles: equation of motion

4-velocity, 4-momentum and 4-force:

$$U = \begin{pmatrix} 1 \\ v \end{pmatrix}, \qquad T = m\,U = \begin{pmatrix} m \\ p \end{pmatrix}, \qquad H = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

Covariant equation of motion

For particles in the gravitation field:

$$\nabla_U T = \dot{T} + \Gamma(U)\,T = H$$

In the Galilean charts: $\dot{m} = 0, \qquad \dot{p} = m\,(g - 2\,\Omega \times v) + F$

[Souriau 1970 Structure des systèmes dynamiques]

Comment: g, Ω are universal, v depends on the particle

Rocket thrust by an exhaust gas with ejection rate $\dot{m}$ and velocity w with respect to the Galilean chart in which the rocket is at rest

$$H = \dot{m} \begin{pmatrix} 1 \\ w \end{pmatrix}$$

Dynamics of particles: Newtonian gravitation

Newtonian gravitation

There exist particular Galilean charts, called **inertial** ou **Newtonian** reference frames, for which the gravitation resulting from a particle of mass $\bar{m}$ passing at $\bar{x}$ à at time t is given by:

$$g = -\frac{k_N\,\bar{m}}{\|\,x-\bar{x}\,\|^2}\,\frac{x-\bar{x}}{\|\,x-\bar{x}\,\|}, \qquad \Omega = 0\ ,$$

- In any another reference frame, the spinning Ω is not in general null
- **Only the experimental observation allows to determine whether a reference frame is inertial** (thanks for instance to Foucault's pendulum or the gyroscope)
- X' being an inertial reference frame, the gravitation in a non inertial one X is

$$g - 2\,\Omega \wedge v = R\,g' + a_t$$

where occurs the acceleration of transport a_t

Equivalence principle (Einstein)

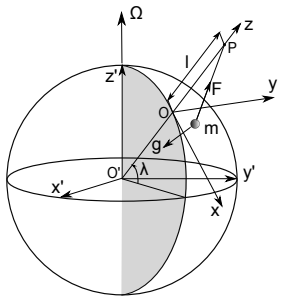
The laws of Nature are such that it is impossible for an observer to distinguish between the effect of the universal attraction and the one of a change of reference frame

Foucault's pendulum

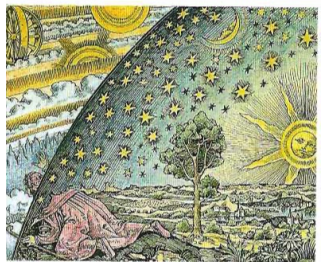
- transformation law of the gravitation (a connection, not a tensor):
  $\Gamma(U) = \underbrace{P\,\Gamma'(U')\,P^{-1}}_{\text{universal attraction}} \underbrace{- \dot{P}\,P^{-1}}_{\text{inertial effect}}$

Foucault's pendulum:
$g = R\,g' + \varpi \wedge (\varpi \wedge x_0)$

component along Oz (at Paris):
$g_z = -\mu/r^2 + r\,\varpi^2 \cos^2\lambda = -9,821\,ms^{-2}$



1st term (universal attraction), $-9,836\,ms^{-2}$, is leading.
2ème (inertial effect), of the order of $10^{-3} \mid g_z \mid$, often neglected wrt Coriolis' force $2\,\Omega \wedge v$, of the order of $10^{-5} \mid g_z \mid$



Foucault's pendulum or the gyroscope
allowing to determine whether a reference frame is inertial.

It is not necessary to invoke reference frames built
from fixed stars (Aristotelian mechanics!!!)

Bargmann's group

The failure of naïve attempts to construct actions on quantum waves is cohomologic

- wave function $\psi \in \mathcal{H}$ of norm 1
  ambiguity $\psi' \sim \psi \Leftrightarrow \exists \alpha \in \mathbb{R},\ \psi' = e^{i\alpha}\psi$
- quotient $\psi \in \mathcal{H} \longrightarrow \boldsymbol{\psi} = [\psi] \in \mathcal{H}/\mathbb{T}$
- observable $\hat{u} \in U(\mathcal{H}) \longrightarrow \hat{\boldsymbol{u}} \in U(\mathcal{H})/\mathbb{T}$
- symmetry group G acting by unitary representation $a \mapsto \hat{\boldsymbol{u}}_a$
  local exponent ξ st $\hat{u}_a\,\hat{u}_{a'} = e^{i\,\xi(a,a')}\hat{u}_{a\,a'} \longrightarrow \hat{\boldsymbol{u}}_a\hat{\boldsymbol{u}}_{a'} = \hat{\boldsymbol{u}}_{a\,a'}$

Bargmann (1954) method to construct local exponents with suitable properties leading to define an extension $\tilde{G} = G \times \mathbb{R}$ with
$(a,\theta)\,(a',\theta') = (a\,a', \theta + \theta' + coc(a,a'))$
where the $\mathbb{R}$-cocycle coc verify an identity to ensure the associativity
Applied to Galileo's group, the extension is a Lie group of dimension 11 called **Bargmann's group**

Souriau (1970) the nature of the cohomological failure is symplectic
de Saxcé & Vallée (2010) IJGP the cocycle as solution of an EDP
de Saxcé & Vallée (2012) IJES also useful for Thermodynamics

Bargmannian transformations

- We consider a line bundle $\pi : \hat{\mathcal{M}} \to \mathcal{M}$ of dimension 5 and a section:

$$\hat{f} : \mathcal{M} \to \hat{\mathcal{M}} \; : \; \boldsymbol{X} \mapsto \hat{\boldsymbol{X}} = \hat{f}\left(\boldsymbol{X}\right)$$

- We built a group of affine transformations $d\hat{X}' \mapsto d\hat{X} = \hat{P}\, d\hat{X}' + \hat{C}$ of $\mathbb{R}^5$ which are Galilean when acting onto the space-time hence of the form:

$$\hat{P} = \begin{pmatrix} P & 0 \\ \Phi & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ v_t & R & 0 \\ \frac{1}{2} \parallel v_t \parallel^2 & v_t^T R & 1 \end{pmatrix} \in \hat{G}$$

  where P is Galilean and the row Φ represents the cocycle

- The 5^{th} dimension $\hat{X}^4$ has the physical meaning of an specific action because

$$d\hat{X}^4 = \frac{e}{m}\, dt = \frac{1}{2} \parallel v_t \parallel^2 dt' + u^T R\, dx' + d\hat{X}'^4$$

  allows to restore the transformation law of the kinetic energy

$$e = \frac{1}{2}\, m \parallel v \parallel^2 = \frac{1}{2}\, m \parallel v_t + R\, v' \parallel^2$$

Bargmann's group versus Galileo's group

The down side of the cards ...

- Galileo's group does not preserve space-time metrics
- Bargmann's group preserves the metrics $ds^2 = \| dx \|^2 - 2\, d\hat{X}^4\, dt$, then the space $\hat{\mathcal{M}}$ is a riemannian manifold and, in this case, the G-structure is not in general integrable, the obstruction being the curvature.
- In other words, we are going to work, up to now, in linear frames which are not associated to local coordinates (moving frames)
- Hence we have to find frames associated to coordinate systems (natural frames)

Potentials of the Galilean gravitation

- JM Souriau 1970 SSD **Maxwell's principle**

  The closure condition $d\omega = 0$ of the presymplectic 2-form gives

  $$curl\, g + 2\,\frac{\partial \Omega}{\partial t} = 0, \qquad \boxed{div\, \Omega = 0}$$

  then there exist **potentials** ϕ, A such that

  $$g = -grad\, \phi - \frac{\partial A}{\partial t}, \quad \Omega = \frac{1}{2}\, curl\, A$$

- The corresponding Lagrangian $\mathcal{L}(t,x,v) = \frac{1}{2}\, m \parallel v \parallel^2 - m\,\phi + m\, A \cdot v$ is covariant provided

  $$\phi' = \phi - A \cdot v_t - \frac{1}{2} \parallel v_t \parallel^2, \quad A' = R^T(A + v_t)$$

  Comment: ϕ is not Newton's potential that does not the expected covariance

Bargmannian coordinates and connections

- To find the natural frames, we must annul the torsion 2-form,

$$\hat{T}(\delta\hat{X}, d\hat{X}) = \hat{\Gamma}(\delta\hat{X})\, d\hat{X} - \hat{\Gamma}(d\hat{X})\, \delta\hat{X} - [\delta, d]\, \hat{X} = 0$$

  in the moving frame, that leads to the change of linear frame

$$dX'^4 = \frac{\mathcal{L}}{m}\, dt = dX^4 - \phi\, dt + A \cdot dx\ , \quad dt' = dt, \quad dx' = dx$$

  Physically, a free particle of which the motion is described in a moving frame is embedded in the gravitation field

- In the new coordinates, the **Bargmannian connection** is

$$\hat{\Gamma}(d\hat{X}) = \begin{pmatrix} 0 & 0 & 0 \\ j(\Omega)\, dx - g\, dt & j(\Omega)\, dt & 0 \\ \left(\dfrac{\partial \phi}{\partial t} - A \cdot g\right) dt & [(grad\, \phi - \Omega \times A)\, dt & 0 \\ + (grad\, \phi - \Omega \times A) \cdot dx & -grad_s A\, dx]^T & \end{pmatrix}$$

Maxwell's principle and symmetry of the Bargmannian connection

[de Saxcé Vallée 2016 Galilean Mechanics and Thermodynamics of Continua]

Theorem

If the Bargmannian connection is symmetric, the Galilean presymplectic 2-form

$$\omega(d\eta, \delta\eta) = m\,[(dv - g\,dt)\cdot(\delta x - v\,\delta t) - (\delta v - g\,\delta t)\cdot(dx - v\,dt) - 2\,\Omega\cdot(\delta x \times dx)]\ .$$

describing the motion of a particle embedded in the gravitation field is closed.

Proof.

$$\omega(d\eta, \delta\eta) = \omega_{free}(d\eta, \delta\eta) + m\,(2\,\Omega\cdot(dx\times\delta x) - g\cdot(dt\,\delta x - \delta t\,dx))$$

$$\omega(d\eta, \delta\eta) = \omega_{free}(d\eta, \delta\eta) + m\,([\delta, d]\,\hat{X}^4 + \hat{T}^4(\delta\hat{X}, d\hat{X}))$$

then owing to Jacobi's identity

$$d\omega = m\,d\,\hat{T}^4\ ,$$

Galilean curvature

The space-time is it flat or curved?

To answer these questions, we have to calculate Riemann-Christoffel curvature tensor

$$R^i_{0j0} = -R^i_{j00} = \frac{\partial g^i}{\partial x^j} + \frac{\partial \Omega^i_j}{\partial t} + \Omega^i_k \, \Omega^k_j,$$

$$R^i_{kj0} = -R^i_{jk0} = \frac{\partial \Omega^i_j}{\partial x^k} - \frac{\partial \Omega^i_k}{\partial x^j}, \qquad R^i_{k0j} = -R^i_{0kj} = \frac{\partial \Omega^i_j}{\partial x^k} \ .$$

A noticeable fact is that $R^i_{jkm} = 0$, that means the slices $t = C^{te}$ are flat subspaces while **the space-time is curved, even in classical mechanics**.

Theorem

If the Galilean curvature is null and there is no other forces, there exists a Galilean coordinate system in which the material particles with constant mass are in uniform straight motion

Spacetime metrics

For weak gravitational fields

- $G = \begin{pmatrix} c^2 + 2\,\phi & -A^T \\ -A & -1_{\mathbb{R}^3} \end{pmatrix} = \epsilon^{-1} \overset{(-1)}{G} + \overset{(0)}{G}$

  where $\epsilon = c^{-2}$, $\overset{(-1)}{G} = \tau \otimes \tau$ and $\overset{(0)}{G}$ are Galilean tensors
- Truncated expansion of

$$G^{-1} \cong \begin{pmatrix} 0 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{pmatrix} + \epsilon \begin{pmatrix} 1 & -A^T \\ -A & A\,A^T \end{pmatrix}$$

  Each term is a Galilean 2-contravariant tensor
- $\sqrt{-\det G} \cong c\,(1 + \epsilon\,(\phi + \frac{\|A\|^2}{2})) = c\,(1 + \epsilon\,I_0)$

  Galilean invariant

  of the metrics

Hilbert-Einstein functional $\mathcal{A} = \int (p_M + p_G)\sqrt{-\det G}\, d^4X$

Curvature tensor

Levi-Civita connection $\Gamma(dX) = (\Gamma^{\mu}_{\alpha\beta} dX^{\alpha}) \cong \overset{(0)}{\Gamma} + \epsilon \overset{(1)}{\Gamma}$

- **Order 0** (Galilean gravitation)

$$\overset{(0)}{\Gamma}{}^{i}_{00} = -g^{i} \quad \text{gravity}$$

$$\overset{(0)}{\Gamma}{}^{j}_{i0} = \Omega^{j}_{i} \quad \text{spinning}$$

- **Order 1**

$$a = \overset{(1)}{\Gamma}{}^{0}_{00} = \frac{\partial \phi}{\partial t} - A \cdot g$$

$$B_i = \overset{(1)}{\Gamma}{}^{0}_{0i} = \delta_{ik} \left(grad\, \phi - \Omega \times A \right)^{k}$$

$$D_{ij} = \overset{(1)}{\Gamma}{}^{0}_{ij} = -\delta_{ik} \left(grad_s A \right)^{k}_{j}$$

May be also obtained from the Bargmannian connection: $\overset{(1)}{\Gamma}{}^{0}_{\alpha\beta} = \hat{\Gamma}^{4}_{\alpha\beta}$

Curvature tensor

Ricci tensor $R'_{\beta\gamma} = R^{\alpha}_{\alpha\beta\gamma}$

expanded as $R' \cong \overset{(0)}{R}{}' + \epsilon \overset{(1)}{R}{}'$

- **Order 0**

$$\overset{(0)}{R}{}' = \begin{pmatrix} 2 \parallel \Omega \parallel^2 - div\, g & (curl\, \Omega)^T \\ curl\, \Omega & 0 \end{pmatrix}$$

- **Order 1** for instance:

$$\overset{(1)}{R}{}'_{lm} = D_{lj}\Omega^j_m + D_{jm}\Omega^j_l + \frac{\partial}{\partial x^j}(A^j D_{lm}) - \frac{\partial D_{lm}}{\partial t} - \frac{\partial}{\partial x^l}(B_m + D_{jm}A^j)$$

$$\dots$$

As expected, symmetric because $\frac{\partial}{\partial x}(B + D\,A)$ is the Hessian of I_0

Hilbert-Einstein Functional

- **Part depending on the scalar curvature** $R = G^{\alpha\beta} R'_{\alpha\beta} \cong -2\,\epsilon\,(I + I_1)$ with the **Galilean invariants** of
  - the metrics and the curvature: $\boxed{I = div\, g - 2 \parallel \Omega \parallel^2 + 2\, A \cdot curl\, \Omega}$
  - only the curvature: $I_1 = -\frac{1}{2}\, Tr\, \left[(D - (Tr\, D)\, 1_{\mathbb{R}^3})^2\right]$

- **Part depending on the matter**
  based on the **Conformation tensor** [Souriau 1964 Géométrie & Relativité]
  generalization of the deformation to the spacetime

4 field equations (not 10!)

- Variation *wrt* ϕ: $\boxed{I + I_1 - \Lambda \quad = -4\,\pi\, k_N \rho}$
- Variation *wrt* A: $\boxed{-2\,(\epsilon^{-1} + I_0)\, curl\, \Omega - (I + I_1 - \Lambda)\, A = -4\,\pi\, k_N \rho\, v}$

Asymptotic expansion of the solution

2nd field equation at order -1: $curl \overset{(0)}{\Omega} = 0$

remember: $div \overset{(0)}{\Omega} = 0$, then $\overset{(0)}{\Omega}$ is harmonic

hypothesis: $\overset{(0)}{\Omega}$ is bounded, then $\boxed{\overset{(0)}{\Omega} = \overset{(0)}{\Omega}(t)}$

Gravitation potentials: classical approximation and correction

$$\phi(x,t) = -\frac{1}{2} \parallel v_t(x,t) \parallel^2 - \int \frac{k_N \rho(\bar{x},t)}{\parallel x - \bar{x} \parallel} d\mathcal{V}(\bar{x}) - \frac{1}{6}\Lambda \parallel x \parallel^2 + O\left(\frac{1}{c^2}\right) ,$$

$$A(x,t) = -v_t(x,t) + \frac{1}{c^2}\int \frac{k_N \rho(\bar{x},t)\, v(\bar{x},t)}{\parallel x - \bar{x} \parallel} d\mathcal{V}(\bar{x}) + O\left(\frac{1}{c^4}\right) .$$

where v_t is the velocity of transport with respect to an inertial frame.

Particular case of interest: rapidly rotating star at $x = 0$: $v_t(x,t) = -\Omega \times x$

[de Saxcé 2020 GERG Asymptotic expansion of General Relativity with Galilean covariance]

Merci !



1. Galilean coordinates
2. Galilean connections
3. Bargmannian coordinates
4. Galilean and Bargmannian connections