Préservation des structures de la géométrie généralisée en calcul numérique 2/3

Vladimir Salnikov







"Paris", 6 Novembre 2020

Groupement de Recherche
Géométrie Différentielle et Mécanique

Philosophy: **Geometry encodes the physics of the system**

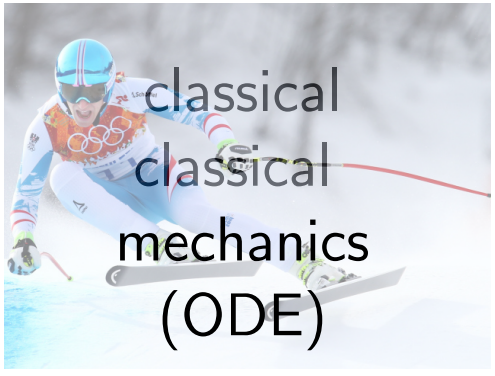
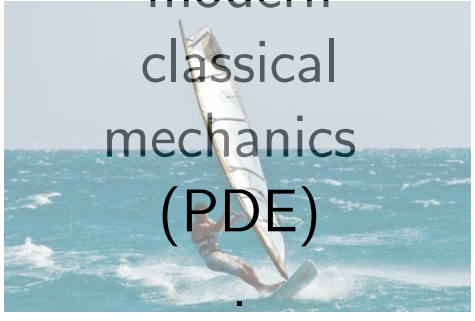
	Mechanical property	Geometric description
 classical mechanics (ODE)	conservation of energy	Poisson / symplectic
	symmetries	Lie groups/algebras, Cartan moving frames
	dissipation / interaction power balance; constraints	(almost) Dirac
	control	(singular) foliations
 modern classical mechanics (PDE)	conservation of energy	multisymplectic
	symmetries	Cartan moving frames
	dissipation / interaction	Stokes–Dirac
	$rot(grad) = 0$, $div(rot) = 0$	$d^2 = 0$ – DEC
	control	foliations

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Geometry encodes the physics of the system

Philosophy:

Razafindralandy

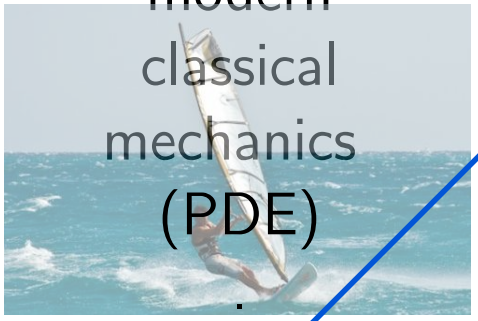
de Saxcé, Carré

Falaize Hélie, Roze

Hubert

Loziienko

Ryvkin, Laurent-G.

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 classical classical mechanics (ODE)	conservation of energy	Poisson / symplectic
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Ryvkin

Ayoub

2 DG/Q variétés

Generalized geometry...

Courant algebroids, Dirac structures

Let us construct on $E = TM \oplus T^*M$
an *exact Courant algebroid structure*.
The symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
the bracket (Courant – Dorfman):

$$[v \oplus \eta, v' \oplus \eta'] = [v, v']_{\mathrm{Lie}} \oplus (\mathcal{L}_v \eta' - \mathrm{d}\eta(v')). \qquad (1)$$

A *Dirac structure* $\mathcal{D}$ is a maximally isotropic (Lagrangian) subbundle of an exact Courant algebroid E closed with respect to the bracket (1).

Trivial example: $\mathcal{D} = TM$.

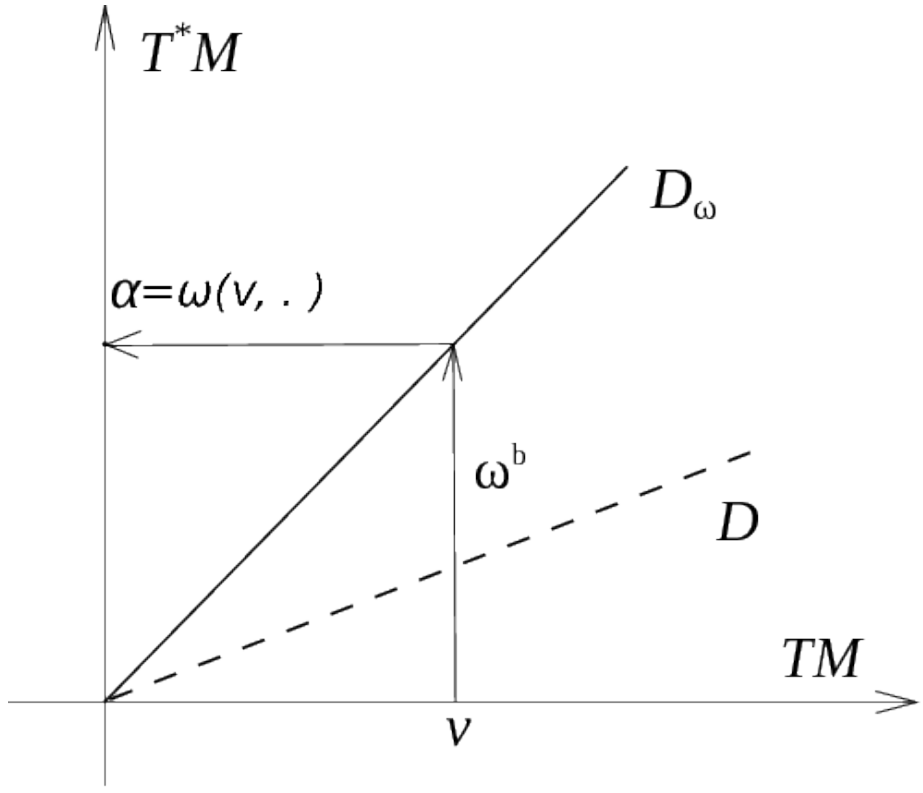
Dirac structures: (pre)symplectic example.

$$\omega = \sum_{i,j} \omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$$

Example. $\mathcal{D} = graph(\omega)$

Isotropy $\Leftrightarrow$
ω_{ij} antisymmetric.

Involutivity $\Leftrightarrow$
ω closed.



$$\mathcal{D}_\omega = \{(v, \iota_v \omega)\}$$

Dirac structures: Poisson example.

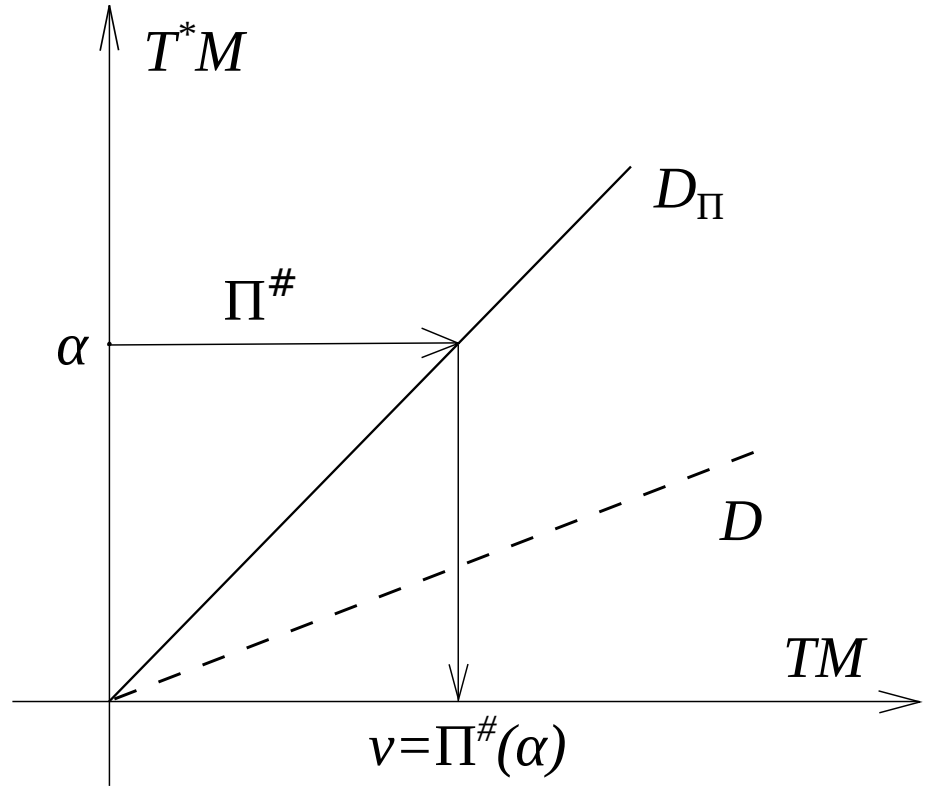
$$\Pi = \sum_{i,j} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \qquad \text{where } \pi^{ij}(x) := \{x^i, x^j\}$$

Example. $\mathcal{D} = \mathit{graph}(\Pi^\sharp)$

Isotropy $\Leftrightarrow$
π^{ij} antisymmetric.

Involutivity $\Leftrightarrow$
Π Poisson.

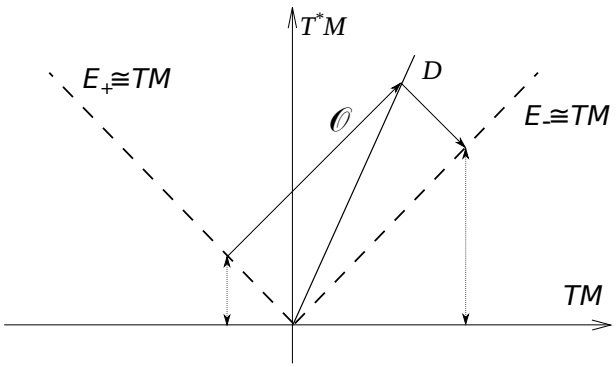




$$\mathcal{D}_\Pi = \{(\Pi^\sharp \alpha, \alpha)\}$$

Dirac structures: general

Choose a metric on $M \Rightarrow TM \oplus T^*M \cong TM \oplus TM$,
Introduce the eigenvalue subbundles $E_\pm = \{v \oplus \pm v\}$
of the involution $(v, \alpha) \mapsto (\alpha, v)$. Clearly, $E_+ \cong E_- \cong TM$.



(Almost) Dirac structure – graph of an orthogonal operator $\mathcal{O} \in \Gamma(\mathrm{End}(TM))$:
$(v, \alpha) = ((\mathrm{id} - \mathcal{O})w, g((\mathrm{id} + \mathcal{O})w, \cdot))$
Dirac structure = almost Dirac + (Jacobi-type) integrability condition:

$$g\left(\mathcal{O}^{-1}\nabla_{(\mathrm{id}-\mathcal{O})\xi_1}(\mathcal{O})\xi_2, \xi_3\right) + cycl(1,2,3) = 0$$

Remark. If the operator $(\mathrm{id} + \mathcal{O})$ is invertible, one recovers D_Π with $\Pi = \frac{\mathrm{id} - \mathcal{O}}{\mathrm{id} + \mathcal{O}}$ (Cayley transform), integrability $\Leftrightarrow [\Pi, \Pi]_{SN} = 0$.

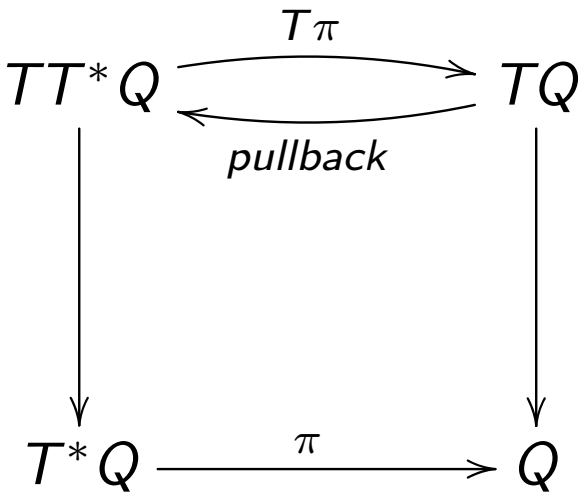
NB: Alternative picture? See soon.

... and mechanics

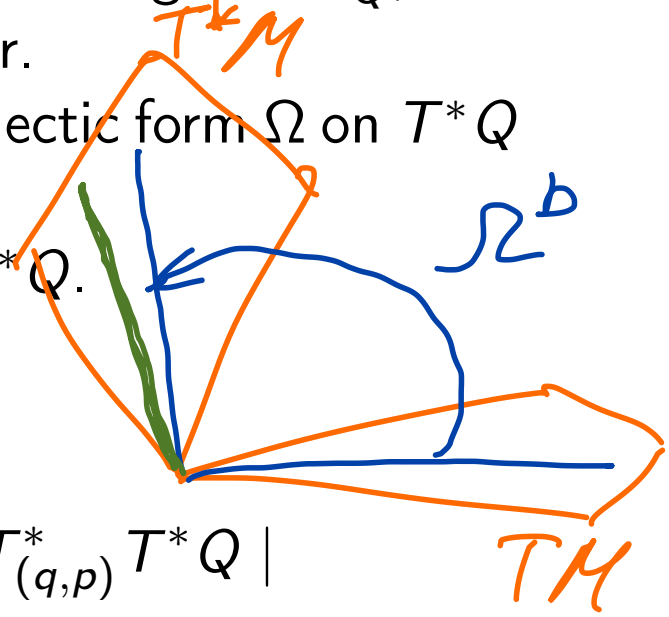
Application 1. Implicit Lagrangian systems / constraints

Tulczyjew (70's), H. Yoshimura, J.E. Marsden (2006).

1. Geometry:



$\Delta_Q \subset TQ$ constraint distribution:
$\Delta_Q(q) = \{v \in T_qQ \mid \omega_q^a(v) = 0\}$.
$\Delta_{T^*Q} \subset TT^*Q$ – preimage of Δ_Q, and
$\Delta^0_{T^*Q}$ its annihilator.
The canonical symplectic form Ω on T^*Q defines a mapping
$\Omega^\flat : TT^*Q \to T^*T^*Q$.



Almost Dirac structure:

$$\mathbb{D}_{\Delta_Q}((q,p)) = \{(w,\alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid$$
$$w \in \Delta_{T^*Q}, \quad \alpha - \Omega^\flat w \in \Delta^0_{T^*Q})\} \qquad (2)$$



2. Dynamics. $L\colon TQ \to \mathbb{R}$ – Lagrangian.
Its differential defines a mapping $\mathrm{d}L\colon TQ \to T^*TQ$.
There are symplectomorphisms $\Omega^\flat\colon TT^*Q \to T^*T^*Q$ as well as $\kappa_Q\colon TT^*Q \to T^*TQ$, then $\gamma_Q := \Omega^\flat \circ \kappa_Q^{-1}\colon T^*TQ \to T^*T^*Q$.

Define the *Dirac differential* $\mathcal{D}L := \gamma_Q \circ \mathrm{d}L\colon TQ \to T^*T^*Q$.
Locally: $(q, v) \to (q, \frac{\partial L}{\partial v}, -\frac{\partial L}{\partial q}, v)$.

Consider a partial vector field X, i.e. a mapping
$X\colon \Delta_Q \oplus Leg(\Delta_Q) \subset TQ \oplus T^*Q \to TT^*Q$.
It can be viewed as $X(q, p)$, where p is given by the Legendre transform, and v is in the constraint distribution.

Algebroïde (cf. Grabowska)

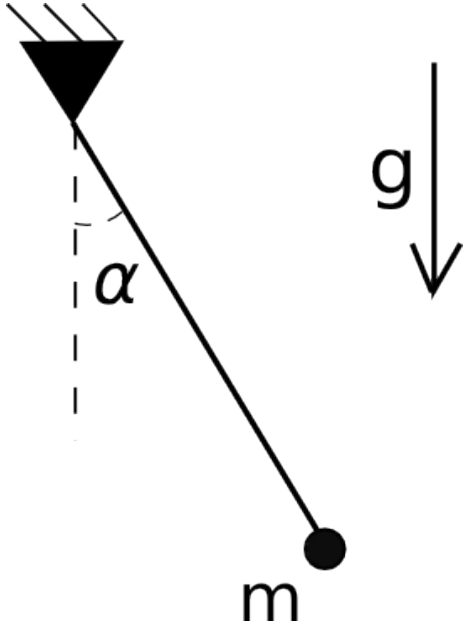
3. All together An *implicit Lagrangian system* is a triple (L, Δ_Q, X), s.t. $(X, \mathcal{D}L) \in \mathbb{D}_{\Delta_Q}$ (eq. 2)

Locally this means $p = \frac{\partial L}{\partial v}$, $\dot{q} = v$, $\dot{q} \in \Delta$, and
$\dot{p} - \frac{\partial L}{\partial q} \in \Delta^0(q) \Leftrightarrow \dot{p} - \frac{\partial L}{\partial q} = \sum_a \lambda_a \omega^a$.

Baby example



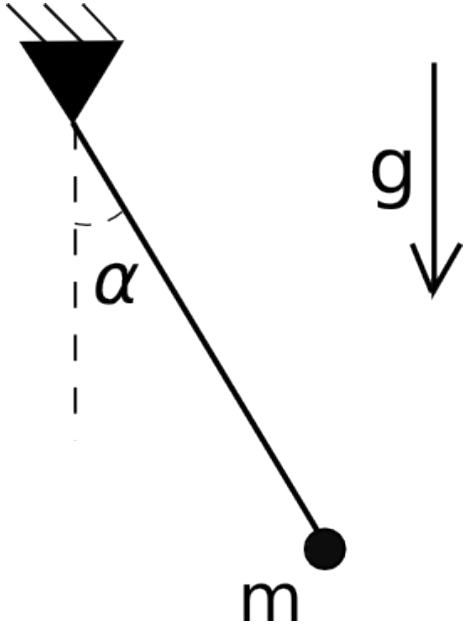




Baby example







Description of Δ_Q and Δ_{T^*Q}:
$Q = \mathbb{R}^2$,
Constraint $\phi(x, y) := x^2 + y^2 - l^2 = 0$.
The distribution Δ_Q is generated by $\xi = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$.
in the kernel of $\psi = \mathrm{d}\phi = 2(x\mathrm{d}x + y\mathrm{d}y)$.

Lagrangian differential and Legendre transform.
The Lagrangian is $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$. The associated Lagrangian differential
$\mathcal{D}L = (q, \frac{\partial L}{\partial v}, -\frac{\partial L}{\partial q}, v) = ((x, y), (m\dot{x}, m\dot{y}), (0, mg), (\dot{x}, \dot{y}))$.

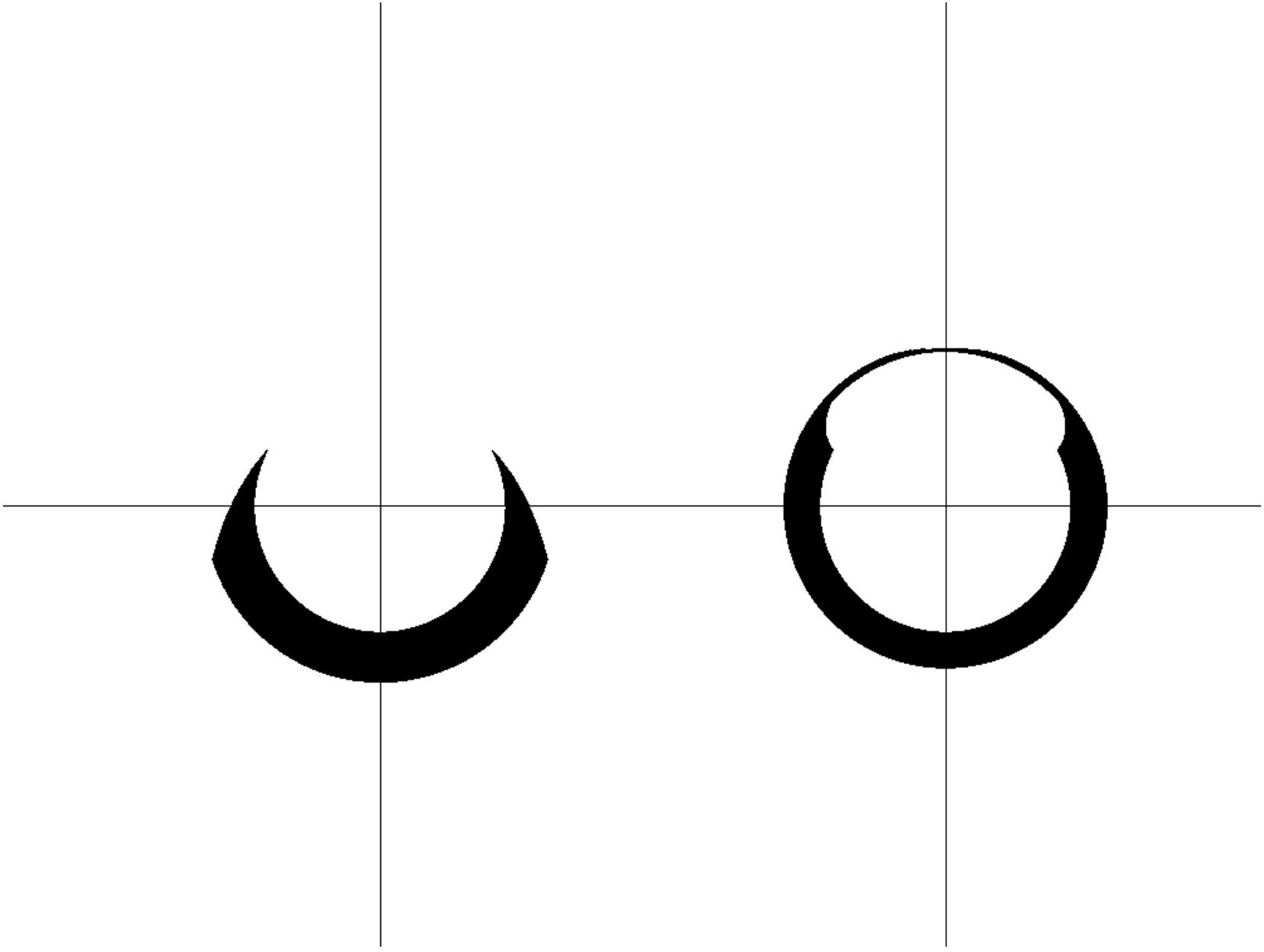
All together

$$\dot{q} \in \Delta_Q, \qquad p = \frac{\partial L}{\partial v}$$

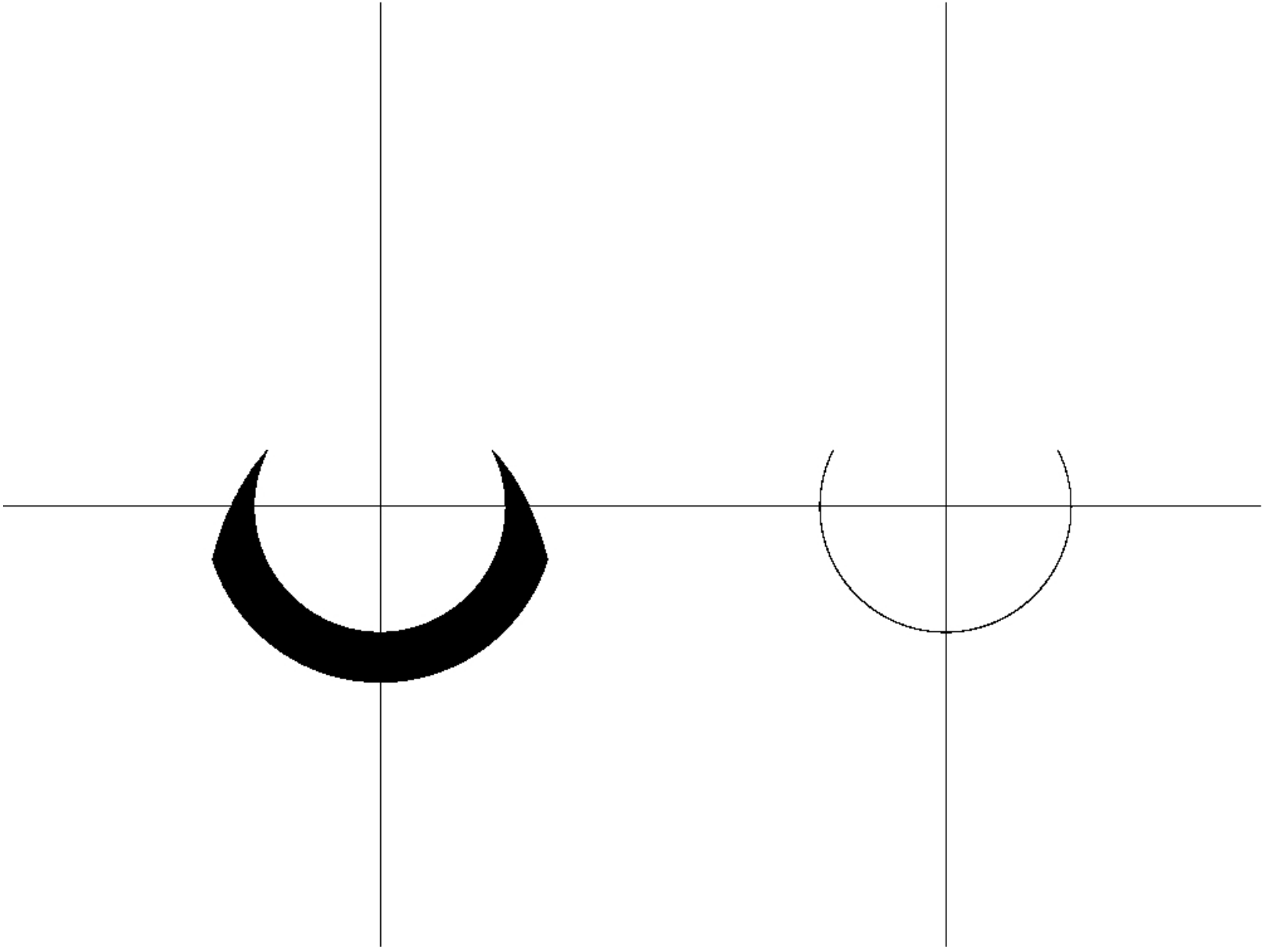
$$\dot{q} = v, \qquad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^0$$

$$x\dot{x} + y\dot{y} = 0$$
$$\ddot{x} = \lambda x$$
$$\ddot{y} = -mg + \lambda y$$

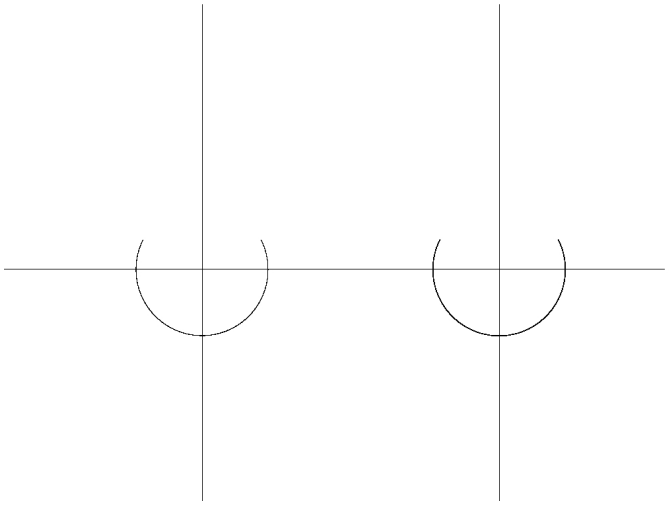
Simulations: Dirac 1 vs Euler



Simulations : Dirac 1 vs Dirac 2

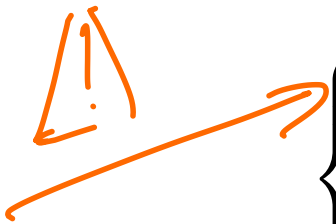


Simulations : Dirac 2 vs All Stars



Dirac order 1(M.Leok, T.Ohsawa)
$\rightarrow$ 0.952115
Dirac order 2 (V.S.)
$\rightarrow$ 0.00204053
Trapezium 2(implicit and cheating)
$\rightarrow$ 0.0128399
Adams–Bashforth 3 (cheating)
$\rightarrow$ 0.000136601
Runge–Kutta 4 (still cheating)
$\rightarrow 9.4 \cdot 10^{-8}$

How to discretize? What is 'cheating'?



$$\begin{cases} \alpha^a(v) = 0, \quad a = 1, \dots, m. \\ \mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}, \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \frac{\partial L}{\partial \mathbf{q}} = \sum_{a=1}^{m} \lambda_a \alpha^a. \end{cases}$$

The discrete Lagrangian $L_d = \Delta t \; L(\mathbf{q}^n, \mathbf{v}^n)$.

Discrete equations:

$$< \alpha_d^a, \mathbf{v}^n >= 0, \quad a = 1, \dots, m; \quad \mathbf{p}^{n+1} = \frac{1}{\Delta t} \frac{\partial L_d}{\partial \mathbf{v}^n}$$

$$\mathbf{p}^n - \frac{1}{\Delta t} \frac{\partial L_d}{\partial \mathbf{v}^n} + \frac{\partial L_d}{\partial \mathbf{q}^n} = \sum_{a=1}^{m} \lambda_a \frac{\partial < \alpha_d^a, \mathbf{v}^n >}{\partial \mathbf{v}^n}$$

Explicitly $\mathbf{p}^n$ and $\mathbf{p}^{n+1}$, and $\mathbf{v}^n$ – approximates the velocity, containing $\mathbf{q}^n$, e.g. $\mathbf{v}^n := \dfrac{\mathbf{q}^{n+1} - \mathbf{q}^{n-1}}{2\Delta t}$.

$2d + m$ equations for $2d + m$ unknowns.

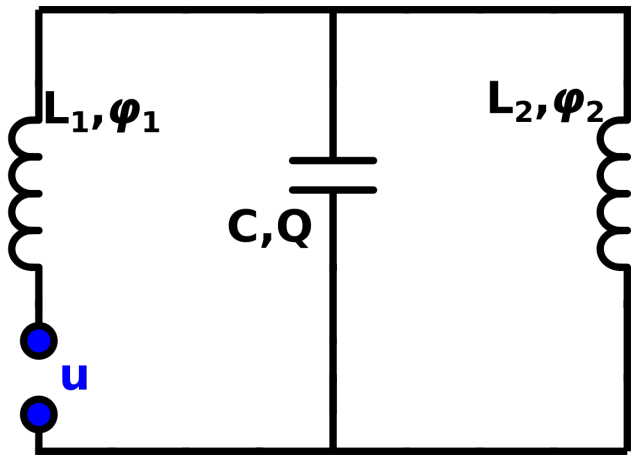
More examples – cf. talk by Daria Loziienko.

A lot of open questions to be answered

-> Part 2 - Graded geometry.

Application 2. Port-Hamiltonian systems.

Example: Electric circuit (L_1, L_2, C) with a controlled port u



$$\begin{cases} \dot{Q} = \varphi_1/L_1 - \varphi_2/L_2 \\ \dot{\varphi}_1 = -Q/C + u \\ \dot{\varphi}_2 = Q/C. \end{cases}$$

$$H = \frac{\varphi_1^2}{2L_1} + \frac{\varphi_2^2}{2L_2} + \frac{Q^2}{2C}$$

Port: input u, output $e = \varphi_1/L_1$.

Port-Hamiltonian system: $\dot{\mathbf{x}} = J(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}} + g(\mathbf{x})\mathbf{f}$ with

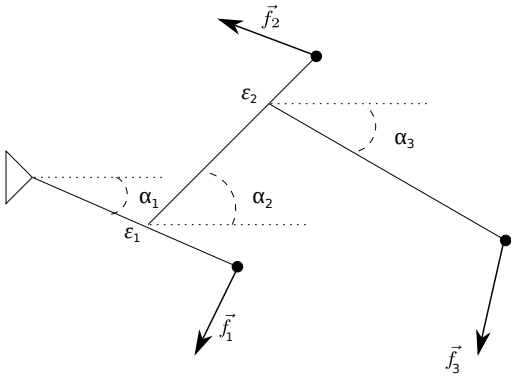
$$\mathbf{x} = \begin{pmatrix} Q \\ \varphi_1 \\ \varphi_2 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{matrix} f = u \\ e = \varphi_1/L_1 \end{matrix}$$

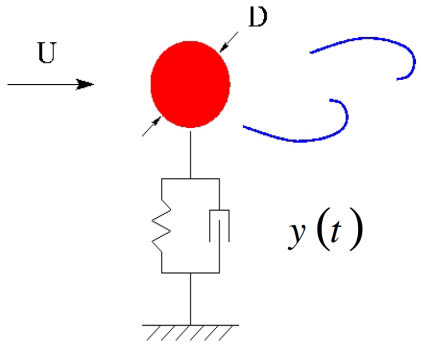
$$f_s := -\dot{\mathbf{x}}, \quad e_s := \begin{pmatrix} p_Q \\ p_{\varphi_1} \\ p_{\varphi_2} \end{pmatrix} \equiv \begin{pmatrix} Q/C \\ \varphi_1/L_1 \\ \varphi_2/L_2 \end{pmatrix}.$$

$$\dot{H} \equiv -e_s^T f_s = u\varphi_1/L_1 \Leftrightarrow e_s^T f_s + ef = 0$$
$$\Rightarrow \textit{almost Dirac}$$

Port-Hamiltonian systems

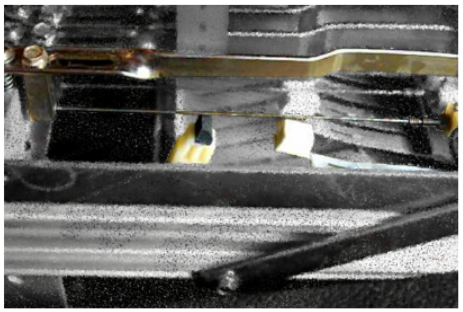
A lot of examples
(ask Antoine Falaize) :





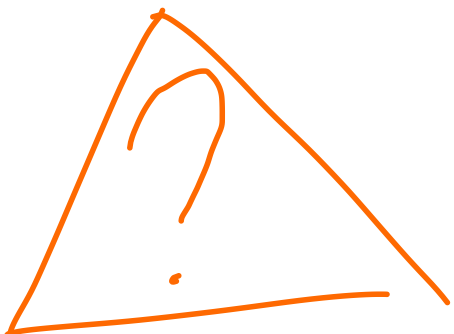








Conjecture (VS): Everything is port-Hamiltonian.
Question: OK for fun, but does it really help?
Answer = real question: How to discretize Dirac?



CNRS 80Prime GraNum project

www.ryvkin.eu/granum/

MEMBRES DU PROJET

- Oscar Cosserat
- Anthony Gravouil
- Aziz Hamdouni
- Camille Laurent-Gengoux (porteur du projet)
- Ruben Louis
- Daria Loziienko
- Cyrille Ospel
- Dina Razafindralandy
- Volodya Roubtsov
- Leonid Ryvkin
- Vladimir Salnikov (porteur du projet)
- Pol Vanhaecke
- Tilmann Wurzbacher



Geometry – episode 2

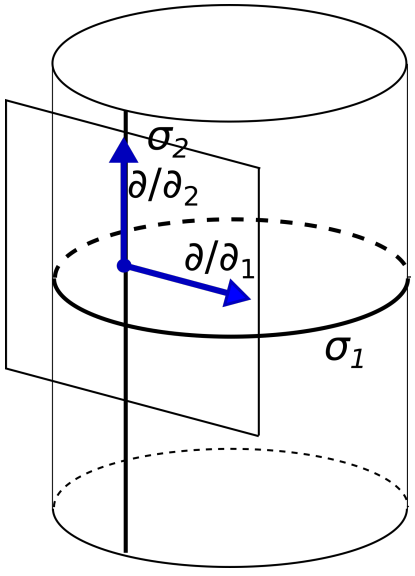
Way 1 (= topic 1 of GraNum) – Foliations

j./w. Alexei Kotov, Camille Laurent-Gengoux, Leonid Ryvkin

Some ideas:

- Holonomic constrains $\rightarrow$ (integrable) Dirac structure. (Proof via pullback of the canonical distribution DS and a gauge transformation).
- Constraint distribution $\rightarrow$ (singular) foliation
  - Poisson integrator
  - Poisson variational principle
  - Convenient description of (a class of) Dirac structures = foliation + extra data on the leaves
  - Discretization of (a class of) Dirac structures

Way 2 (= "Gra" of GraNum) – Graded geometry



Consider functions on $T[1]\Sigma$.

$\sigma^1, \ldots, \sigma^d$ – coordinates on Σ:
$deg(\sigma^\mu) = 0$, $\sigma^{\mu_1}\sigma^{\mu_2} = \sigma^{\mu_2}\sigma^{\mu_1}$.
$deg(h(\sigma^1, \ldots, \sigma^d)) = 0$.

$\theta^1, \ldots, \theta^d$ – fiber linear coordinates:
$deg(\theta^\mu) := 1$, $\theta^{\mu_1}\theta^{\mu_2} = -\theta^{\mu_2}\theta^{\mu_1}$

Arbitrary homogeneous function on $T[1]\Sigma$ of $deg = p$:
$f = \sum f_{\mu_1 \ldots \mu_p}(\sigma^1, \ldots, \sigma^d)\theta^{\mu_1} \ldots \theta^{\mu_p}$.

Graded commutative product: $f \cdot g = (-1)^{deg(f)deg(g)} g \cdot f$

$f \leftrightarrow \omega = \sum f_{\mu_1 \ldots \mu_p} \mathrm{d}\sigma^{\mu_1} \wedge \cdots \wedge \mathrm{d}\sigma^{\mu_p} \in \Omega(\Sigma)$

$\rightarrow$ *"Definition" of a graded manifold*
– manifold with a ($\mathbb{Z}$-)grading defined on the sheaf of functions.

Graded manifolds – details

"...graded manifolds are just manifolds with a few bells and whistles..." (D. Roytenberg)

Graded geometry: definitions (do not read)

- Graded vector space V is a collection of vector spaces $V = \oplus V_i$ ($i \in \mathbb{Z}$ or $i \in \mathbb{Z}_{\geq 0}$); if $v \in V_i$, $deg(v) = i$.
- Homomorphism shifting the grading by p: $(V[p])_i = V_{i-p}$.
- Assume the base to be of degree 0, the dual vector space $(V_i)^*$ is defined as $(V^*)_{-i}$.
- Graded algebra structure $\cdot\colon V \otimes V \to V$, s.t. $V_p \otimes V_q \to V_{p+q}$.
- Graded commutator $[a, b] = ab - (-1)^{deg(a)deg(b)}ba$.
- Graded symmetric algebra over V: $S(V) = \mathrm{Tensor}(V)/[\cdot, \cdot]$

Definition. Graded manifold M is a couple $(M_0, \mathcal{O}_M)$, where M_0 is a smooth manifold and the sheaf of functions $\mathcal{O}_M$ is locally isomorphic to $C^\infty(U_0) \otimes S(V)$, where U_0 is an open subset of M_0.

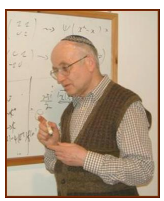
- Top degree of the generators of $\mathcal{O}_M$ – is called degree of M.

Standard abuse of notations: V_k-vector bundle or sheaf of sections.

Graded manifolds, super manifolds

History


Felix Berezin


Joseph Bernstein


Pierre Deligne


Dimitry Leites

Philosophy


Dmitry Roytenberg

"...graded manifolds are just manifolds with a few bells and whistles..."

Graded manifolds

D. Roytenberg: "...bells and whistles..."

Prop. (D.Roytenberg) Given a non-negatively graded manifold $(M, \mathcal{O}_M)$ there is a tower of fibrations

$$M = M_n \to M_{n-1} \to \cdots \to M_1 \to M_0,$$

where any M_k is a graded manifold of degree at most k, for $k > 0$ $M_{k+1} \to M_k$ is an affine bundle.

Remark. Gradings can be encoded in the Euler vector field $\epsilon = deg(q^\alpha) q^\alpha \frac{\partial}{\partial q^\alpha}$; V_i corresponds to the i-eigenspace of ϵ.

Remark. Gradings can be encoded in the homogeneity structure $h\colon \mathbb{R}_+ \times \mathcal{M} \to \mathcal{M}$ such that $(q^1, \ldots, q^N) \mapsto h_t(q^1, \ldots, q^N) \equiv (t^{deg(q^1)}q^1, \ldots, t^{deg(q^N)}q^N)$.

Q-manifolds (DG-manifolds)

Motivating example:
$T[1]\Sigma$, $deg(\sigma^\mu) = 0$, $deg(\theta^\mu) = 1$,
Functions of the form: $f = \sum f_{\mu_1 \dots \mu_p}(\sigma^1, \dots, \sigma^d)\theta^{\mu_1} \dots \theta^{\mu_p}$

Consider a vector field $Q = \sum \theta^\mu \frac{\partial}{\partial \sigma^\mu}$

- $\deg Q = 1$
- $Q(f \cdot g) = (Qf) \cdot g + (-1)^{1 \cdot deg(f)} f \cdot (Qg)$
- $[Q, Q] \equiv 2Q^2 = 0$

Q-manifolds (DG-manifolds)

Motivating example:
$T[1]\Sigma$, $deg(\sigma^\mu) = 0$, $deg(\theta^\mu) = 1$,
Functions of the form: $f = \sum f_{\mu_1 \dots \mu_p}(\sigma^1, \dots, \sigma^d)\theta^{\mu_1} \dots \theta^{\mu_p}$

Consider a vector field $Q = \sum \theta^\mu \frac{\partial}{\partial \sigma^\mu}$

$$\left.\begin{array}{l} \deg Q = 1 \\ Q(f \cdot g) = (Qf) \cdot g + (-1)^{1 \cdot deg(f)} f \cdot (Qg) \\ [Q, Q] \equiv 2Q^2 = 0 \end{array}\right\} \leftarrow \mathrm{d}_{\text{de Rham}}$$

Poisson manifold $\to (T^*[1]M, Q_\pi)$

Consider a Poisson manifold M,
$\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$.

A Poisson bracket can be written as $\{f,g\}=\pi(\mathrm{d}f,\mathrm{d}g)$, where $\pi\in\Gamma(\Lambda^2 TM)$ is a bivector field. $\pi^{ij}(x)=\{x^i,x^j\}$.

Consider $T^*[1]M$ (coords. $x^i(0), p_i(1)$), with a $deg=1$ vector field

$$Q_\pi=\left\{\frac{1}{2}\pi^{ij}p_ip_j,\cdot\right\}_{T^*M}=\pi^{ij}(x)p_j\frac{\partial}{\partial x^i}-\frac{1}{2}\frac{\partial\pi^{jk}}{\partial x^i}p_jp_k\frac{\partial}{\partial p_i}$$

Jacobi identity for π: $Q_\pi^2=0\Leftrightarrow$

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0\Leftrightarrow$$

$$\frac{\partial\pi^{ij}(x)}{\partial x^l}\pi^{lk}(x)+\frac{\partial\pi^{ki}(x)}{\partial x^l}\pi^{lj}(x)+\frac{\partial\pi^{jk}(x)}{\partial x^l}\pi^{li}(x)=0$$

Derived bracket construction

Let $(\mathcal{M}, Q)$ be a Q-manifold, and
$\mathcal{G}$ be degree -1 vector fields ε on $\mathcal{M}$.
Define the Q-derived bracket: $[\varepsilon, \varepsilon']_Q := [\varepsilon, [Q, \varepsilon']]$.

Remark. Good for equivariant Q-cohomology.
V.S. "Graded geometry in gauge theories and beyond", JGP, 2015.

Example 1. $(T^*[1]M, Q_\pi)$
$\varepsilon = \varepsilon_i(x)\frac{\partial}{\partial p_i} \leftrightarrow \varepsilon_i(x)dx^i \in \Omega^1(M)$.
If ε is exact, i.e. $\varepsilon_i dx^i = \epsilon_{,i} dx^i$, then $[\varepsilon, \varepsilon']_Q = \{\epsilon, \epsilon'\}_{,i}\frac{\partial}{\partial p_i}$

Example 2. Symplectic, Poisson, Dirac structures, algebroids, foliations and more.



Example from physics. (Part of) the Standard Model

CIRM?



$\Rightarrow\Rightarrow$
$\Rightarrow\Rightarrow$

Gauging

$\Rightarrow\Rightarrow$
$\Rightarrow\Rightarrow$

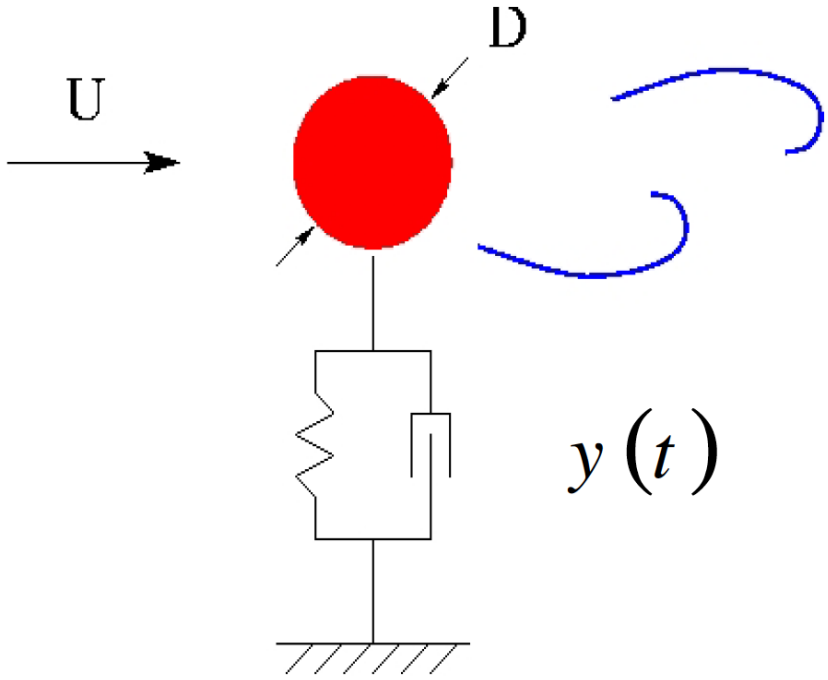


Quarks
$SU(3)$ symmetry

8 connection 1-forms
Gluons

(j/w Thomas Strobl – Lyon, Alexei Kotov – Hradec Králové)

Example: vortex induced vibrations. skip



Simplified model of fluid–structure interaction (cf. T. Leclercq, E. de Langre, Journal of Fluids and Structures, 80:2018)

The phenomenon is modelled by a harmonic oscillator coupled to the Van der Pol system:

$$\begin{aligned} \ddot{y} &+ y = m\Omega^2 q \\ \ddot{q} &- \varepsilon(1-q^2)\dot{q} + \Omega^2 q = A\ddot{y} \end{aligned}$$

Oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + m\Omega q \end{aligned}$$

Port-Hamiltonian framework: $H = \frac{1}{2}(x_1^2 + x_2^2)$,

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$f_s = -\dot{X}, \quad e_s = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad f = M\Omega^2 q, \quad e = \dot{x}_1,$$

Energy evolution: $\dot{H} = -e_s^T f_s = ef$.

Associated Dirac structure

Consider a manifold M, with coordinates (x_1, x_2),
then $f_s \in \Gamma(TM), e_s \in \Gamma(T^*M)$,
and the inputs–outputs: $f \in \Gamma(\mathcal{F})$, $e \in \Gamma(\mathcal{F}^*)$.
$\mathbb{TM} = (TM \times \mathcal{F}) \oplus (T^*M \times \mathcal{F}^*)$, where one considers $\mathcal{F}$ as a bundle over a point The (almost) Dirac is defined by $e_s^T f_s + ef = 0$ – a subbundle of rank 3, given by

$$\begin{pmatrix} f_s \\ f \\ e_s \\ e \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 - M\Omega^2 q \\ m\Omega^2 q \\ x_1 \\ x_2 \\ x_2 \end{pmatrix}$$

Van der Pol system

Rewrite the second equation as

$$\ddot{q} - \varepsilon(1 - q^2)\dot{q} + (\Omega^2 - Am\Omega^2)q = -Ay,$$

where $(\Omega^2 - Am\Omega^2) =: \tilde{\Omega}^2$ and $\varepsilon(1 - q^2) =: a$.

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= -\tilde{\Omega}^2 q_1 + a q_2 - Ay, \end{aligned}$$

Port-Hamiltonian structure: $H = \frac{1}{2}(\tilde{\Omega} q_1^2 + q_2^2)$,

$$Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$f_s = -\dot{Q}, e_s = \begin{pmatrix} \tilde{\Omega}^2 q_1 \\ q_2 \end{pmatrix}, \quad f_i = -Ay, e_i = \dot{q}_1, \quad f_d = -\dot{q}_1, e_d = -a\dot{q}_1.$$

Energy evolution: $\dot{H} = -e_s^T f_s = e_i f_i + e_d f_d$.

Associated Dirac structure

By abuse of notation, a manifold M with coordinates (q_1, q_2); $f_s \in \Gamma(TM), e_s \in \Gamma(T^*M)$. Inputs–outputs: $(f_i, f_d) \in \Gamma(\mathcal{F})$, $(e_i, e_d) \in \Gamma(\mathcal{F}^*)$.
$\mathbb{TM} = (TM \times \mathcal{F}) \oplus (T^*M \times \mathcal{F}^*)$, almost Dirac structure $e_s^T f_s + e_i f_i + e_d f_d = 0$ – a subbundle of rank 4, given by

$$\begin{pmatrix} f_s \\ f_i \\ f_d \\ e_s \\ e_i \\ e_d \end{pmatrix} = \begin{pmatrix} -q_2 \\ \tilde{\Omega}^2 q_1 - a q_2 + A y \\ -A y \\ -q_2 \\ \tilde{\Omega}^2 q_1 \\ q_2 \\ q_2 \\ -a q_2 \end{pmatrix}$$

Coupling

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + m\Omega q \\
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= -\tilde{\Omega}^2 q_1 + a q_2 - A y
\end{aligned}$$

Port–Hamiltonian formalism: $H = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}(\tilde{\Omega} q_1^2 + q_2^2)$,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ q_1 \\ q_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ m\Omega^2 q_1 \\ 0 \\ a(q_1) q_2 - A x_1 \end{pmatrix},$$

$$f_s = -\dot{X}, e_s = \begin{pmatrix} x_1 \\ x_2 \\ \tilde{\Omega}^2 q_1 \\ q_2 \end{pmatrix}, \quad e_i = 1 \in \mathbb{R}^1, f_i = m\Omega^2 q_1 x_2 - A x_1 q_2 + a(q_1) q_2^2.$$

Evolution of energy: $\dot{H} = -e_s^T f_s = e_i f_i$.

Graded description

Manifold M, with coordinates: (x_1, x_2, q_1, q_2),
$f_s \in \Gamma(TM)$, $e_s \in \Gamma(T^*M)$,
inputs–outputs: $f_i \in \Gamma(\mathcal{F})$, $e_i \in \Gamma(\mathcal{F}^*)$.
$\mathbb{TM} = (TM \times \mathcal{F}) \oplus (T^*M \times \mathcal{F}^*)$.
The (almost) Dirac structure is given by $e_s^T f_s + e_i f_i$ – a subbundle of rank 5, or by

$$\begin{pmatrix} f_s \\ f_i \end{pmatrix} = D \begin{pmatrix} e_s \\ e_i \end{pmatrix},$$

where $D\colon\ T_X^*M \times \mathcal{F}^* \to T_XM \times \mathcal{F}$ – a bivector, in components:

$$D = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -m\Omega^2 q_1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -a(q_1)q_2 + Ax_1 \\ 0 & m\Omega^2 q_1 & 0 & aq_2 - Ax_1 & 0 \end{pmatrix}.$$

Q-structure

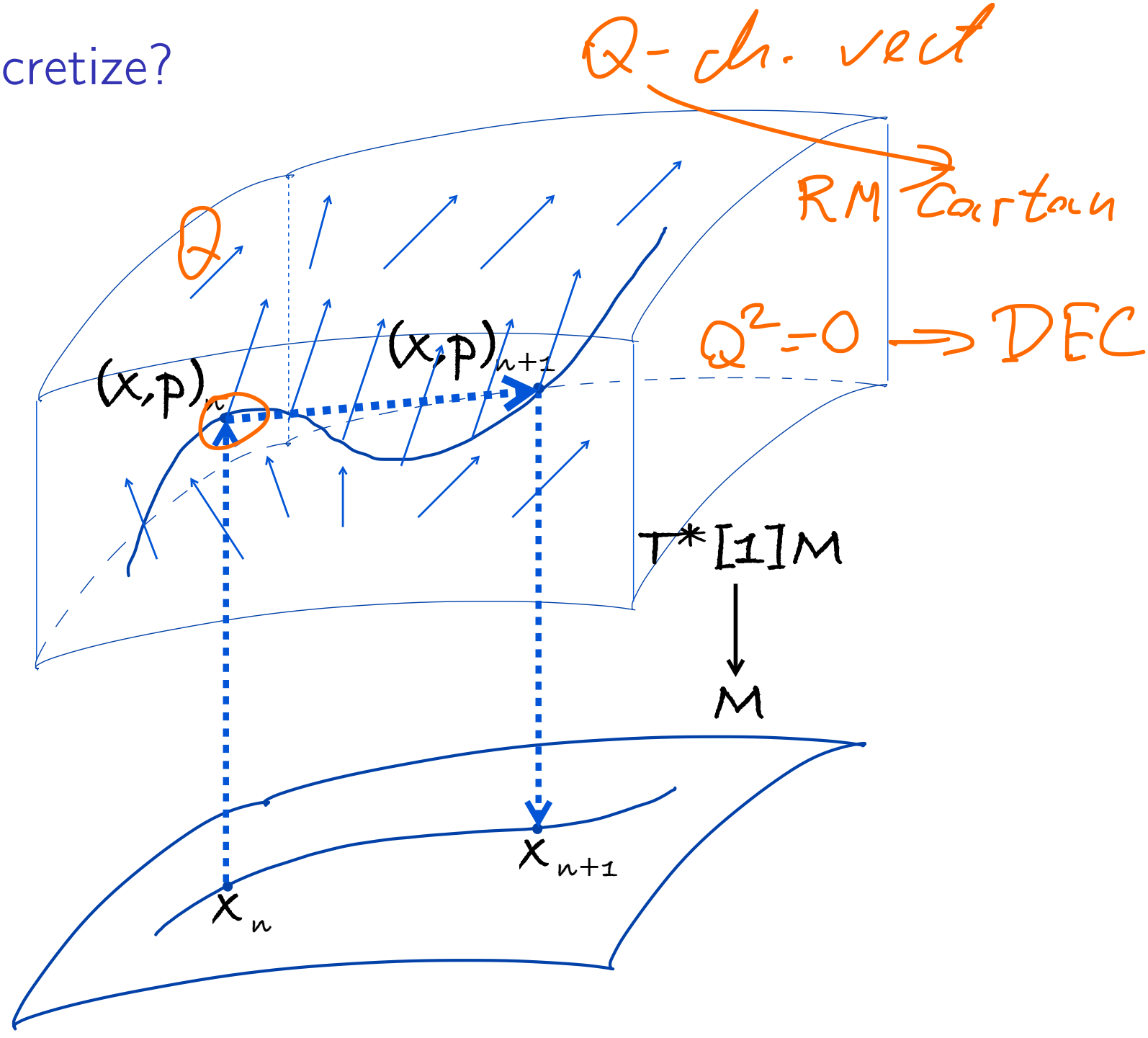
For the *graded description* consider the graded manifold $T^*[1]\mathcal{M}$ with coordinates x^i (of degree 0), et p_i (of degree 1). The degree 1 vector field constructed from D is

$$Q = -p_2\frac{\partial}{\partial x^1} + (p_1 - p_5 m\Omega^2 x^3)\frac{\partial}{\partial x^2} - p_4\frac{\partial}{\partial x^3} +$$
$$+(p_3 - p_5 a(x^3)x^4 - Ax^1)\frac{\partial}{\partial x^4} +$$
$$+(p_2 m\Omega^2 x^3 + p_4 a(x^3)x^4 - Ax^1)\frac{\partial}{\partial x^5} +$$
$$+Ap_4p_5p_5\frac{\partial}{\partial p_1} + (-m\Omega^2 p_2 p_5 + 2\varepsilon x^3 x^4 p_4 p_5)\frac{\partial}{\partial p_3} - a(x^3)p_4p_5\frac{\partial}{\partial p_4}$$

[1] V. Salnikov, A. Hamdouni, Geometric integrators in mechanics - the need for computer algebra tools, Proc. of The third International Conference "Computer algebra 2019, Moscow.

[2] V. Salnikov, A. Hamdouni, Géométrie généralisée et graduée pour la mécanique, actes du CFM, 2019, Brest, France.

How to discretize?



Open questions

Near at hand:

- Poisson integrators
- Dynamics for general Dirac structures
- Application to control theory

Slightly further away

- Field theories / sigma models for mechanics and beyond.
  E.g. Poisson Sigma Model via n-plectic geometry.
- PDEs in general
- Discretization and symbolic calculus in the graded world

Thanks for your attention.