Dynamique en grande déformation et grand déplacement des poutres tri-dimensionnelles de Timoshenko

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Cachan, France

Principe of beams

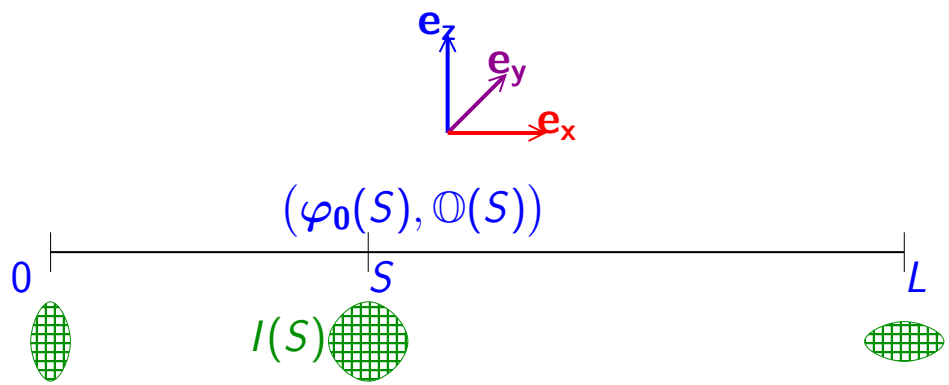
$\Rightarrow$ Asymptotical model for elongated body $L \gg \sqrt{A}$

- One dimensional body: $(S, t) \in [0, L] \times \mathbb{R}$
- Immersion of a 1D material manifold in a 3D euclidean space
- Two methodologies
  - Asymptotics from a 3 dimensional analysis
  - Kinematical hypothesis: rigid section

Characterization of beams

Material coordinates

$$S \in [0, L], \qquad (\xi_1, \xi_2) \in I(S)$$



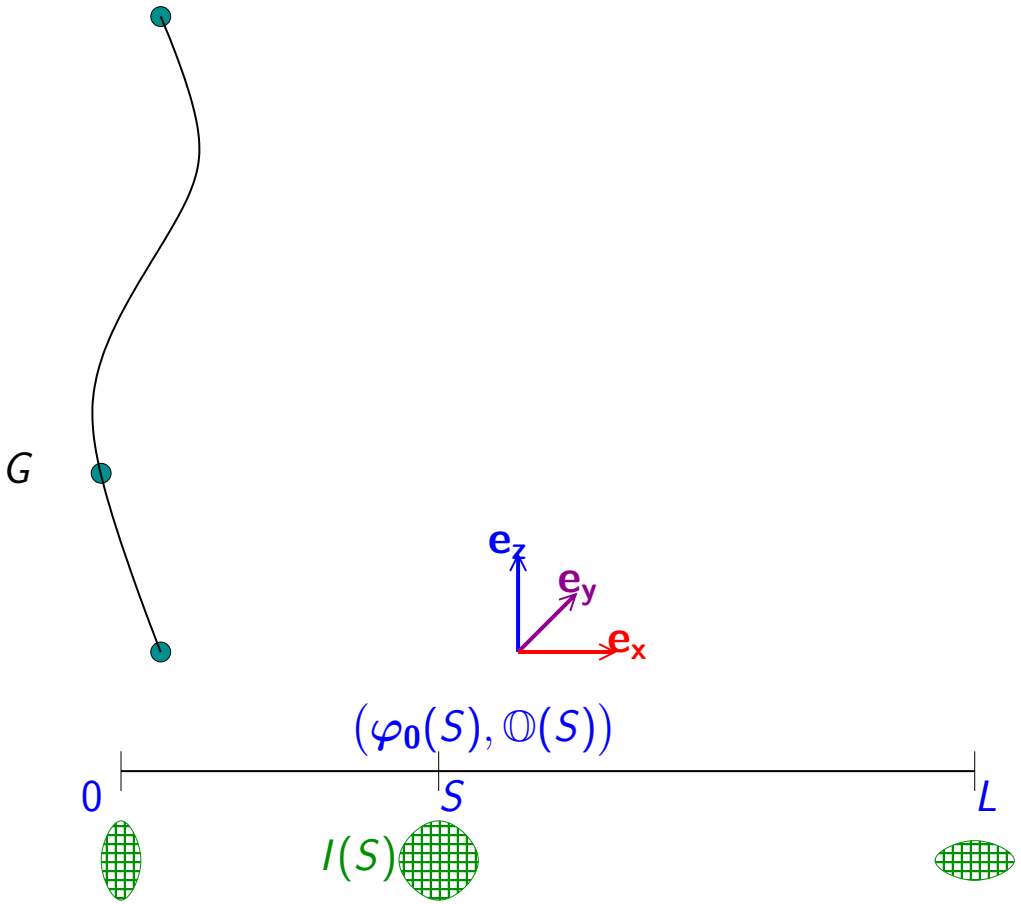
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Configuration in $\mathcal{E}$

$$\varphi_0(S) = \mathbf{OG}$$



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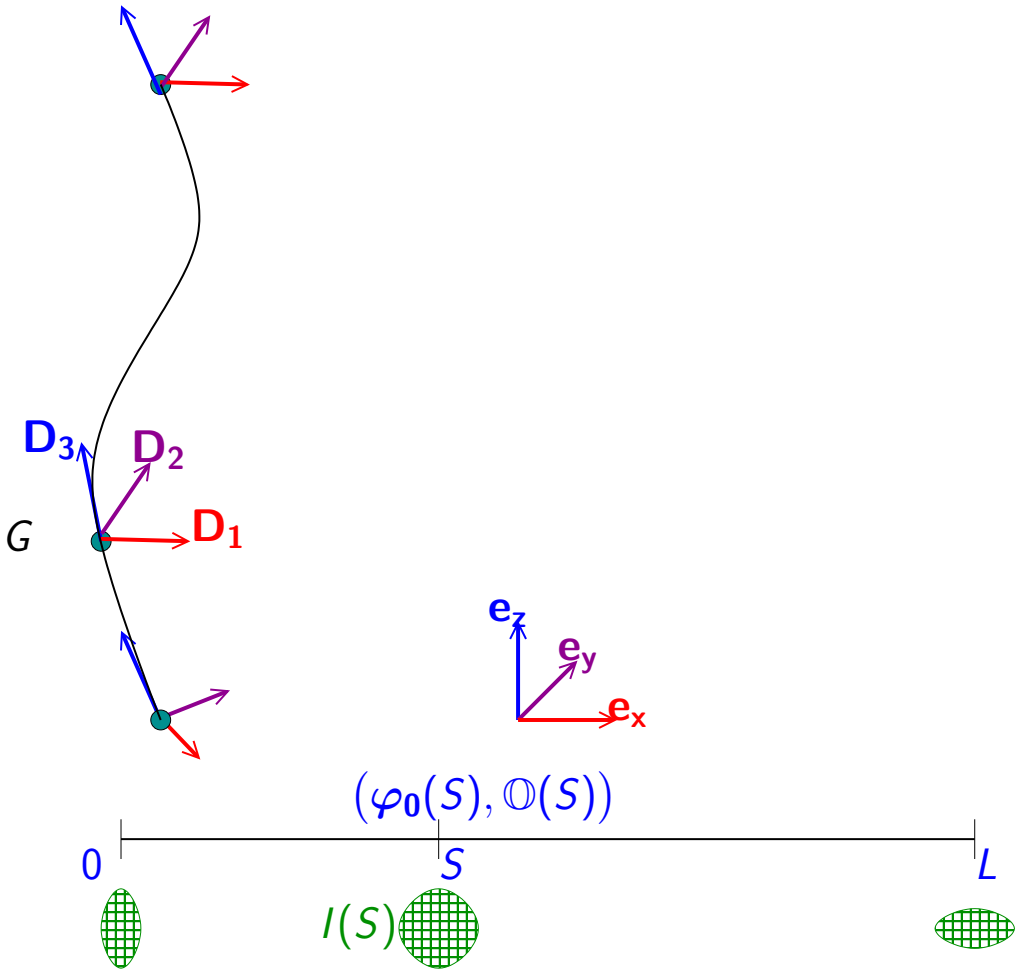
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$$\mathbb{O}(S) \in SO(3) \quad | \quad \mathbf{D_i} = \mathbb{O}\mathbf{e_i}$$



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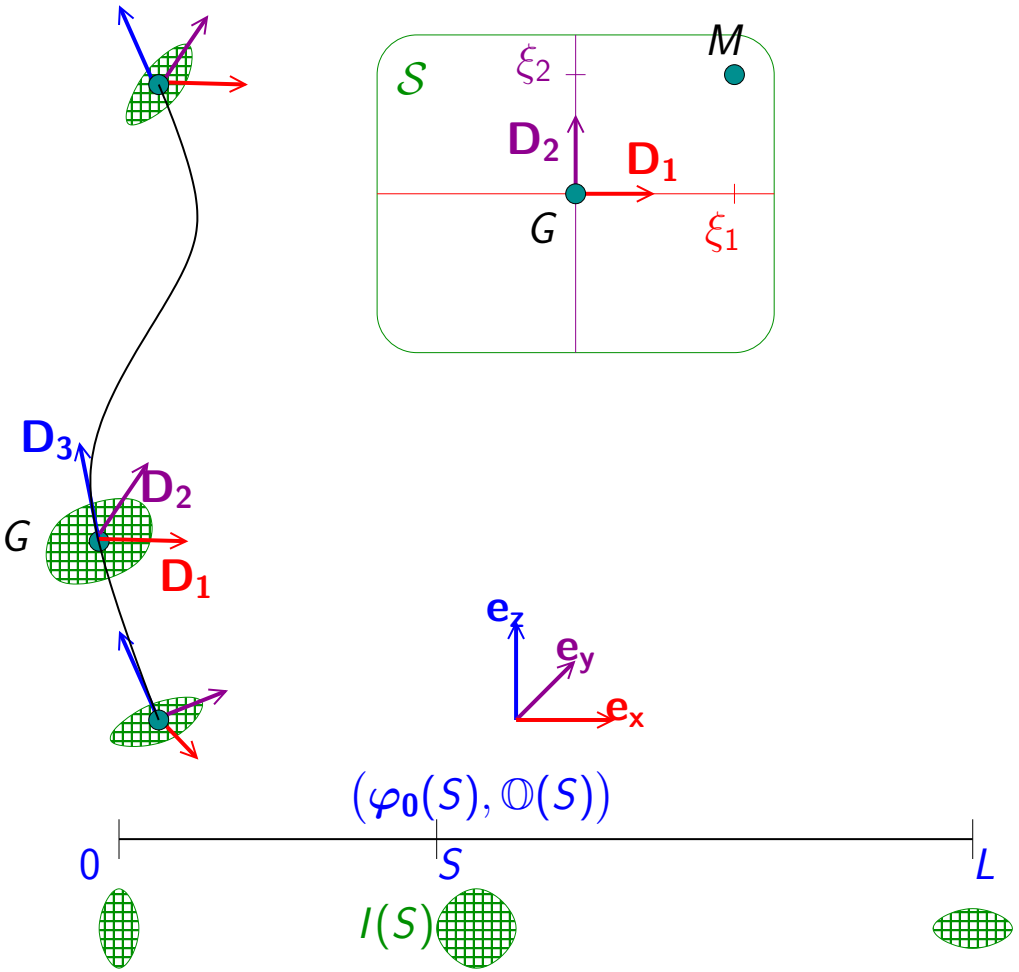
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Body $\mathcal{B} = \bigcup_{S\in[0,L]} I(S)$

Placement of any point M of $\mathcal{B}$:

$$(\xi_1, \xi_2, S) \longrightarrow \mathbf{OM} = \mathbf{OG} + \mathbf{GM}$$
$$\mathbf{GM} = \xi_1 \mathbf{D_1} + \xi_2 \mathbf{D_2}$$



Characterization of beams

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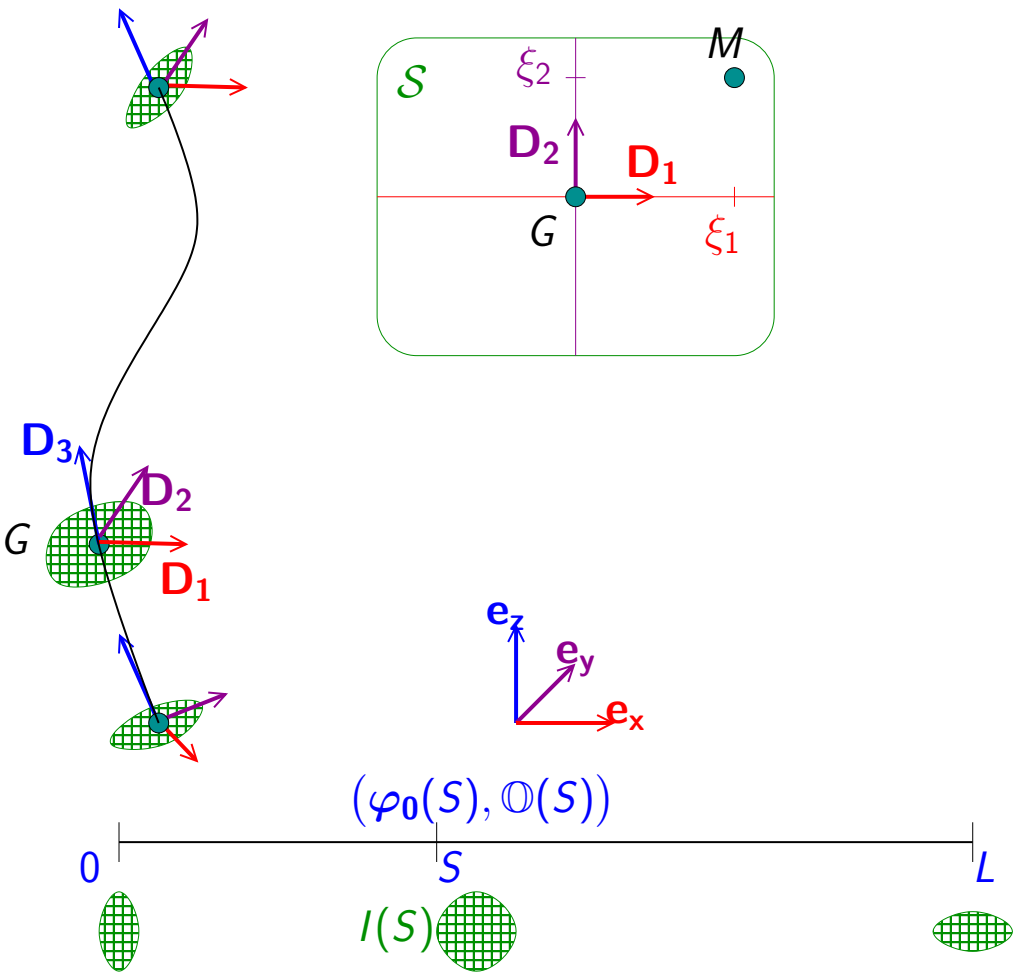
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Mass density $\rho_0(\xi_1, \xi_2, S)$



G is the center of mass of $\mathcal{S}$: $\int_{\mathcal{S}} \mathbf{GM}\ \rho_0 \mathrm{d}A = 0$

$\{\mathbf{D_i}\}$ principal basis $\int_{\mathcal{S}} (\mathbf{D_i} \wedge \mathbf{GM}) \cdot (\mathbf{D_j} \wedge \mathbf{GM})\ \rho_0 \mathrm{d}A = I_i \delta_{ij}$

Kinematics of a Timoshenko beam-like structure

Material coordinates

$$S \in [0, L], \qquad (\xi_1, \xi_2) \in I(S)$$

Configuration in $\mathcal{E}$

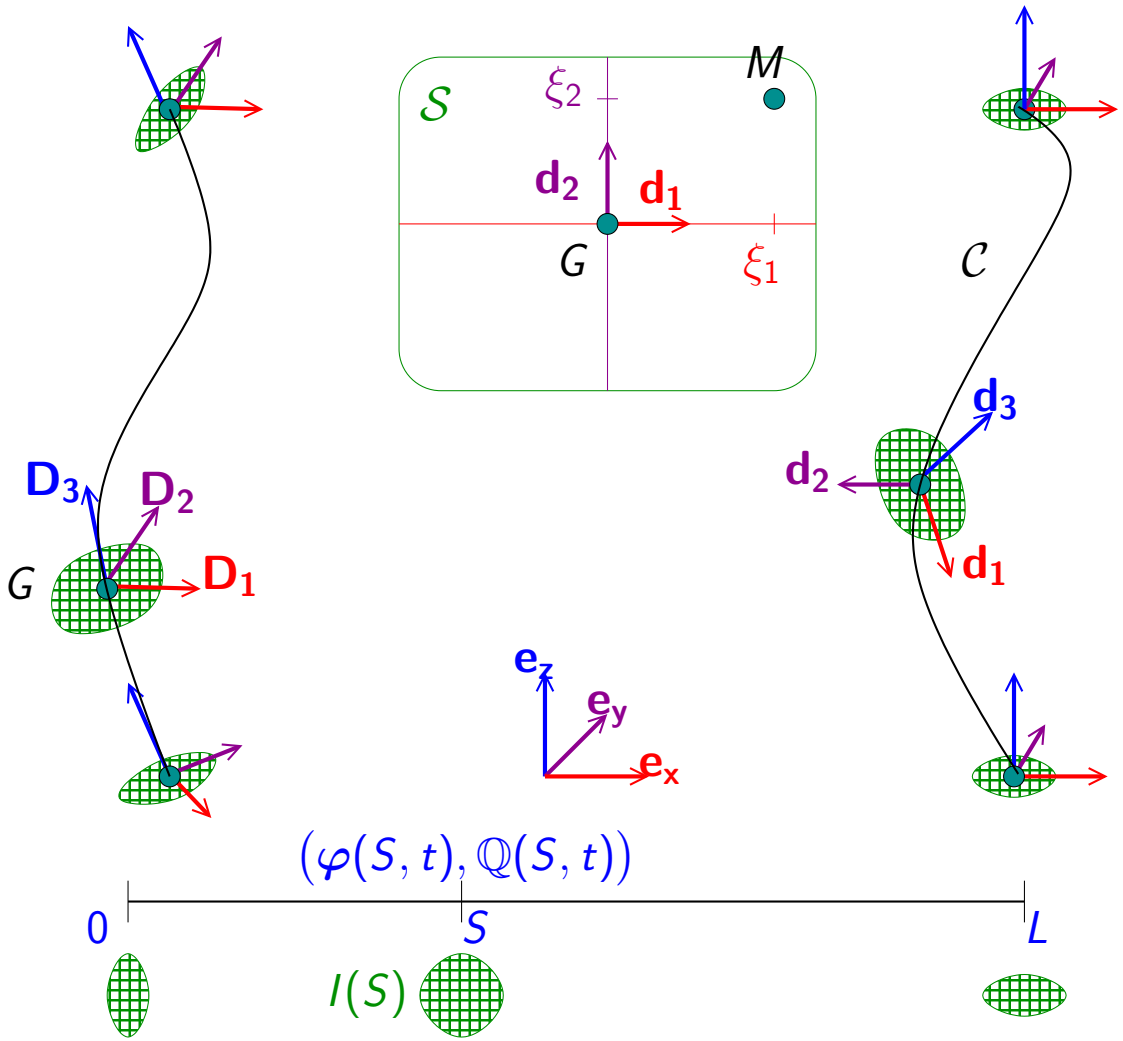
$$\varphi(S, t) = \mathbf{OG}$$
$$\mathbb{Q}(S, t) \in SO(3) \mid \mathbf{d_i}(S, t) = \mathbb{Q}\mathbf{D_i}$$

Body $\mathcal{B} = \bigcup_{S \in [0,L]} I(S)$

Placement ϕ of any point M of $\mathcal{B}$

$$(\xi_1, \xi_2, S, t) \rightarrow \mathbf{OM} = \mathbf{OG} + \mathbf{GM}$$
$$\mathbf{GM} = \xi_1 \mathbf{d_1} + \xi_2 \mathbf{d_2}$$

Section $\mathcal{S}$ is **rigidly** transformed



$\mathbf{d_3}$ is **not** tangent to $\mathcal{C}$

Gradient of the transformation: *How to compute* $D\phi \equiv \mathbf{F}$ *?*

Curvature

$$\exists\, \boldsymbol{\kappa} \;\mid\; \frac{\partial \mathbf{d_i}}{\partial S} = \boldsymbol{\kappa} \wedge \mathbf{d_i}$$

Spin

$$\exists\, \boldsymbol{\omega} \;\mid\; \frac{\partial \mathbf{d_i}}{\partial t} = \boldsymbol{\omega} \wedge \mathbf{d_i}$$

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No commutation (Maurer-Cartan)

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} - \frac{\partial \boldsymbol{\omega}}{\partial S} = \boldsymbol{\omega} \wedge \boldsymbol{\kappa}$$

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$$D\phi : \; T\mathcal{B}_0 \;\rightarrow\; T\mathcal{B}_t, \qquad \text{with} \qquad D\phi = \frac{\partial x^i}{\partial X^J}\mathbf{e_i}\otimes\mathbf{E_J} = \frac{\partial x^i}{\partial X^J}\frac{\partial \phi}{\partial x^i}\otimes\mathbf{E_J} = \frac{\partial \phi}{\partial X^J}\otimes\mathbf{E_J}$$

Classical mechanics		Timoshenko-Cosserat
$\{\mathbf{E_J}\}$	reference local basis in $\mathcal{B}_0$	$\{\mathbf{D_J}\}$
(X^1, X^2, X^3)	material coordinate	(ξ_1, ξ_2, S)
$\{\mathbf{e_i}\}$	current local basis in $\mathcal{B}_t$	$\{\mathbf{d_i}\}$
$\phi = x_i\mathbf{e_i}$	placement within current basis	$\xi_1\mathbf{d_1}(S) + \xi_2\mathbf{d_2}(S) + \boldsymbol{\varphi}(S)$

$$\begin{aligned} D\phi &= \frac{\partial \phi}{\partial \xi_1}\otimes\mathbf{D_1} + \frac{\partial \phi}{\partial \xi_2}\otimes\mathbf{D_2} + \frac{\partial \phi}{\partial S}\otimes\mathbf{D_3} \\ &= \mathbf{d_1}\otimes\mathbf{D_1} + \mathbf{d_2}\otimes\mathbf{D_2} + \left(\xi_1\frac{\partial \mathbf{d_1}}{\partial S} + \xi_2\frac{\partial \mathbf{d_2}}{\partial S} + \frac{\partial \boldsymbol{\varphi}}{\partial S}\right)\otimes\mathbf{D_3} \\ &= \mathbf{d_i}\otimes\mathbf{D_i} + \left(\boldsymbol{\kappa}\wedge\mathbf{GM} + \frac{\partial \boldsymbol{\varphi}}{\partial S} - \mathbf{d_3}\right)\otimes\mathbf{D_3} \end{aligned}$$

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Classical mechanics with rigid section hypothesis on a non-convected current basis

Strain, stress, energy

Deformation gradient

$$\mathbf{F} = \mathbf{d_i} \otimes \mathbf{D_i} + \left(\boldsymbol{\kappa} \wedge \mathbf{GM} + \frac{\partial \boldsymbol{\varphi}}{\partial S} - \mathbf{d_3} \right) \otimes \mathbf{D_3}$$

Green-Lagrange strain

$$\mathbf{E}(\xi_1, \xi_2, S) = \frac{1}{2} \left(\mathbf{F}^T \mathbf{F} - \mathbf{I} \right), \qquad \mathbf{E} = E^{IJ} \mathbf{D_I} \otimes \mathbf{D_J}$$

governed by

$$\boldsymbol{\kappa} = \kappa_i \mathbf{d_i} \quad \text{Curvature (flexion, torsion)}$$

$$\boldsymbol{\varepsilon} = \frac{\partial \boldsymbol{\varphi}}{\partial S} - \mathbf{d_3} = \varepsilon_i \mathbf{d_i} \quad \text{Strain-vector (gliding, extension)}$$

Strain, stress, energy

Deformation gradient

$$\mathbf{F} = \mathbf{d_i} \otimes \mathbf{D_i} + \left(\boldsymbol{\kappa} \wedge \mathbf{GM} + \frac{\partial \boldsymbol{\varphi}}{\partial S} - \mathbf{d_3} \right) \otimes \mathbf{D_3}$$

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Free-energy (hyper-elastic material)

$$\Psi(\mathbf{E}) = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$$

Second Piola-Kirchhoff stress tensor

$$\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}}, \quad \mathbf{S} = \frac{\partial \Psi}{\partial E_{IJ}} \mathbf{D_I} \otimes \mathbf{D_J}$$

Internal virtual power

$$\mathcal{P}_i(\Phi, \delta\Phi) = \int_{\mathcal{B}} \mathbf{S} : \delta\mathbf{E}\, \mathrm{d}V = \int_{\mathcal{B}} \frac{\partial \Psi}{\partial E_{IJ}} \delta E_{IJ}\, \mathrm{d}V = \int_{\mathcal{B}} \frac{\partial \Psi}{\partial \varepsilon_i} \delta\varepsilon_i + \frac{\partial \Psi}{\partial \kappa_i} \delta\kappa_i\, \mathrm{d}V$$

Other formulation

First Piola-Kirchhoff stress tensor: $\mathbf{P} = \mathbf{FS}$

Force acting on $\mathcal{S}$:

$$\mathrm{N} = \int_{\mathcal{S}} \mathbf{P}(\mathbf{D_3})\,\mathrm{d}A := N_i \mathbf{d_i}$$

Moment acting on $\mathcal{S}$:

$$\mathrm{M} = \int_{\mathcal{S}} \mathbf{GM} \wedge \mathbf{P}(\mathbf{D_3})\,\mathrm{d}A := M_i \mathbf{d_i}$$

Internal virtual power

$$\mathcal{P}_i(\Phi, \delta\Phi) = \int_{\mathcal{B}} \mathbf{P} : \delta\mathbf{F}\,\mathrm{d}V$$

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Virtual perturbation $\delta\Phi$ $\quad \delta\mathbf{F} = \delta\mathbf{d}_\alpha \otimes \mathbf{D}_\alpha + \delta\dfrac{\partial \mathbf{OM}}{\partial S} \otimes \mathbf{D_3}$

$$\begin{array}{rcl|rcl} \delta\mathbf{d_i} & = & \delta\boldsymbol{\theta} \wedge \mathbf{d_i} & \delta\mathbf{OG} & = & \delta\boldsymbol{\varphi} \\ \delta\mathbf{GM} & = & \delta\boldsymbol{\theta} \wedge \mathbf{GM} & \delta\mathbf{OM} & = & \delta\boldsymbol{\varphi} + \delta\boldsymbol{\theta} \wedge \mathbf{GM} \end{array}$$

$$\begin{aligned} \mathcal{P}_i(\Phi, \delta\Phi) &= \int_{\mathcal{B}} \mathbf{P}(\mathbf{D_3}) \cdot \left(\frac{\partial \delta\boldsymbol{\varphi}}{\partial S} - \delta\boldsymbol{\theta} \wedge \frac{\partial \boldsymbol{\varphi}}{\partial S} \right) + (\mathbf{GM} \wedge \mathbf{P}(\mathbf{D_3})) \cdot \frac{\partial \delta\boldsymbol{\theta}}{\partial S}\,\mathrm{d}V \\ &= \int_{\mathcal{C}} \mathrm{N} \cdot \left(\frac{\partial \delta\boldsymbol{\varphi}}{\partial S} - \delta\boldsymbol{\theta} \wedge \frac{\partial \boldsymbol{\varphi}}{\partial S} \right) + \mathrm{M} \cdot \frac{\partial \delta\boldsymbol{\theta}}{\partial S}\,\mathrm{d}S \end{aligned}$$

Comparison formulation

$$\mathcal{P}_i(\Phi, \delta\Phi) = \int_{\mathcal{C}} \mathsf{N} \cdot \left(\frac{\partial \delta\boldsymbol{\varphi}}{\partial S} - \delta\boldsymbol{\theta} \wedge \frac{\partial \boldsymbol{\varphi}}{\partial S} \right) + \mathsf{M} \cdot \frac{\partial \delta\boldsymbol{\theta}}{\partial S} \, \mathrm{d}S$$

Corrotational perturbation of $\mathbf{u} = u_i \mathbf{d_i}$

$$\begin{aligned} \delta^R \mathbf{u} &= \delta\mathbf{u} - \delta\boldsymbol{\theta} \wedge \mathbf{u} \\ &= (\delta u_i)\mathbf{d_i} + u_i(\delta\boldsymbol{\theta} \wedge \mathbf{d_i}) - \delta\boldsymbol{\theta} \wedge \mathbf{u} \\ &= (\delta u_i)\mathbf{d_i} \end{aligned}$$

$$\frac{\partial \delta\boldsymbol{\varphi}}{\partial S} - \delta\boldsymbol{\theta} \wedge \frac{\partial \boldsymbol{\varphi}}{\partial S} = \delta^R \frac{\partial \boldsymbol{\varphi}}{\partial S} = \delta^R \boldsymbol{\varepsilon}$$

$$\frac{\partial \delta\boldsymbol{\theta}}{\partial S} = \delta^R \boldsymbol{\kappa}$$

$$\mathcal{P}_i(\Phi, \delta\Phi) = \int_{\mathcal{C}} \mathsf{N} \cdot \delta^R \boldsymbol{\varepsilon} + \mathsf{M} \cdot \delta^R \boldsymbol{\kappa} \, \mathrm{d}S$$

Dual vectors associated with forces and torques are corotational strain and curvature

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$$\frac{\partial \delta\boldsymbol{\theta}}{\partial S} = \delta^R \boldsymbol{\kappa}$$

$$\mathcal{P}_i(\Phi, \delta\Phi) = \int_{\mathcal{C}} \mathsf{N} \cdot \delta^R \boldsymbol{\varepsilon} + \mathsf{M} \cdot \delta^R \boldsymbol{\kappa} \, \mathrm{d}S$$

Dual vectors associated with forces and torques are corotational strain and curvature

P : δF

$$\int_{\mathcal{C}} N_i \delta\varepsilon_i + M_i \delta\kappa_i \, \mathrm{d}S$$

S : δE

$$\int_{\mathcal{C}} \int_{\mathcal{S}} \frac{\partial \Psi}{\partial \varepsilon_i} \mathrm{d}A \; \delta\varepsilon_i + \int_{\mathcal{S}} \frac{\partial \Psi}{\partial \kappa_i} \mathrm{d}A \; \delta\kappa_i \, \mathrm{d}S$$

$$N_i = \int_{\mathcal{S}} \frac{\partial \Psi}{\partial \varepsilon_i} \, \mathrm{d}A \qquad M_i = \int_{\mathcal{S}} \frac{\partial \Psi}{\partial \kappa_i} \, \mathrm{d}A$$

Components are related to energy variation along a given generalized strain.

Weak formulation

$$\mathcal{P}_i(\Phi, \delta\Phi) = \int_{\mathcal{C}} \mathsf{N} \cdot \left(\frac{\partial \delta\boldsymbol{\varphi}}{\partial S} - \delta\boldsymbol{\theta} \wedge \frac{\partial \boldsymbol{\varphi}}{\partial S} \right) + \mathsf{M} \cdot \frac{\partial \delta\boldsymbol{\theta}}{\partial S} \, \mathrm{d}S$$

- Integration by parts

$$\mathcal{P}_i(\Phi, \delta\Phi) = \Big[\mathsf{N} \cdot \delta\boldsymbol{\varphi} + \mathsf{M} \cdot \delta\boldsymbol{\theta} \Big]_0^L - \int_{\mathcal{C}} \frac{\partial \mathsf{N}}{\partial S} \cdot \delta\boldsymbol{\varphi} + \left(\mathsf{M} + \frac{\partial \boldsymbol{\varphi}}{\partial S} \wedge \mathsf{N} \right) \cdot \delta\boldsymbol{\theta} \, \mathrm{d}S$$

- Principle of Virtual Power:

$$\mathcal{P}_{cin}(\Phi, \delta\Phi) + \mathcal{P}_i(\Phi, \delta\Phi) = \mathcal{P}_{ext}(\Phi, \delta\Phi)$$

- Weak formulation $\forall \ (\delta\boldsymbol{\varphi}, \delta\boldsymbol{\theta}) \in W_\Phi$:

$$\Big[\mathsf{N} \cdot \delta\boldsymbol{\varphi} + \mathsf{M} \cdot \delta\boldsymbol{\theta} \Big]_0^L = \int_{\mathcal{C}} \left(\frac{\partial \mathsf{N}}{\partial S} + \mathbf{q} - \rho_0 A \, \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} \right) \cdot \delta\boldsymbol{\varphi} \, \mathrm{d}S \ + \ldots$$

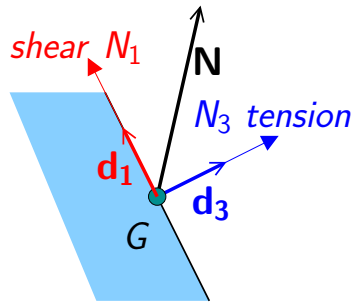
$$\int_{\mathcal{C}} \left(\frac{\partial \mathsf{M}}{\partial S} + \frac{\partial \boldsymbol{\varphi}}{\partial S} \times \mathsf{N} + \mathbf{m} - \rho_0 \frac{\partial}{\partial t} (\mathbb{I} \, \boldsymbol{\omega}) \right) \cdot \delta\boldsymbol{\theta} \, \mathrm{d}S$$

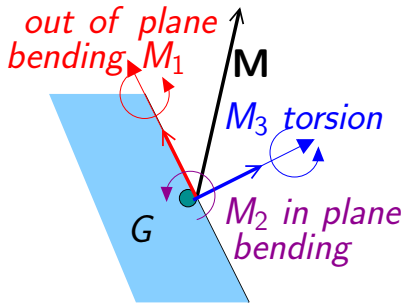
Analysis of the model

Strong formulation

$$\begin{aligned} \frac{\partial \mathrm{N}}{\partial S} + \mathbf{q} &= \rho_0 A \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} \\ \frac{\partial \mathrm{M}}{\partial S} + \frac{\partial \boldsymbol{\varphi}}{\partial S} \times \mathrm{N} + \mathbf{m} &= \rho_0 \frac{\partial}{\partial t}(\mathbb{I}\,\boldsymbol{\omega}) \end{aligned}$$

Depends on $\frac{\partial \boldsymbol{\varphi}}{\partial S}$, curvature $\boldsymbol{\kappa}$ and their spatial derivative | $\frac{\partial \boldsymbol{\varphi}}{\partial t}$, spin $\boldsymbol{\omega}$ and their time derivative

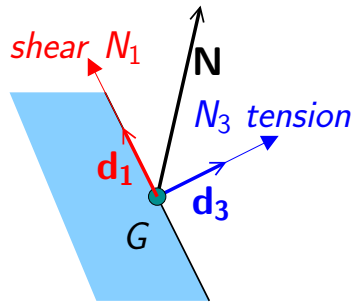


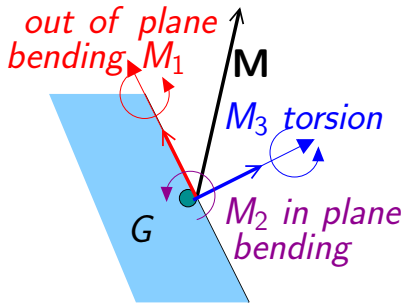


Strong formulation

$$\frac{\partial \mathbf{N}}{\partial S} + \mathbf{q} = \rho_0 A \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}$$
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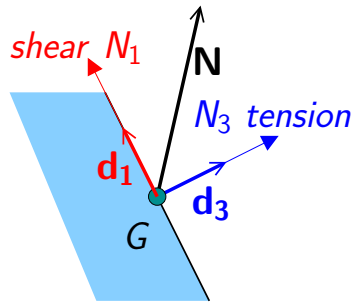




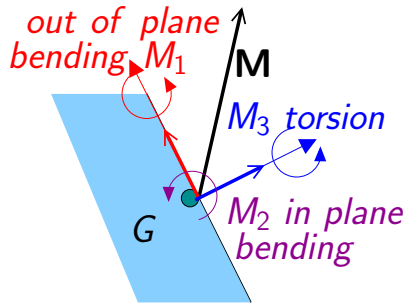
- Any shape, geometry: beams, archs, rings
- Any material, energy function
- Small or large displacement, strain, slow or fast evolution

Strong formulation

$$\frac{\partial \mathrm{N}}{\partial S} + \mathbf{q} = \rho_0 A \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}$$
$$\frac{\partial \mathrm{M}}{\partial S} + \frac{\partial \boldsymbol{\varphi}}{\partial S} \times \mathrm{N} + \mathbf{m} = \rho_0 \frac{\partial}{\partial t}(\mathbb{I}\,\boldsymbol{\omega})$$



Depends on $\frac{\partial \boldsymbol{\varphi}}{\partial S}$, curvature $\boldsymbol{\kappa}$ and their spatial derivative | $\frac{\partial \boldsymbol{\varphi}}{\partial t}$, spin $\boldsymbol{\omega}$ and their time derivative



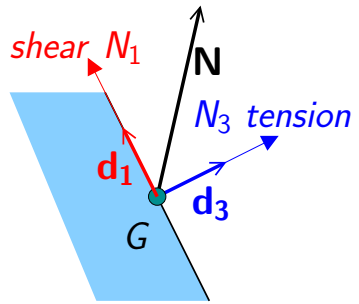
- Any shape, geometry: beams, archs, rings
- Any material, energy function
- Small or large displacement, strain, slow or fast evolution

	static	**linear**	**non-linear**
rigid	ok	ok	spin-top / rocket
linear	ok	vibration	
large	elastica		

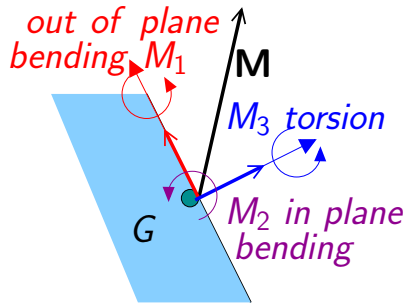
Include bending, tension, torsion, shear with coupling

Strong formulation

$$\frac{\partial \mathrm{N}}{\partial S} + \mathbf{q} = \rho_0 A \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}$$
$$\frac{\partial \mathrm{M}}{\partial S} + \frac{\partial \boldsymbol{\varphi}}{\partial S} \times \mathrm{N} + \mathbf{m} = \rho_0 \frac{\partial}{\partial t}(\mathbb{I}\,\boldsymbol{\omega})$$



Depends on $\frac{\partial \boldsymbol{\varphi}}{\partial S}$, curvature $\boldsymbol{\kappa}$ and their spatial derivative | $\frac{\partial \boldsymbol{\varphi}}{\partial t}$, spin $\boldsymbol{\omega}$ and their time derivative



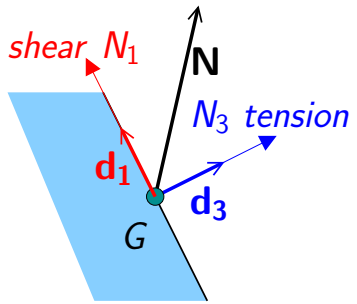
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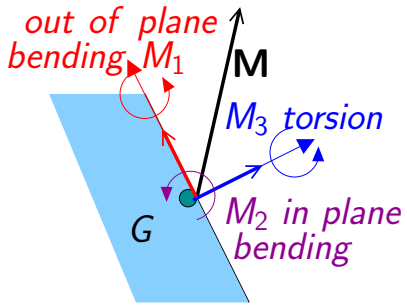
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Depends on $\frac{\partial \boldsymbol{\varphi}}{\partial S}$, curvature $\boldsymbol{\kappa}$ and their spatial derivative | $\frac{\partial \boldsymbol{\varphi}}{\partial t}$, spin $\boldsymbol{\omega}$ and their time derivative



- Any shape, geometry: beams, archs, rings
- Any material, energy function
- Small or large displacement, strain, slow or fast evolution

	static	**linear**	**non-linear**
rigid	ok	ok	spin-top / rocket
linear	ok	vibration	rotor blade vib.
large	elastica	string	difficult

Include bending, tension, torsion, shear with coupling

No commutation

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} - \frac{\partial \boldsymbol{\omega}}{\partial S} = \boldsymbol{\omega} \wedge \boldsymbol{\kappa}$$

Non-linearities

Example for static problem (no external load)

$$\frac{\partial \mathrm{N}}{\partial S} = 0$$
$$\frac{\partial \mathrm{M}}{\partial S} + \frac{\partial \boldsymbol{\varphi}}{\partial S} \times \mathrm{N} = 0$$

Remarks:

$\mathrm{N} = N_i \mathbf{d_i}$ then $\frac{\partial \mathrm{N}}{\partial S} = N_i' \mathbf{d_i} + \boldsymbol{\kappa} \wedge \mathrm{N}$

$\frac{\partial \boldsymbol{\varphi}}{\partial S} = \mathbf{d_3} + \boldsymbol{\varepsilon}$

Geometricaly non-linear (archs, rings)

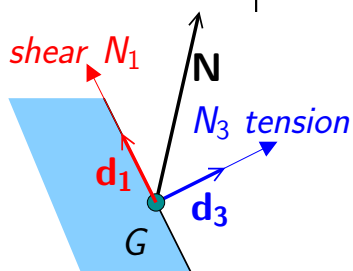
$$N_1' + \kappa_2 N_3 - \kappa_3 N_2 = 0$$
$$N_2' + \kappa_3 N_1 - \kappa_1 N_3 = 0$$
$$N_3' + \kappa_1 N_2 - \kappa_2 N_1 = 0$$
$$M_1' - N_2 + \kappa_2 M_3 - \kappa_3 M_2 + \varepsilon_2 N_3 - \varepsilon_3 N_2 = 0$$
$$M_2' + N_1 + \kappa_3 M_1 - \kappa_1 M_3 + \varepsilon_3 N_1 - \varepsilon_1 N_3 = 0$$
$$M_3' + \kappa_1 M_2 - \kappa_2 M_1 + \varepsilon_1 N_2 - \varepsilon_2 N_1 = 0$$

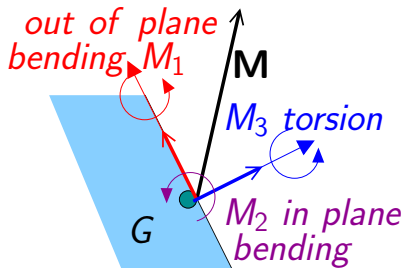
Geometricaly linear (straight)

$$N_1' = 0$$
$$N_2' = 0$$
$$N_3' = 0$$
$$M_1' - N_2 = 0$$
$$M_2' + N_1 = 0$$
$$M_3' = 0$$

Materialy non-linear (constitutive law)

Example: $N_3 = AE\varepsilon_3(1+\varepsilon_3)(1+\frac{\varepsilon_3}{2})$





Vibrations

3D vibrations of straight beam

Linearized motion (Eulerian / Lagrangian equivalence) $\{\mathbf{D_i}\} \sim \{\mathbf{d_i}\} \sim \{\mathbf{e_i}\}$

- Micro-rotation: $\boldsymbol{\theta}(S,t) \in so(3) \quad | \; \mathbf{d_i} = \mathbf{e_i} + \boldsymbol{\theta} \wedge \mathbf{e_i}$

$$\boldsymbol{\kappa} = \theta_i' \mathbf{d_i} \qquad \boldsymbol{\omega} = \dot{\theta}_i \mathbf{d_i} \qquad \frac{\partial \boldsymbol{\kappa}}{\partial t} - \frac{\partial \boldsymbol{\omega}}{\partial S} = 0$$

- Displacement: $\boldsymbol{\varphi}(S,t) = \mathbf{u} + S\mathbf{e_3}$

- Strain

$$\boldsymbol{\varepsilon} = \frac{\partial \boldsymbol{\varphi}}{\partial S} - \mathbf{d_3} \simeq u_i' \mathbf{d_i} + \mathbf{e_3} - \mathbf{d_3} = \left| \begin{array}{l} \varepsilon_1 = u_1' - \theta_2 \\ \varepsilon_2 = u_2' + \theta_1 \\ \varepsilon_3 = u_3' \end{array} \right. \qquad \boldsymbol{\kappa} = \left| \begin{array}{l} \kappa_1 = \theta_1' \\ \kappa_2 = \theta_2' \\ \kappa_3 = \theta_3' \end{array} \right.$$

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Kirchhoff-Saint Venant hyperelastic material

$$\Psi = \frac{\lambda}{2} \left(\mathrm{Tr}(\mathbf{E}) \right)^2 + \mu \mathrm{Tr}(\mathbf{E}^2)$$

Linear constitutive relation

$$\begin{array}{lll} N_1 = \mu A \varepsilon_1 & N_2 = \mu A \varepsilon_2 & N_3 = (\lambda + 2\mu) A \varepsilon_3 \\ M_1 = (\lambda + 2\mu) I_{11} \kappa_1 & M_2 = (\lambda + 2\mu) I_{22} \kappa_2 & M_3 = \mu I_{33} \kappa_3 \end{array}$$

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$$\begin{array}{lll} N_1 = GA(u_1' - \theta_2) & N_2 = GA(u_2' + \theta_1) & N_3 = EAu_3' \\ M_1 = EI_{11}\theta_1' & M_2 = EI_{22}\theta_2' & M_3 = G(I_{11} + I_{22})\theta_3' \end{array}$$

Equation of vibrations

$$N_1 = GA(u_1' - \theta_2) \qquad N_2 = GA(u_2' + \theta_1) \qquad N_3 = EAu_3'$$
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$$N_1' = \rho A\ddot{u}_1 \qquad N_2' = \rho A\ddot{u}_2 \qquad N_3' = \rho A\ddot{u}_3$$
$$M_1' - N_2 = \rho I_{11}\ddot{\theta}_1 \qquad M_2' + N_1 = \rho I_{22}\ddot{\theta}_2 \qquad M_3' = \rho(I_{11} + I_{22})\ddot{\theta}_3$$

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$$
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M_1' - N_2 = \rho I_{11}\ddot{\theta}_1 & M_2' + N_1 = \rho I_{22}\ddot{\theta}_2 & M_3' = \rho(I_{11} + I_{22})\ddot{\theta}_3
\end{array}
$$

Torsion

$$G(I_{11} + I_{22})\theta_3'' = \rho(I_{11} + I_{22})\ddot{\theta}_3$$

Extension

$$EA\, u_3'' = \rho A\, \ddot{u}_3$$

Bending $(\mathbf{e}_1, \mathbf{e}_3)$-plane

$$
\begin{aligned}
GA(u_1'' - \theta_2') &= \rho A\, \ddot{u}_1 \\
EI_{22}\, \theta_2'' + GA(u_1' - \theta_2) &= \rho I_{22}\, \ddot{\theta}_2
\end{aligned}
$$

Bending $(\mathbf{e}_2, \mathbf{e}_3)$-plane

$$
\begin{aligned}
GA(u_2'' + \theta_1') &= \rho A\, \ddot{u}_2 \\
EI_{11}\, \theta_1'' - GA(u_2' + \theta_1) &= \rho I_{11}\, \ddot{\theta}_1
\end{aligned}
$$

Equation of vibrations

$$
\begin{array}{lll}
N_1 = GA(u_1' - \theta_2) & N_2 = GA(u_2' + \theta_1) & N_3 = EAu_3' \\
M_1 = EI_{11}\theta_1' & M_2 = EI_{22}\theta_2' & M_3 = G(I_{11} + I_{22})\theta_3' \\
N_1' = \rho A\ddot{u}_1 & N_2' = \rho A\ddot{u}_2 & N_3' = \rho A\ddot{u}_3 \\
M_1' - N_2 = \rho I_{11}\ddot{\theta}_1 & M_2' + N_1 = \rho I_{22}\ddot{\theta}_2 & M_3' = \rho(I_{11} + I_{22})\ddot{\theta}_3
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Bending $(\mathbf{e}_1, \mathbf{e}_3)$-plane

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\end{aligned}
$$

Bending $(\mathbf{e}_2, \mathbf{e}_3)$-plane

$$
\begin{aligned}
GA(u_2'' + \theta_1') &= \rho A\, \ddot{u}_2 \\
EI_{11}\, \theta_1'' - GA(u_2' + \theta_1) &= \rho I_{11}\, \ddot{\theta}_1
\end{aligned}
$$

$\Rightarrow$ 6 degrees of freedom $u_i(S,t)$ and $\theta_i(S,t)$

$\Rightarrow$ 7 independent parameters: ρ, I_{11}, I_{22}, A, L, E and G

Non-dimensional form

Typical dimension

gyration radius $[m]$ $\quad \varrho := \sqrt{\dfrac{I_{11}+I_{22}}{2A}}$ $\quad$ *eccentricity* $[\sim]$ $\quad e = \dfrac{I_{11}+I_{22}}{2I_{11}}$

characteristic time $[s]$ $\quad t_c := \varrho\sqrt{\dfrac{\rho}{G}}$ $\quad$ *material-ratio* $[\sim]$ $\quad g := \dfrac{E}{G}$

Re-scaling

space: $\quad s = \dfrac{S}{\varrho}$ $\qquad$ displacement: $\quad \mathbf{u}(s,\tau) = \dfrac{\mathbf{u}^*(S,t)}{\varrho}$

time: $\quad \tau = \dfrac{t}{t_c}$ $\qquad$ rotation: $\quad \boldsymbol{\theta}(s,\tau) = \boldsymbol{\theta}^*(S,t)$

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Torsion

$$\theta_3'' = \ddot{\theta}_3$$

Extension

$$g\, u_3'' = \ddot{u}_3$$

Bending $(\mathbf{e}_1, \mathbf{e}_3)$-plane

$$u_1'' - \theta_2' = \ddot{u}_1$$
$$g\,\theta_2'' + u_1' - \theta_2 = \ddot{\theta}_2$$

Bending $(\mathbf{e}_2, \mathbf{e}_3)$-plane

$$u_2'' + \theta_1 = \ddot{u}_2$$
$$g\,\theta_1'' - e(u_2' + \theta_1) = \ddot{\theta}_1$$

Dispersion relation

Harmonic vibrations: $\mathbf{u}(s,\tau) = \mathbf{U}e^{i(ks-\omega\tau)}$, $\boldsymbol{\theta}(s,\tau) = \boldsymbol{\theta}e^{i(ks-\omega\tau)}$

Torsion

$$\theta_3'' = \ddot{\theta}_3$$

Extension

$$g\, u_3'' = \ddot{u}_3$$

Bending $(\mathbf{e}_1, \mathbf{e}_3)$-plane

$$u_1'' - \theta_2' = \ddot{u}_1$$
$$g\,\theta_2'' + u_1' - \theta_2 = \ddot{\theta}_2$$

Bending $(\mathbf{e}_2, \mathbf{e}_3)$-plane

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$$g\,\theta_1'' - e(u_2' + \theta_1) = \ddot{\theta}_1$$

Dispersion relation

Harmonic vibrations: $\mathbf{u}(s,\tau) = \mathbf{U}e^{i(ks-\omega\tau)}$, $\boldsymbol{\theta}(s,\tau) = \boldsymbol{\theta}e^{i(ks-\omega\tau)}$

Torsion

$$k^2 - \omega^2 = 0$$

Extension

$$gk^2 - \omega^2 = 0$$

Bending $(\mathbf{e}_1, \mathbf{e}_3)$-plane

$$(k^2 - \omega^2)(gk^2 - \omega^2) - \omega^2 = 0$$

Bending $(\mathbf{e}_2, \mathbf{e}_3)$-plane

$$(k^2 - \omega^2)(gk^2 - \omega^2) - e\,\omega^2 = 0$$

Dispersion relation

Harmonic vibrations: $\mathbf{u}(s,\tau) = \mathbf{U}e^{i(ks-\omega\tau)}$, $\boldsymbol{\theta}(s,\tau) = \boldsymbol{\theta}e^{i(ks-\omega\tau)}$

Torsion

$$k^2 - \omega^2 = 0$$

Extension

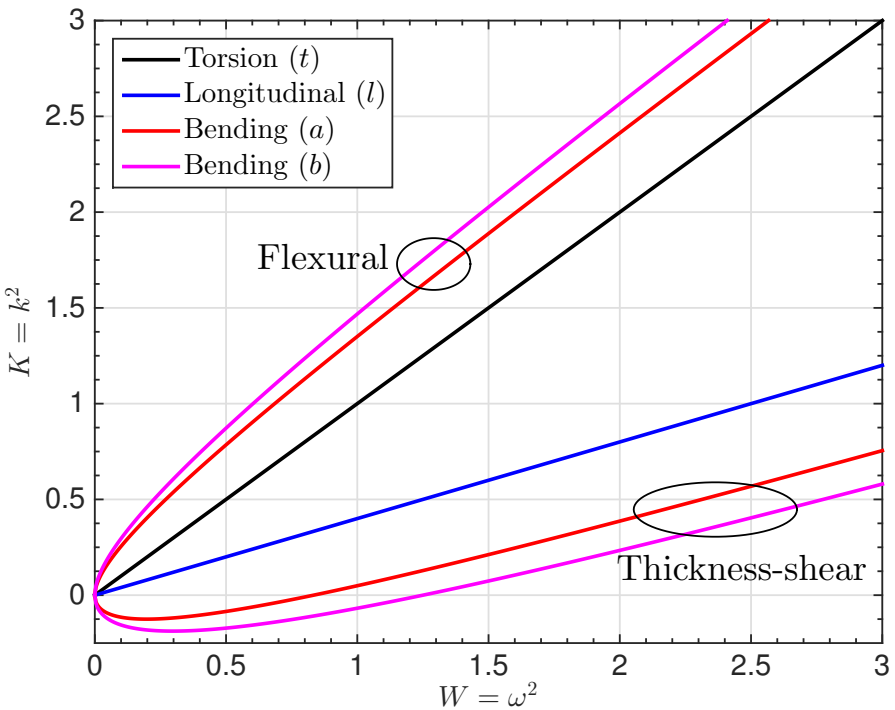
$$gk^2 - \omega^2 = 0$$

Bending $(\mathbf{e}_1, \mathbf{e}_3)$-plane

$$(k^2 - \omega^2)(gk^2 - \omega^2) - \omega^2 = 0$$

Bending $(\mathbf{e}_2, \mathbf{e}_3)$-plane

$$(k^2 - \omega^2)(gk^2 - \omega^2) - e\omega^2 = 0$$



Dispersion relation

Harmonic vibrations: $\mathbf{u}(s,\tau) = \mathbf{U}e^{i(ks-\omega\tau)}$, $\boldsymbol{\theta}(s,\tau) = \boldsymbol{\theta}e^{i(ks-\omega\tau)}$

Torsion

$$k^2 - \omega^2 = 0$$

Extension

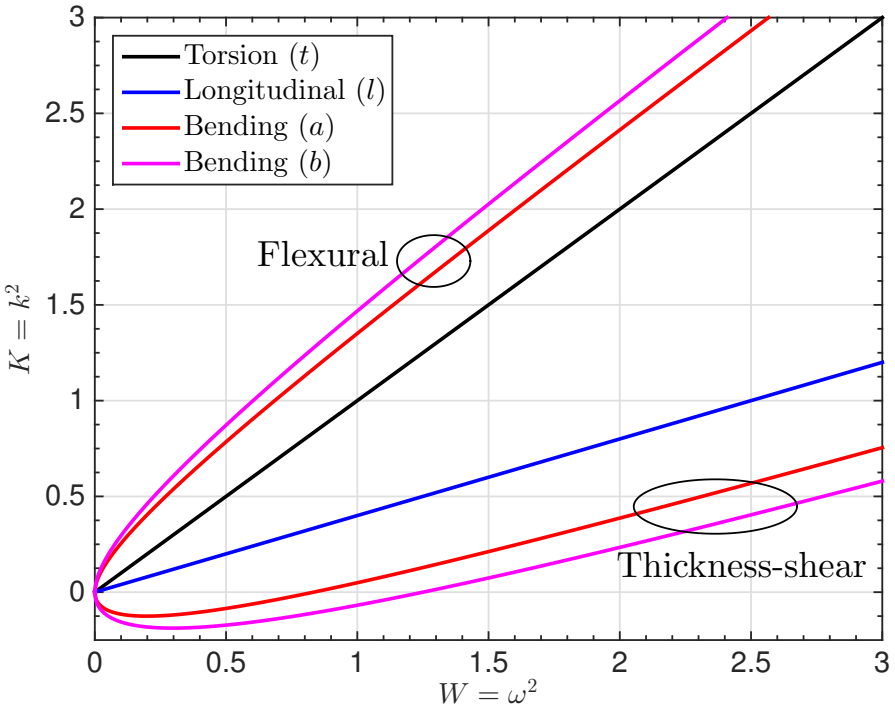
$$g k^2 - \omega^2 = 0$$

Bending $(\mathbf{e}_1, \mathbf{e}_3)$-plane

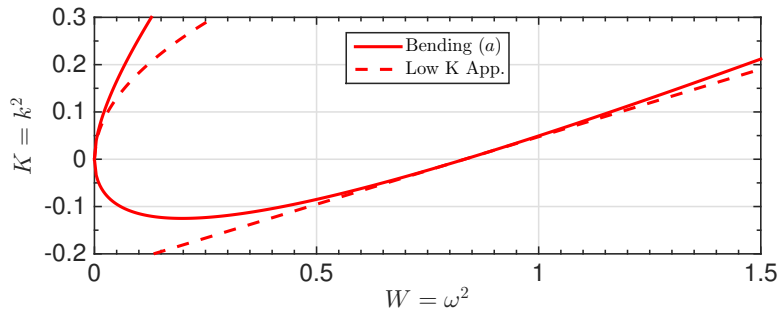
$$(k^2 - \omega^2)(g k^2 - \omega^2) - \omega^2 = 0$$

Bending $(\mathbf{e}_2, \mathbf{e}_3)$-plane

$$(k^2 - \omega^2)(g k^2 - \omega^2) - e\,\omega^2 = 0$$



Low-wavenumber approximation: Euler-Bernoulli model



$$\omega^2 \simeq g\, k^4$$

Modes and eigenvectors

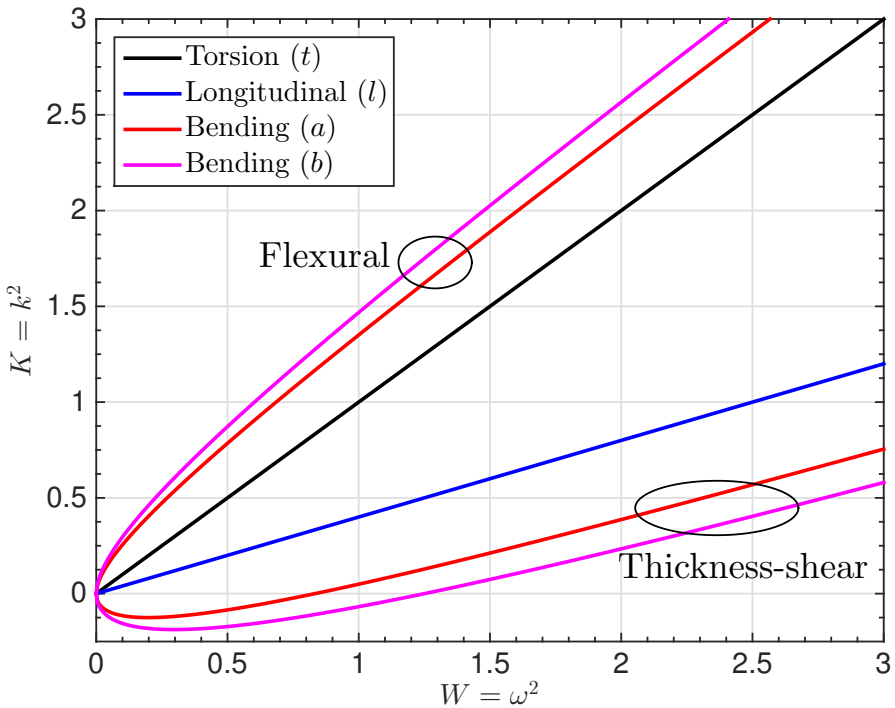
Ex: bending

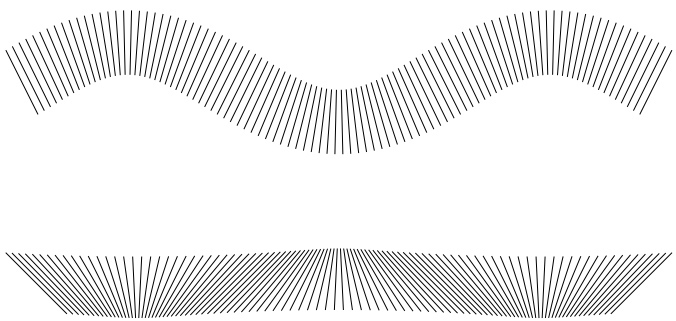
$$u_1(s,\tau) = U_1 e^{i(ks-\omega\tau)} \qquad \theta_2(s,\tau) = \theta_2 e^{i(ks-\omega\tau)},$$

Eigenvalue problem

$$\begin{aligned} u_1'' - \theta_2' &= \ddot{u}_1 \\ g\,\theta_2'' + u_1' - \theta_2 &= \ddot{\theta}_2 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} -k^2 & -ik \\ ik & -gk^2 - 1 \end{pmatrix} \left| \begin{matrix} U_1 \\ \theta_2 \end{matrix} \right. = -\omega^2 \left| \begin{matrix} U_1 \\ \theta_2 \end{matrix} \right.$$

- ω defines the eigenvalues and modes are eigenvectors.





Small vibrations $\Delta\Phi$ super-imposed on finite motion Φ

Small vibrations $\Delta\Phi$ super-imposed on finite motion Φ

Principe of Virtual Power

$$\begin{aligned}
\text{Without vib.} \quad & \mathcal{P}_{cin}(\Phi, \delta\Phi) + \mathcal{P}_i(\Phi, \delta\Phi) = \mathcal{P}_{ext}(\Phi, \delta\Phi) \\
\text{With vib.} \quad & \mathcal{P}_{cin}(\Phi + \Delta\Phi, \delta\Phi) + \mathcal{P}_i(\Phi + \Delta\Phi, \delta\Phi) = \mathcal{P}_{ext}(\Phi + \Delta\Phi, \delta\Phi) \\
\text{Then} \quad & \Delta\mathcal{P}_{cin}(\Phi, \delta\Phi) + \Delta\mathcal{P}_i(\Phi, \delta\Phi) = \Delta\mathcal{P}_{ext}(\Phi, \delta\Phi)
\end{aligned}$$

where Gâteau-Fréchet derivative is used

$$\Delta\mathcal{P}_i(\Phi, \delta\Phi) := \lim_{h \to 0} \left(\frac{1}{h} \left(\mathcal{P}_i(\Phi + h\Delta\Phi, \delta\Phi) - \mathcal{P}_i(\Phi, \delta\Phi) \right) \right) [\Delta\Phi]$$

Perturbation of internal Virtual Power induced by vibrations

$$\Delta\mathcal{P}_i(\Phi, \delta\Phi) = \int_{\mathcal{B}} \underbrace{\mathbf{S} : (\Delta\mathbf{F}^T \delta\mathbf{F})}_{\textit{Geometry}} + \underbrace{\delta\mathbf{E} : \mathbf{C}(\Delta\mathbf{E})}_{\textit{Material}} \, \mathrm{d}V$$

- Symmetry is respected
- The evaluation of $\mathbf{C}$ is performed around the Φ-state
- Integration over the beam section, is possible... but tricky

Dead load / following load

For the **external virtual power**, the linearisation writes:

$$\Delta\mathcal{P}_{ext}(\Phi, \delta\Phi) = \int_{\mathcal{C}} \Delta\mathbf{q} \cdot \delta\boldsymbol{\varphi} + \Delta\mathbf{m} \cdot \delta\boldsymbol{\theta} \, \mathrm{d}S$$

$$\Delta\mathbf{q} = \int_{\mathcal{S}} \Delta\mathbf{b} \, \mathrm{d}A + \int_{\partial\mathcal{S}} \Delta\mathbf{f} \, \mathrm{d}L$$

$$\Delta\mathbf{m} = \int_{\mathcal{S}} \mathbf{GM} \times \Delta\mathbf{b} + (\Delta\boldsymbol{\theta} \times \mathbf{GM}) \times \mathbf{b} \, \mathrm{d}A + \int_{\partial\mathcal{S}} \mathbf{GM} \times \Delta\mathbf{f} + (\Delta\boldsymbol{\theta} \times \mathbf{GM}) \times \mathbf{f} \, \mathrm{d}L$$

Dead load / following load

For the **external virtual power**, the linearisation writes:

$$\Delta\mathcal{P}_{ext}(\Phi, \delta\Phi) = \int_{\mathcal{C}} \Delta\mathbf{q} \cdot \delta\boldsymbol{\varphi} + \Delta\mathbf{m} \cdot \delta\boldsymbol{\theta} \, \mathrm{d}S$$

$$\Delta\mathbf{q} = \int_{\mathcal{S}} \Delta\mathbf{b} \, \mathrm{d}A + \int_{\partial\mathcal{S}} \Delta\mathbf{f} \, \mathrm{d}L$$

$$\Delta\mathbf{m} = \int_{\mathcal{S}} \mathbf{GM} \times \Delta\mathbf{b} + (\Delta\boldsymbol{\theta} \times \mathbf{GM}) \times \mathbf{b} \, \mathrm{d}A + \int_{\partial\mathcal{S}} \mathbf{GM} \times \Delta\mathbf{f} + (\Delta\boldsymbol{\theta} \times \mathbf{GM}) \times \mathbf{f} \, \mathrm{d}L$$

Dead load / following load

For the **external virtual power**, the linearisation writes:

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Dead load, then $\Delta\mathbf{f} = 0$

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Following load, then $\Delta\mathbf{f} = \Delta\boldsymbol{\theta} \wedge \mathbf{f}$

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May be used to define *natural* and *essential* boundary conditions !

Small vibrations $\Delta\Phi$ super-imposed on finite motion Φ

Example of Finite Pre-stretch

Buckling

String

Static finite pre-stretch Φ

- Pre-stretch

$$\boldsymbol{\varepsilon} = \varepsilon_3 \mathbf{d_3} = \epsilon\, \mathbf{d_3}$$

$\epsilon < 0$ compression
$\epsilon > 0$ tension

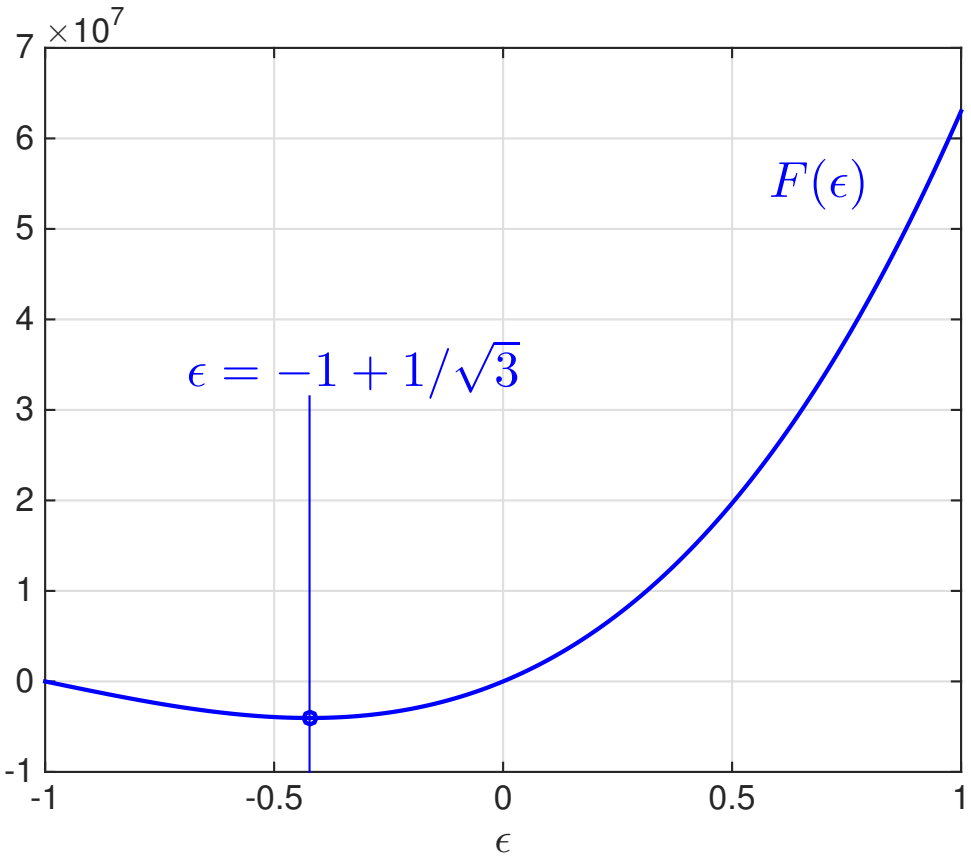
- Kinematics $\mathbf{d_3} = \mathbf{e_3}$ $\quad \boldsymbol{\varphi} = S(1+\epsilon)\mathbf{e_3}$
- Kirchhoff-Saint Venant hyperelastic material

$$\Psi = \frac{\lambda}{2}\left(\mathrm{Tr}(\mathbf{E})\right)^2 + \mu \mathrm{Tr}(\mathbf{E}^2)$$

$$\Psi(\epsilon) = \frac{\lambda + 2\mu}{2}\epsilon^2(1+\frac{\epsilon}{2})^2$$

Longitudinal static force
$F(\epsilon)$

$$\mathbf{N} = A(\lambda+2\mu)\epsilon(1+\epsilon)(1+\frac{\epsilon}{2})\mathbf{d}_3$$



Vibration $\Delta\Phi$ super-imposed on finite pre-strech Φ

- Pre-stretch of straight beam $\Phi = \left(\boldsymbol{\varphi} = S(1+\epsilon)\mathbf{e_3},\ \mathbf{D_3} = \mathbf{e_3}\right)$
- Super-imposed vibration $\Delta\Phi = (\mathbf{u}, \boldsymbol{\theta})$
- Virtual motion $\delta\Phi = (\delta\mathbf{u}, \delta\boldsymbol{\theta})$
- Virtual power, geometrical and material contribution

$$\int_{\mathcal{B}} \mathbf{S} : (\Delta\mathbf{F}^T \delta\mathbf{F})\ \mathrm{d}V = \epsilon(1+\frac{\epsilon}{2})\left((\lambda+2\mu)\int_{\mathcal{C}} A\frac{\partial\mathbf{u}}{\partial S}\cdot\frac{\partial\delta\boldsymbol{u}}{\partial S} + (\mathbb{I}\frac{\partial\boldsymbol{\theta}}{\partial S})\cdot\frac{\partial\delta\boldsymbol{\theta}}{\partial S}\ \mathrm{d}S + \lambda A\int_{\mathcal{C}} \boldsymbol{\theta}\cdot\delta\boldsymbol{\theta} + \theta_3\delta\theta_3\ \mathrm{d}S\right)$$

$$\begin{aligned}\int_{\mathcal{B}} \delta\mathbf{E} : \mathbf{C}(\Delta\mathbf{E})\ \mathrm{d}V &= \int_{\mathcal{C}} \mu\, I_3 \frac{\partial\theta_3}{\partial S}\frac{\partial\delta\theta_3}{\partial S} + (\lambda+2\mu)(1+\epsilon)^2\sum_{\alpha=1}^{2}\mathbb{I}_\alpha\frac{\partial\theta_\alpha}{\partial S}\frac{\partial\delta\theta_\alpha}{\partial S}\ \mathrm{d}S \\ &+ \mu A\int_{\mathcal{C}}\left(\frac{\partial u_1}{\partial S} - (1+\epsilon)\theta_2\right)\left(\frac{\partial\delta u_1}{\partial S} - (1+\epsilon)\delta\theta_2\right)\ \mathrm{d}S \\ &+ \mu A\int_{\mathcal{C}}\left(\frac{\partial u_2}{\partial S} + (1+\epsilon)\theta_1\right)\left(\frac{\partial\delta u_2}{\partial S} + (1+\epsilon)\delta\theta_1\right)\ \mathrm{d}S \\ &+ (\lambda+2\mu)A(1+\epsilon)^2\int_{\mathcal{C}}\frac{\partial u_3}{\partial S}\frac{\partial\delta u_3}{\partial S}\ \mathrm{d}S\end{aligned}$$

Equation for vibrations are *still* uncoupled

- Bending in $(\mathbf{d_1}, \mathbf{d_3})$-plane

$$\begin{aligned} \widetilde{G}A\,u_1'' - \widehat{G}A\,\theta_2' &= \rho A\,\ddot{u}_1 \\ \widetilde{E}I_2\,\theta_2'' + \widehat{G}A\,u_1' - \widetilde{G}A\,\theta_2 &= \rho I_2\,\ddot{\theta}_2 \end{aligned}$$

- Longitudinal vibes along $\mathbf{d_3}$

$$\widetilde{E}A\,u_3'' = \rho A\,\ddot{u}_3$$

- Torsional vibes around $\mathbf{d_3}$

$$\widetilde{G}I_3\,\theta_3'' - 2gA\,\theta_3 = \rho\,I_3\ddot{\theta}_3$$

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- Effective rigidity moduli with respect to the finite strain:

$$\widetilde{E}(\epsilon) = (\lambda+2\mu)\big(1+3\epsilon(1+\frac{\epsilon}{2})\big) \qquad \widetilde{G}(\epsilon) = \mu\big(1+g\epsilon(1+\frac{\epsilon}{2})\big) \qquad \widehat{G}(\epsilon) = \mu(1+\epsilon)$$

with a bulk-shear ratio

$$g = \frac{\lambda + 2\mu}{\mu} = \frac{2(1+\nu)}{1-2\nu}$$

Buckling

Impose harmonic displacement $u_1(S) = U_1 e^{ikS}$ and angle $\theta_2(S) = Y_2 e^{ikS}$
Stiffness matrix (symetric):

$$\mathbb{K} = \begin{pmatrix} -k^2 \widetilde{G} A & -ik \widehat{G} A \\ ik \widehat{G} A & -k^2 \widetilde{E} I_2 - \widetilde{G} A \end{pmatrix}$$

Non-trivial solution appears for $\det(\mathbb{K}) = 0$, *ie* for

$$\widetilde{G}(\epsilon)\widetilde{E}(\epsilon) I_2 k^2 - A(\widehat{G}(\epsilon)^2 - \widetilde{G}(\epsilon)^2) = 0$$

For pinned boundary conditions $k = \pi/\ell$, the critical longitudinal strain ϵ may be computed, and then the critical load F^b can be computed

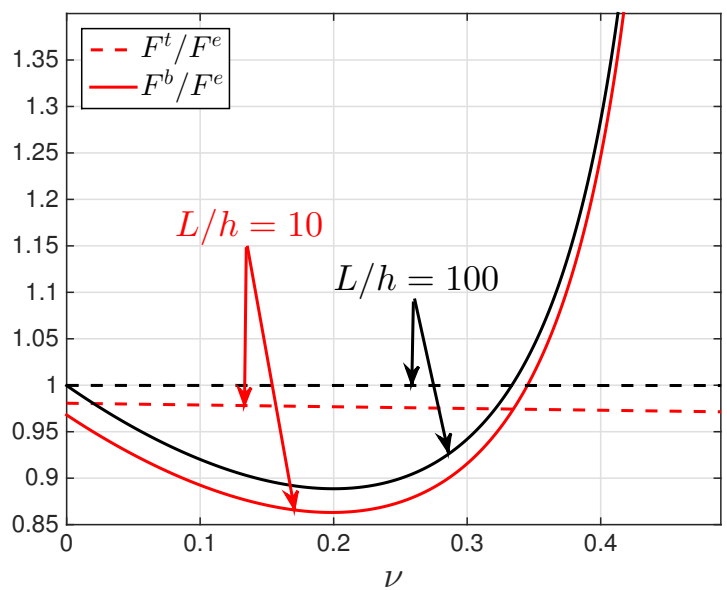
$$F^b(\epsilon) = A(\lambda+2\mu)\epsilon(1+\epsilon)(1+\frac{\epsilon}{2})$$

- Euler prediction

$$F^e = -EI_2 k^2$$

- Timoshenko and Gere estimation

$$F^t = -\frac{EI_2 k^2}{1 + \frac{EI_2}{GA} k^2}$$



String approximation

- Vibration of a string, having a pre-tension $F = (\lambda + 2\mu)\, A\, \epsilon$

$$F\, u_1'' = \rho A\, \ddot{u}_1$$

- Bending of pre-stretch beam

$$\begin{aligned} \widetilde{G} A\, u_1'' - \widehat{G} A\, \theta_2' &= \rho A\, \ddot{u}_1 \\ \widetilde{E} I_2\, \theta_2'' + \widehat{G} A\, u_1' - \widetilde{G} A\, \theta_2 &= \rho I_2\, \ddot{\theta}_2 \end{aligned}$$

Effective rigidity moduli with respect to the finite strain:

$$\begin{aligned} \widetilde{E}(\epsilon) &= (\lambda + 2\mu)\Big(1 + 3\epsilon(1 + \tfrac{\epsilon}{2})\Big) \\ \widetilde{G}(\epsilon) &= \mu\Big(1 + \tfrac{\lambda+2\mu}{\mu}\epsilon(1 + \tfrac{\epsilon}{2})\Big) \\ \widehat{G}(\epsilon) &= \mu(1 + \epsilon) \end{aligned}$$

Asymptotics

- Vibration of a string, having a pre-tension $F(\epsilon) = (\lambda + 2\mu)\, A\, \epsilon$

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- Using some asymptotic the bending problem becomes

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Hyp3 Neglecting gliding angle $|u_1' - \theta_2| \ll 1 \quad \Rightarrow \quad \widetilde{F}(\epsilon) = \frac{1}{2(1-\nu)}F(\epsilon)$

Conclusion

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- Light hypotheses: on the geometry at rest, on the material
- Strong hypothesis: rigid cross-section, one dimensional Cosserat body

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2^{nd} step ... then integration over the cross-section is performed

- Second variation gives a panorama of analytical problems
  - Instabilities: buckling
  - Super-imposed vibration on
    - large strain: string
    - large displacement: elastica, spin-tops, rockets, rotor blade
  - Difficult problems especially if:

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} - \frac{\partial \Delta\boldsymbol{\omega}}{\partial S} = \Delta\boldsymbol{\omega} \wedge \boldsymbol{\kappa} \neq 0 \qquad \frac{\partial \Delta\boldsymbol{\kappa}}{\partial t} - \frac{\partial \boldsymbol{\omega}}{\partial S} = \boldsymbol{\omega} \wedge \Delta\boldsymbol{\kappa} \neq 0$$