# Propagation d'ondes dans un milieux défectueux : l'apport de la géométrie

Loïc Le Marrec
with L. Rakotomanana, G. Casale, VH. Nguyen, M. Hariz, G Futhazar







June, 2019

Réunion du GDR-GDM *Géométrie Différentielle et Mécanique*
La Rochelle, France

# Dynamics in a material a concrete's case

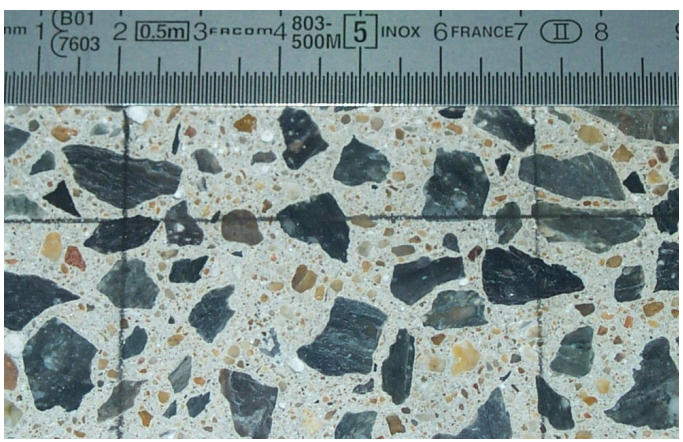
## Low frequency

- Vibration of structure
- Large wavelength
- Averaged information

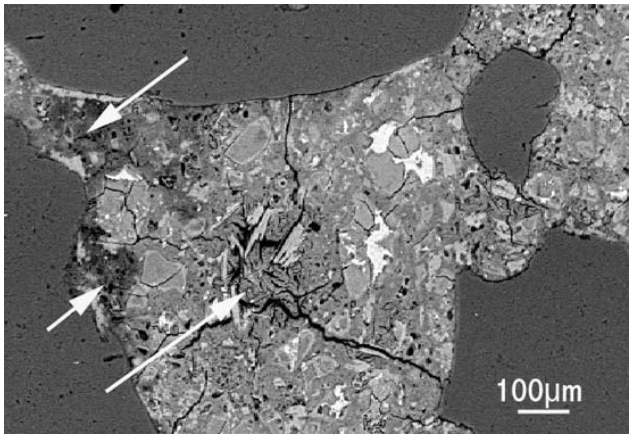


## High frequency

- Wave propagation
- Small wavelength
- Local information on the microstructure

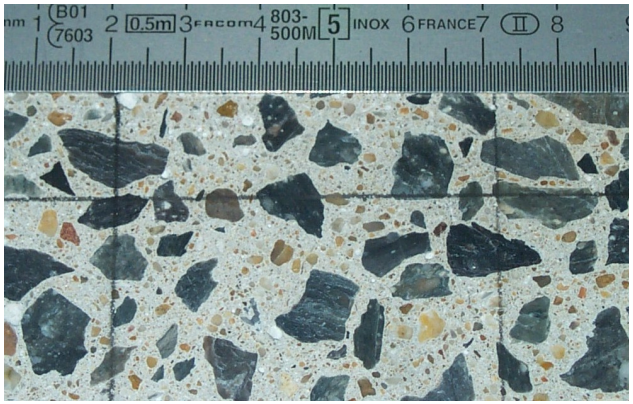


# Which model ?



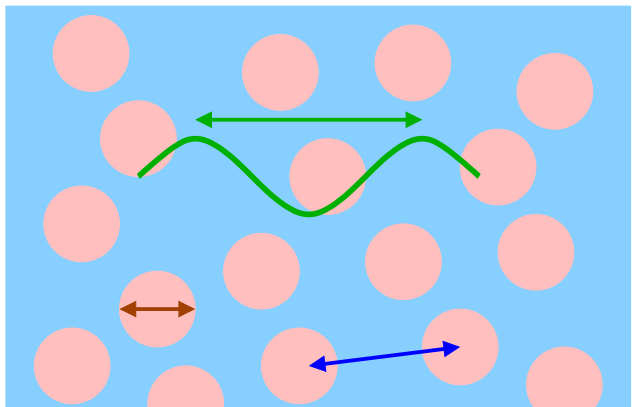

Real medium

- Complex material: poro-elastic and heterogenous
- Micro-cracks
- Large size's distribution for pores and heterogeneities



At centimetric wavelength: diphasic media

- Mortar: elastic, homogeneous
- Granulates: elastic, homogeneous
- Monodisperse and circular inclusions



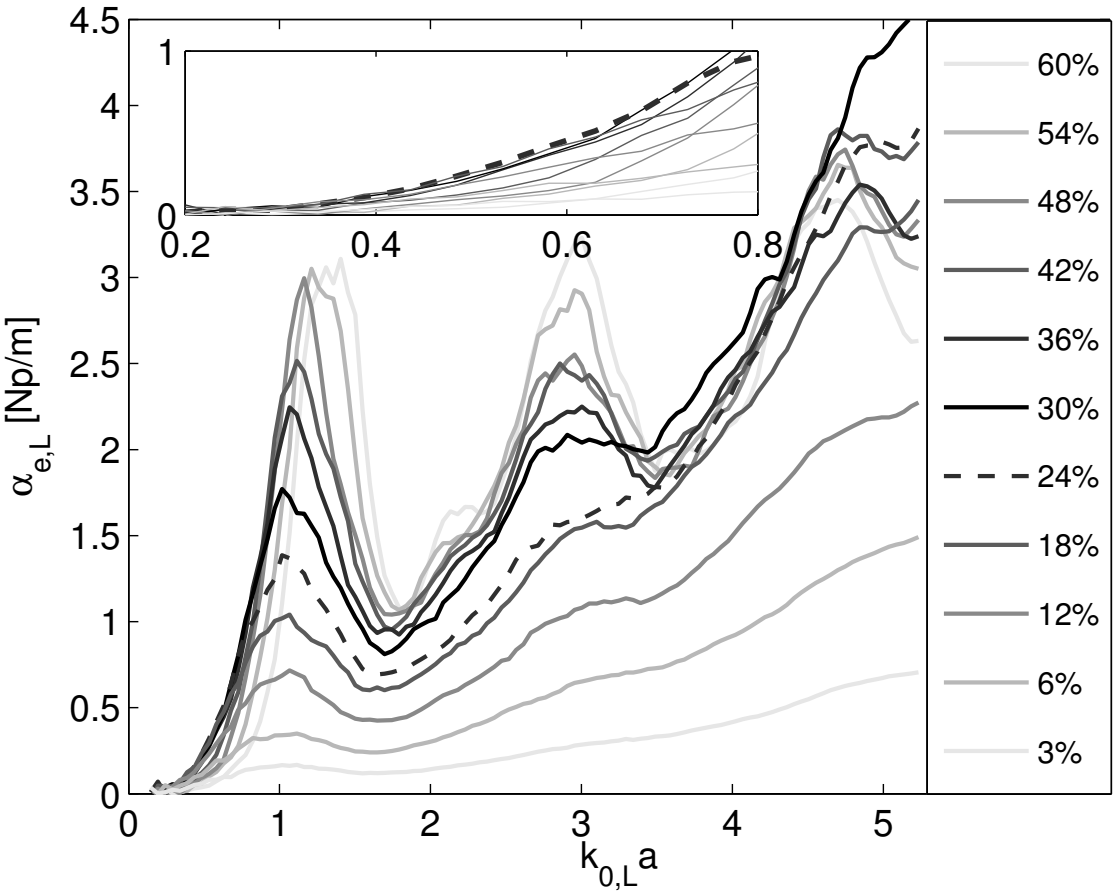
Critical dimensions

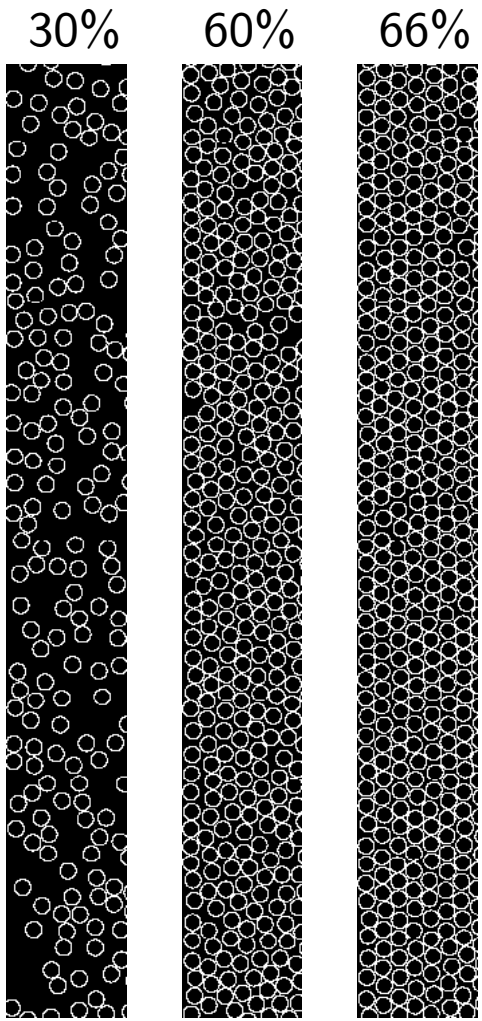
- Scattering cross-section: contrast
- Radius $a$: size of the exclusion domain
- Wavelength $L$: frequency domain
- **Typical distance:** mean-free-path, scatterer's density

# Coherent wave properties

$$u(\omega, x) = A(\omega) e^{i\frac{\omega}{c}x - \alpha x}$$

Attenuation : $\alpha(\omega)$ *vs* $ka$



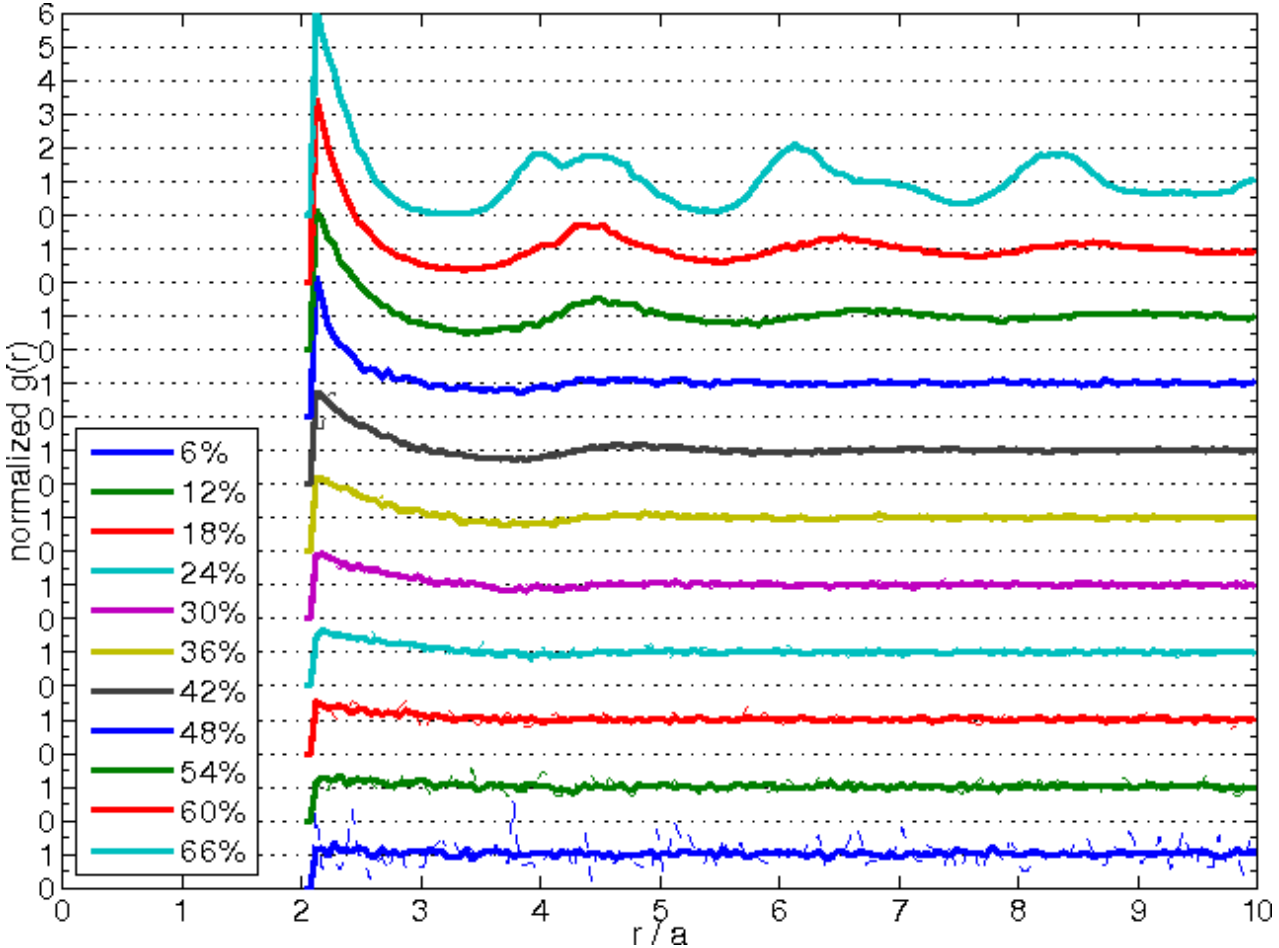



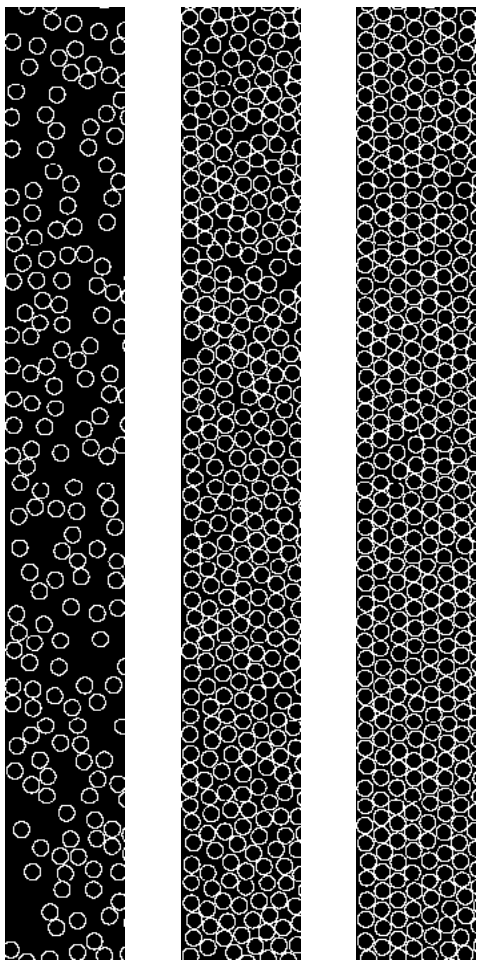

Wave in Random and Complex Media (2012)

# Resonant scattering *or* density effect ?

$$g(|\mathbf{r}_1 - \mathbf{r}_2|) = V\, p(\mathbf{r}_2|\mathbf{r}_1)$$

Normalized distribution of neighbors



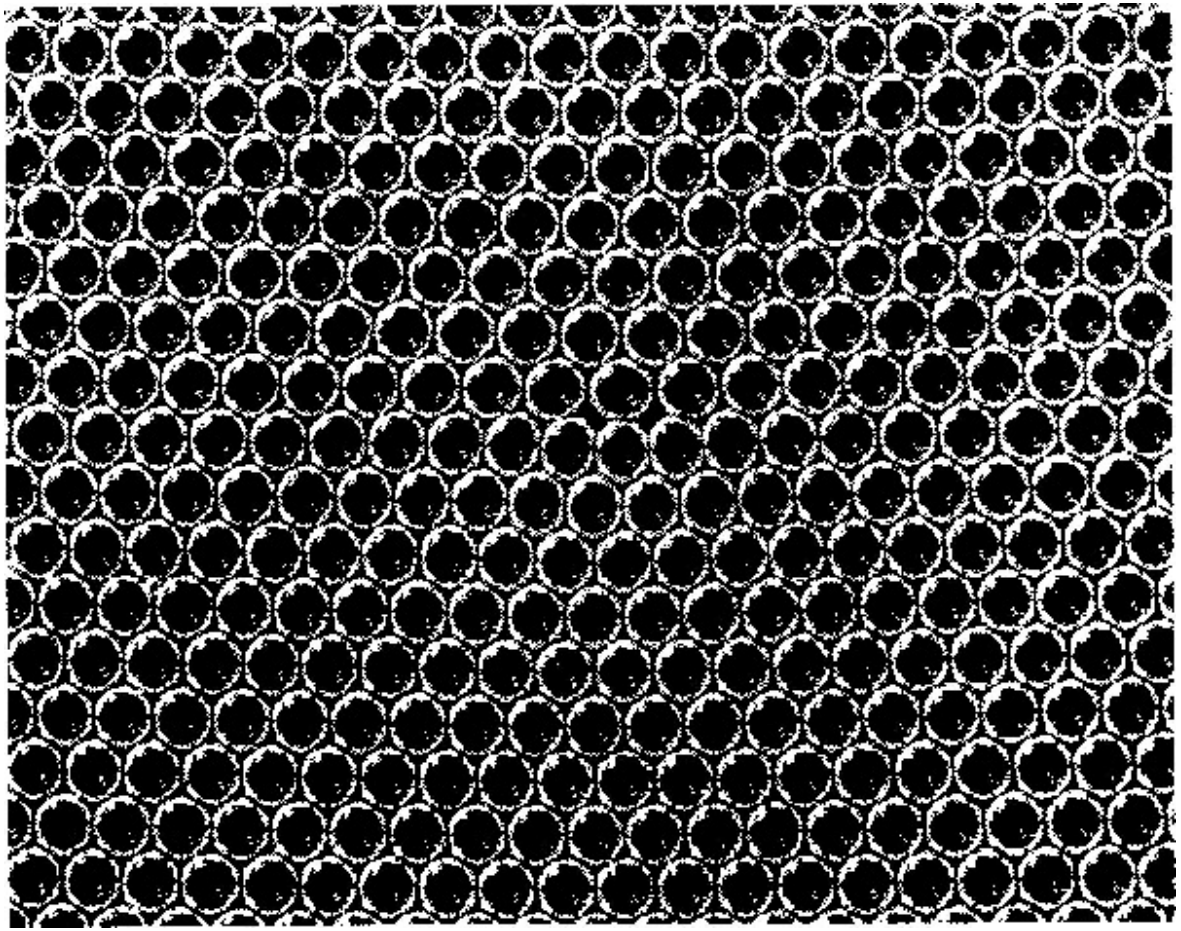




S. Torquato, 2002

# Distribution in solids (dense medium)

## Single dislocation

Bragg,Nye (1947)

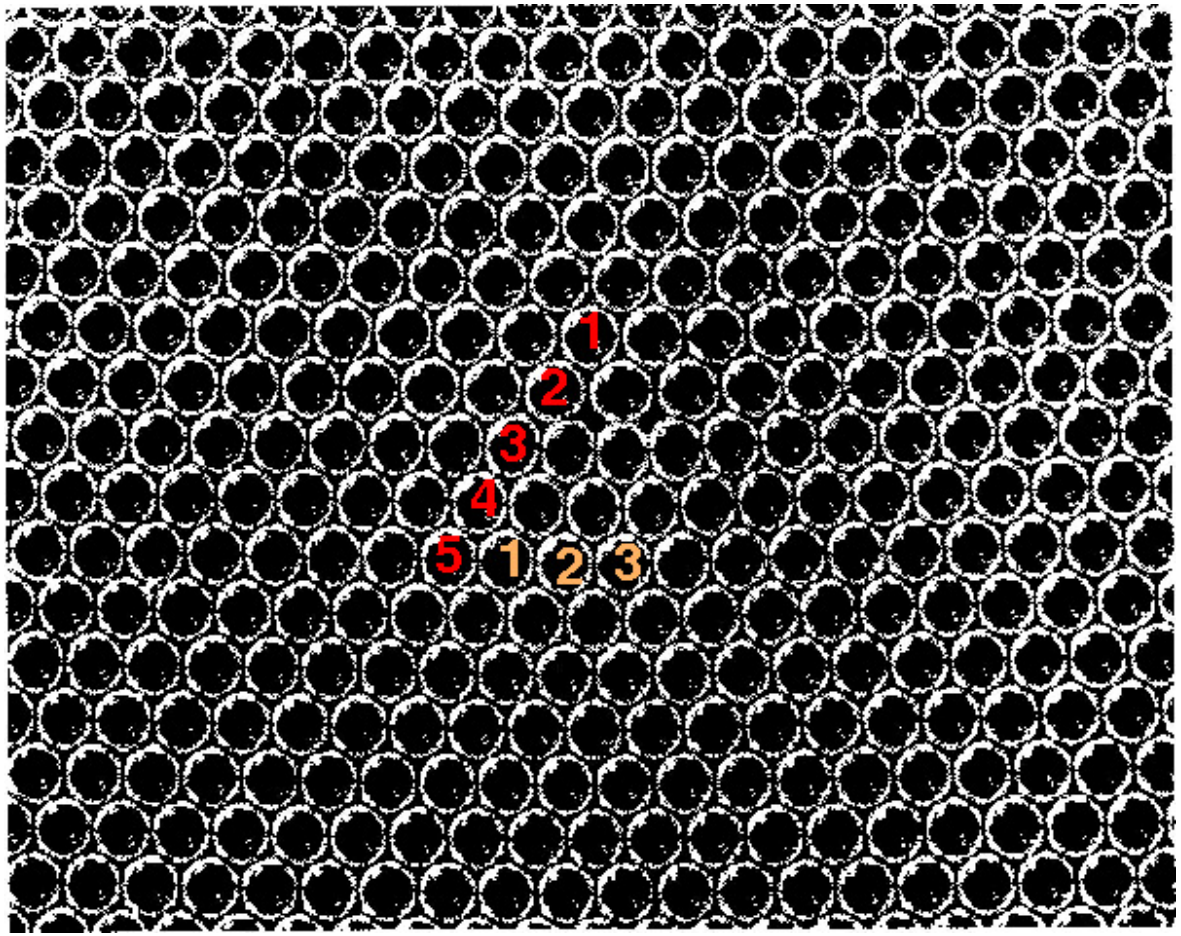


- Physical effects induce new patterns in natural dense medium.
- Very high-order correlations predominate.
- Previous statistical interpretation fails to take into consideration these localized fracture of the crystal.

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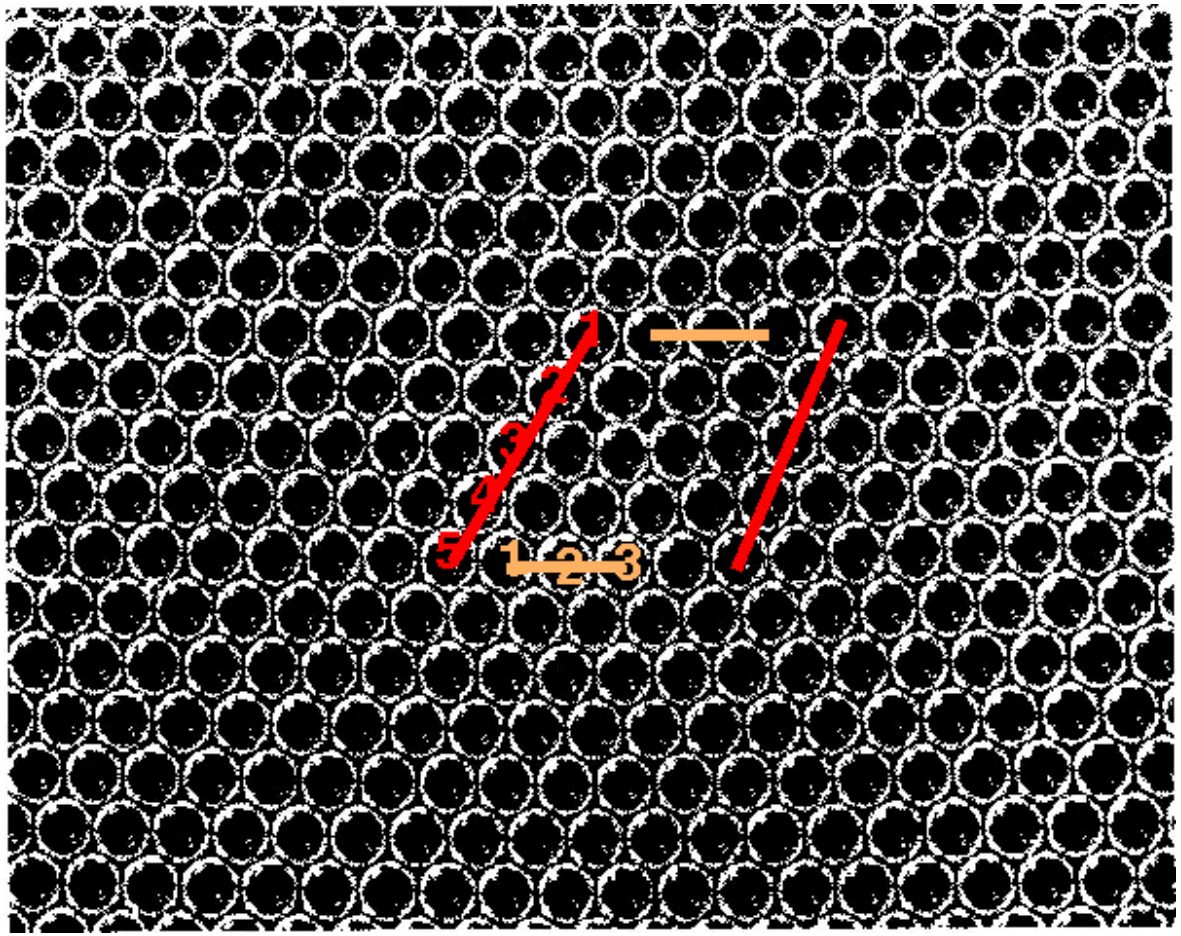



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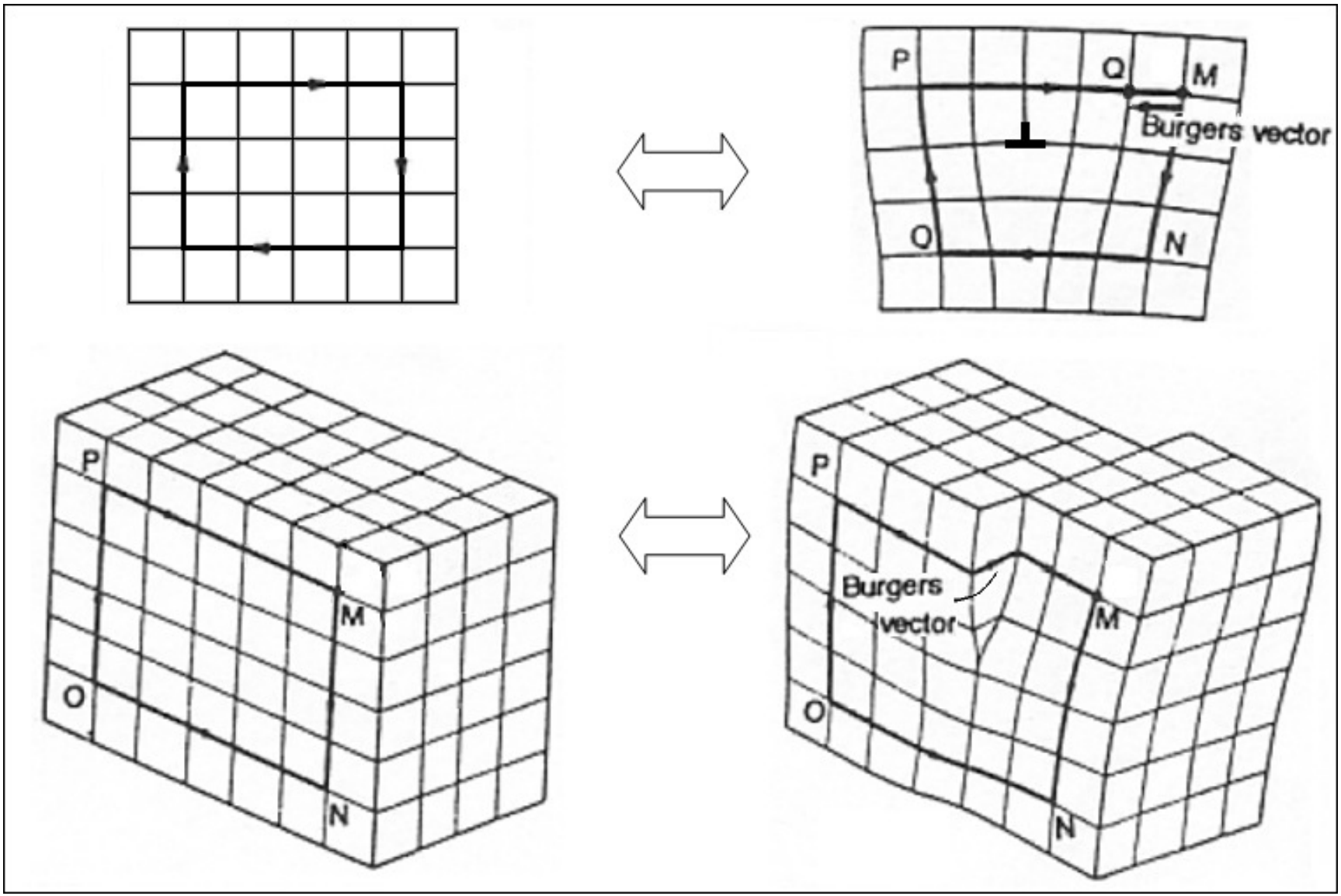
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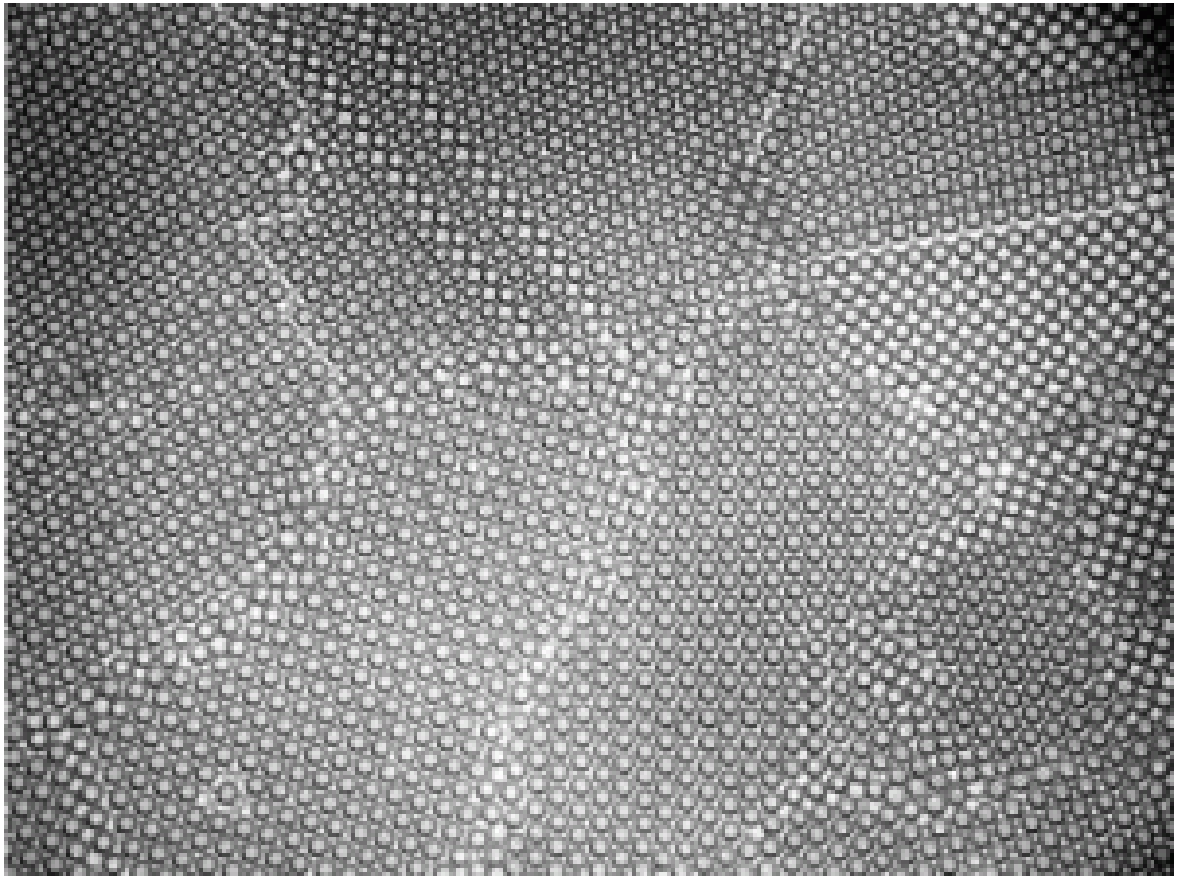
# Example of dislocations



# Distribution in solids (dense medium)

## Grain boundary

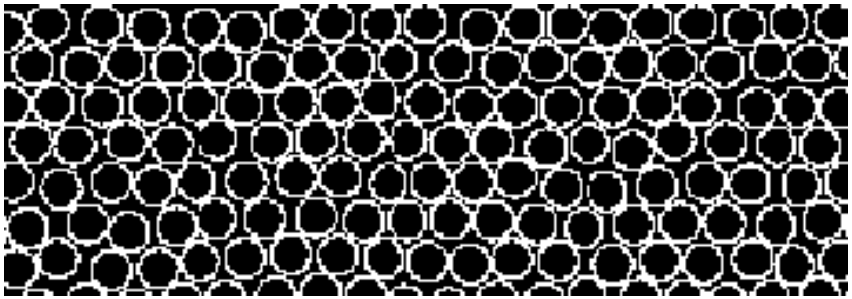
Van der Net *et al* (2006)



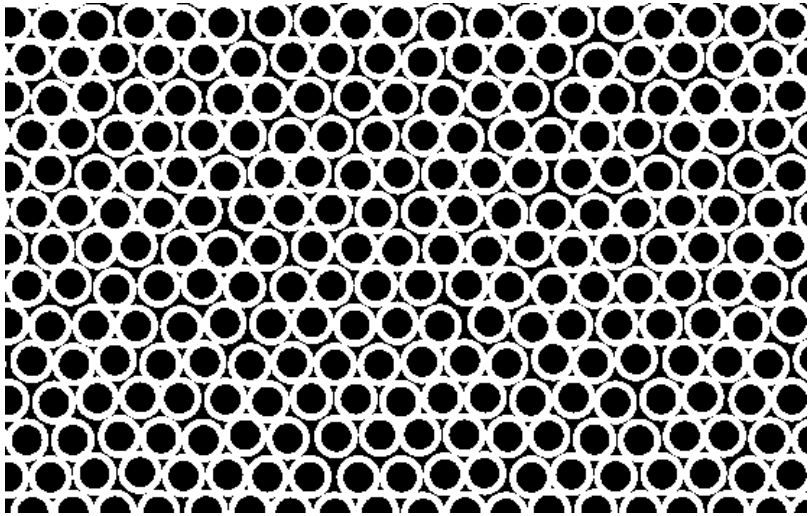
- The structure is mainly described by discontinuity along lines or surfaces
- The average density of defects varies smoothly and slowly in the medium.

# Human / natural distribution of dense medium

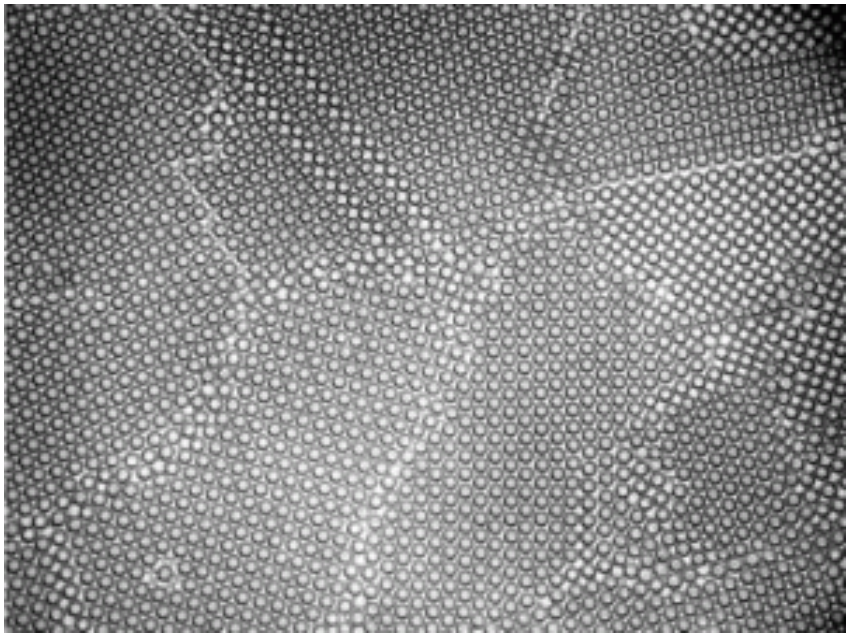
Dynamical computation (J.Piraux LMA)



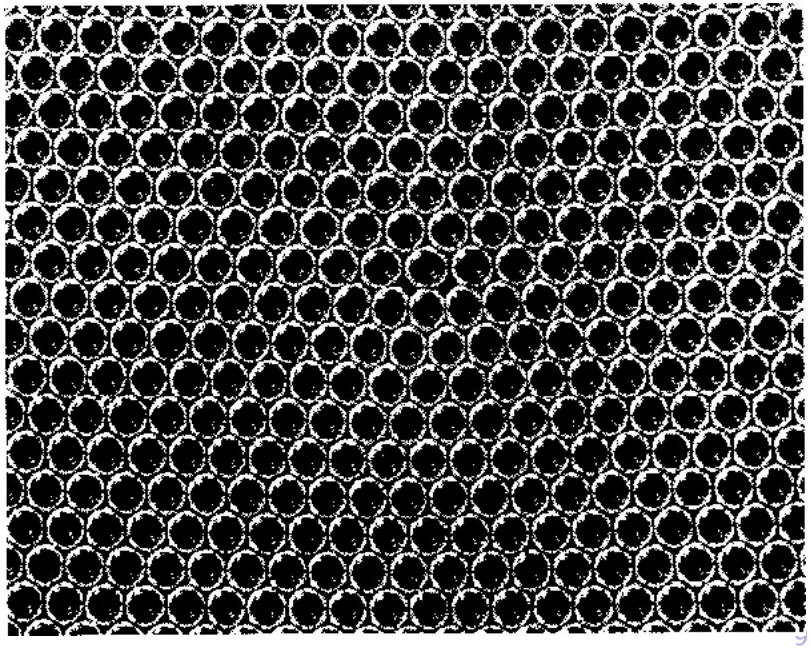
Quasi periodic simulation



Grain boundary (Van der Net *et al* 2006)

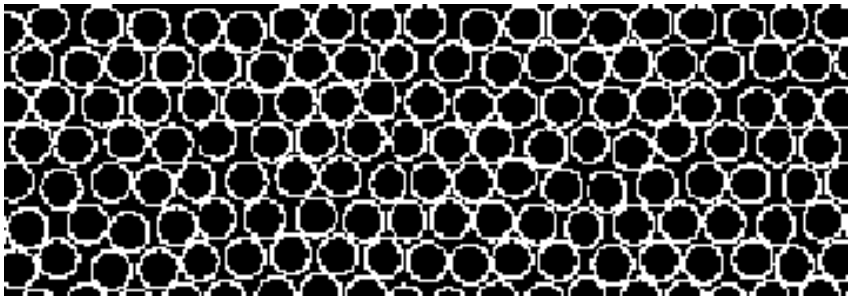


Single dislocation (Bragg,Nye 1947)

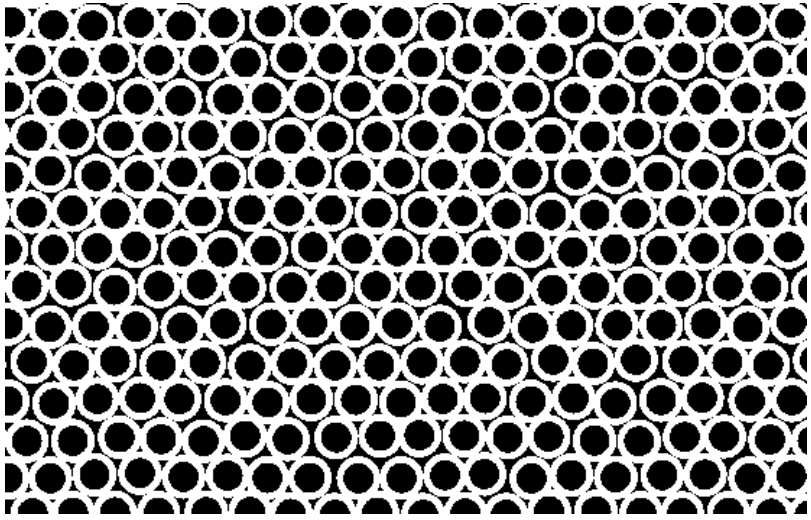


# Human / natural distribution of dense medium

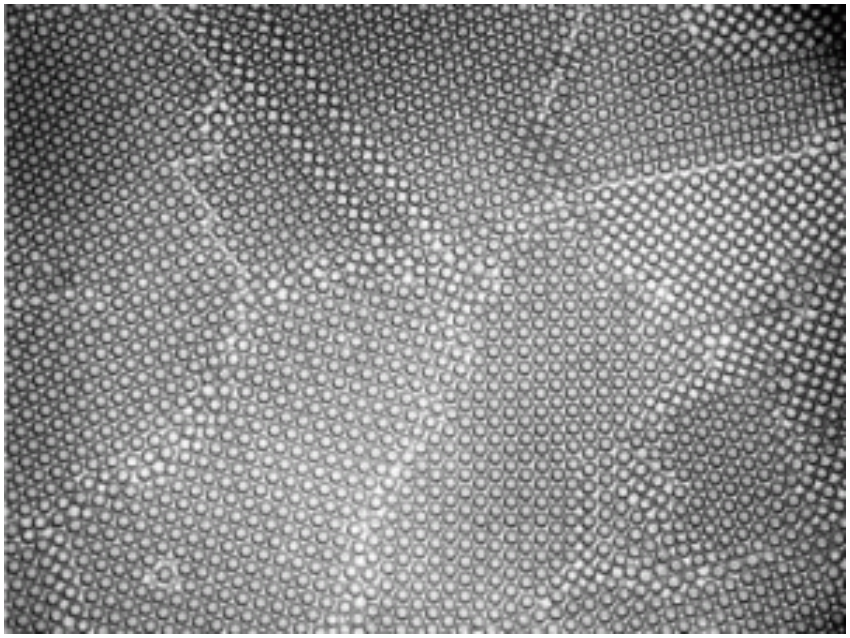
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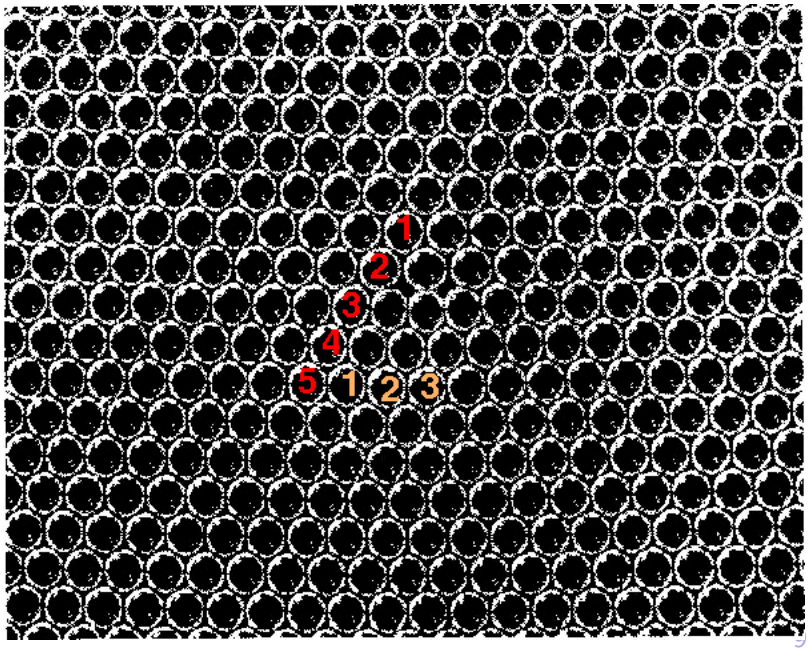
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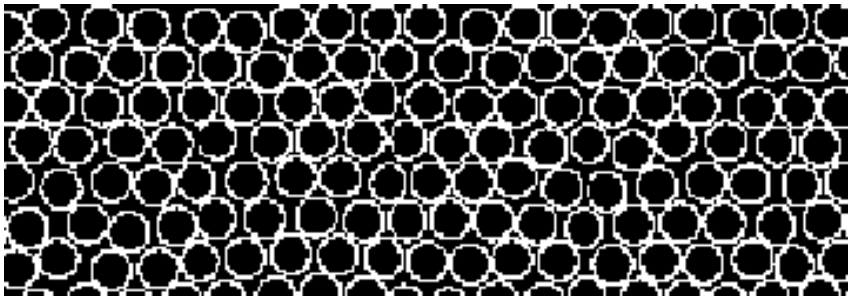


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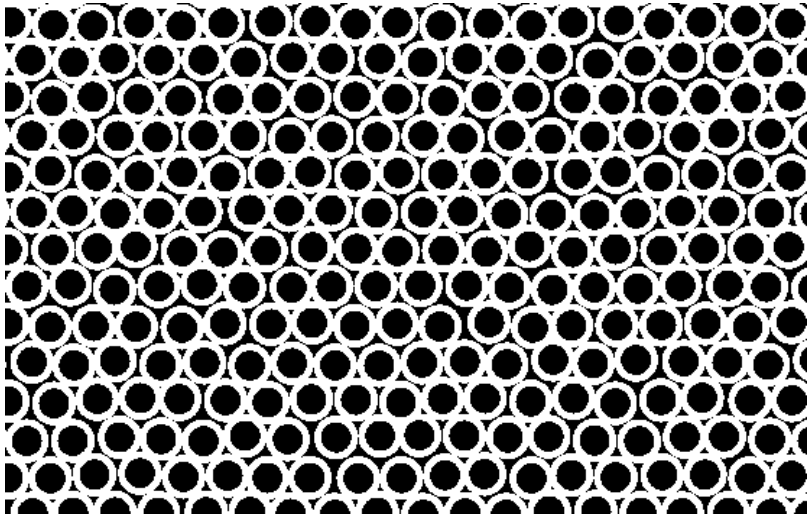


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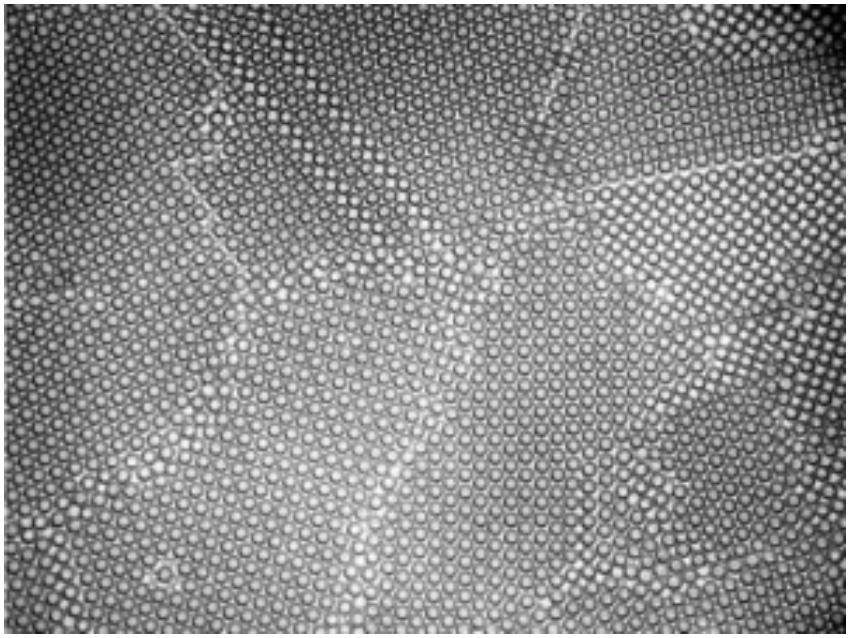
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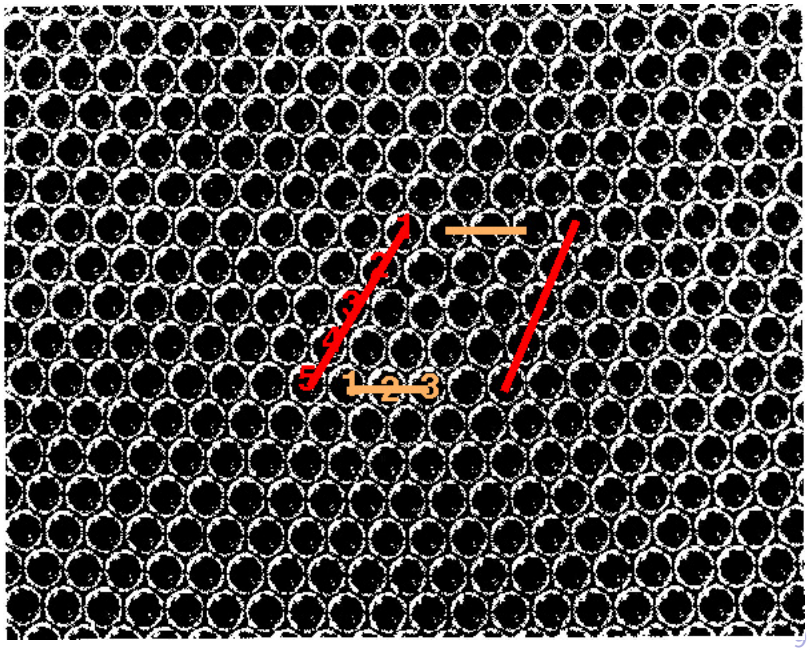
Quasi periodic simulation



Grain boundary (Van der Net *et al* 2006)

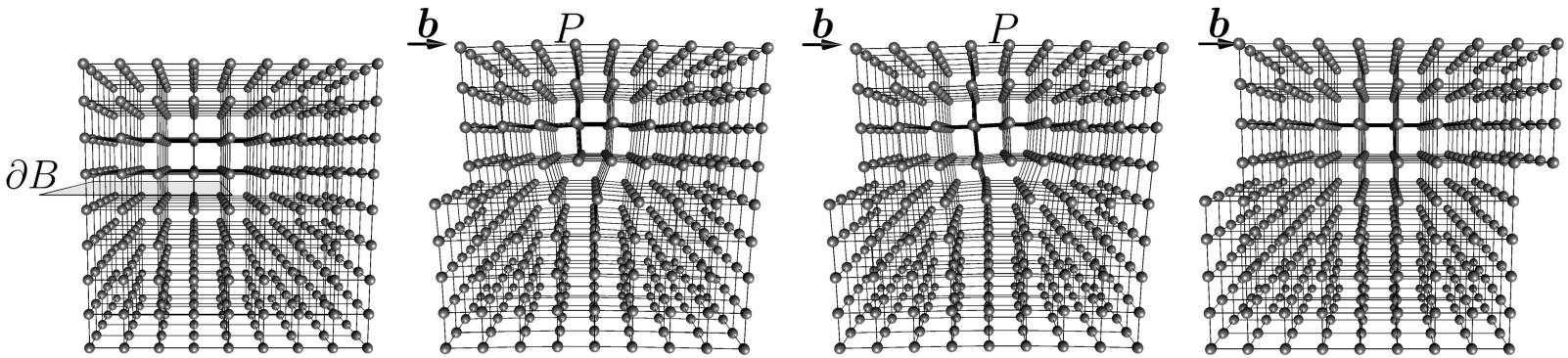


Single dislocation (Bragg,Nye 1947)



# Torsion and plasticity

Effect of a moving dislocation $\Rightarrow$ plasticity




At a macro-scale

- These dislocations are present in all the material.
- The density of dislocation varies smoothly at the macroscale.
- Torsion is a continuous tensor
- It's a good tools for continuous distribution of meso defects

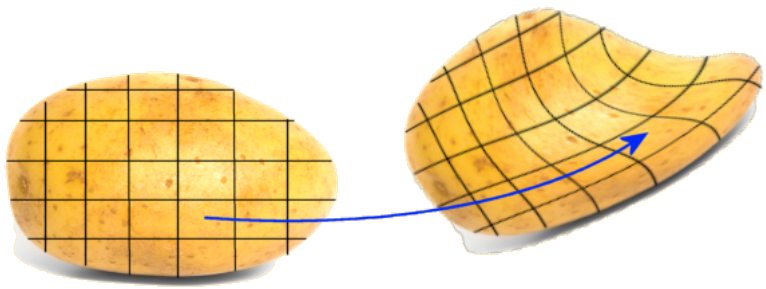
We can create a continuous distribution of torsion by imposing a plastic transformation on a sample.

# Generalized Continuum Mechanics

The body's transformation and vector transformations

## Classical elasticity

Marsden & Hughes 1968



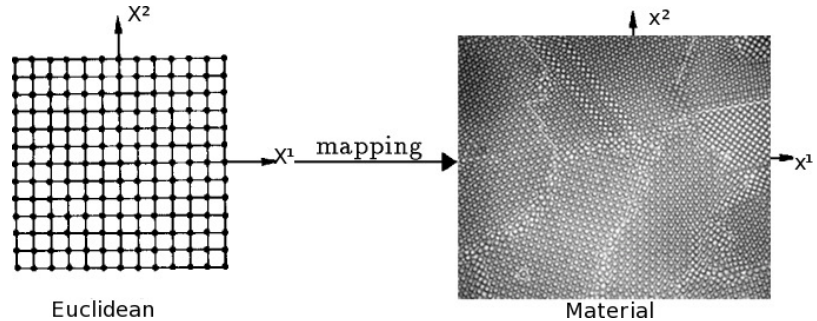
Smooth mapping $\varphi$

$$\begin{aligned} \varphi: \quad X &\longrightarrow x = \varphi(X) \\ \mathbb{F} = D\varphi: \quad \mathbf{V} &\longrightarrow \mathbf{v} = \mathbb{F}\,\mathbf{V} \end{aligned}$$

$$\boxed{v^\alpha = \frac{\partial \varphi^\alpha}{\partial X^i} V^i = F_i^\alpha V^i}$$

## Generalized mechanics

Katanaev 2005, Rakotomanana 2003




Piecewise $\varphi$ and smooth **triads** $\Psi$

$$\begin{aligned} \varphi: \quad X &\longrightarrow x = \varphi(X) \\ \Psi: \quad \mathbf{V} &\longrightarrow \mathbf{v} = \Psi\,\mathbf{V} \end{aligned}$$

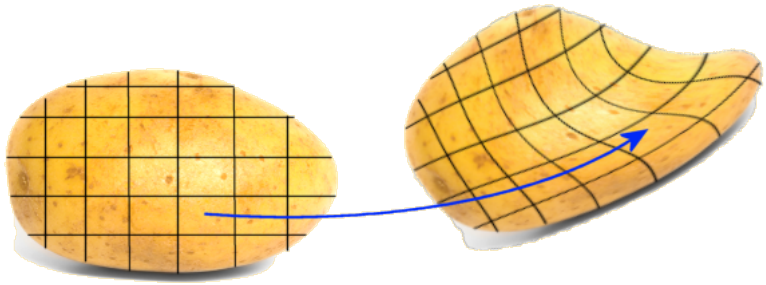
$$\boxed{v^\alpha = \Psi_i^\alpha V^i}$$

$$\Psi_i^\alpha \Psi_\alpha^j = \delta_i^j$$

**Defects** if $\Psi \neq D\varphi$ : plasticty, dislocations...

## Classical elasticity

Marsden & Hughes 1968

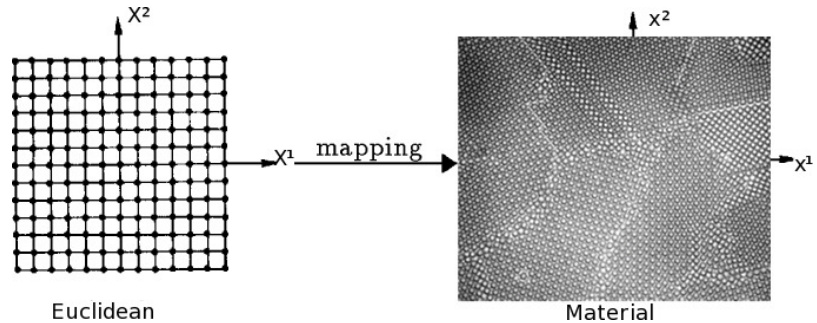


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Katanaev 2005, Rakotomanana 2003




Piecewise continuous $\varphi$ and **triads** $\Psi$

$$\begin{aligned} \varphi: \quad X &\longrightarrow x = \varphi(X) \\ \Psi: \quad \mathbf{V} &\longrightarrow \mathbf{v} = \Psi\,\mathbf{V} \end{aligned}$$

Material metric in the new configuration $\varphi_*(G)$

$$g_{\alpha\beta} = F^i_\alpha F^j_\beta G_{ij} \qquad g_{\alpha\beta} = \Psi^i_\alpha \Psi^j_\beta G_{ij}$$

Strain

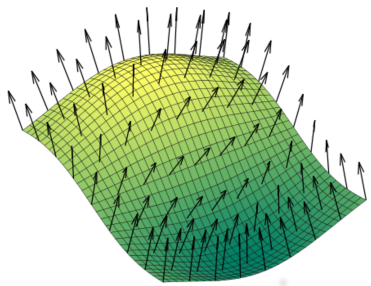
$$\boxed{\varepsilon_{\alpha\beta} = \frac{1}{2}\left(g_{\alpha\beta} - \delta_{\alpha\beta}\right)}$$

The strain looks the same !

# Connection, covariante derivative and torsion

The connection $\nabla$ compares vectors defined on two different points.
The covariante derivative

$$\begin{aligned} \nabla_\alpha \mathbf{e}_\beta &= \Gamma^\gamma_{\alpha\beta}\mathbf{e}_\gamma \\ \nabla_\alpha \mathbf{V} &= \frac{\partial V^\beta}{\partial x^\alpha}\mathbf{e}_\beta + V_\beta \Gamma^\gamma_{\alpha\beta}\mathbf{e}_\gamma \end{aligned}$$



The connection's coefficients $\Gamma^\gamma_{\alpha\beta}$ are defined by vector transformation

**Classical:** $\nabla_\alpha \mathbf{e}_\beta = \nabla_\beta \mathbf{e}_\alpha$

$$\mathbb{F} = D\varphi : \ \mathbf{V} \longrightarrow \mathbf{v} = \mathbb{F}\,\mathbf{V}$$

$$\begin{aligned} \Gamma^\gamma_{\alpha\beta} &= F^\gamma_i \frac{\partial F^i_\beta}{\partial x^\alpha} \quad \text{with} \quad F^i_\beta = \frac{\partial X^i}{\partial x^\beta} \\ &= F^\gamma_i \frac{\partial^2 X^i}{\partial x^\alpha \partial x^\beta} \\ &= \Gamma^\gamma_{\beta\alpha} \end{aligned}$$

**Generalized:** $\nabla_\alpha \mathbf{e}_\beta \neq \nabla_\beta \mathbf{e}_\alpha$

$$\Psi : \ \mathbf{V} \longrightarrow \mathbf{v} = \Psi\,\mathbf{V},$$

$$\begin{aligned} \Gamma^\gamma_{\alpha\beta} &= \Psi^\gamma_i \frac{\partial \Psi^i_\beta}{\partial x^\alpha} \\ &\neq \Gamma^\gamma_{\beta\alpha} \end{aligned}$$

Derivations do not commutes !

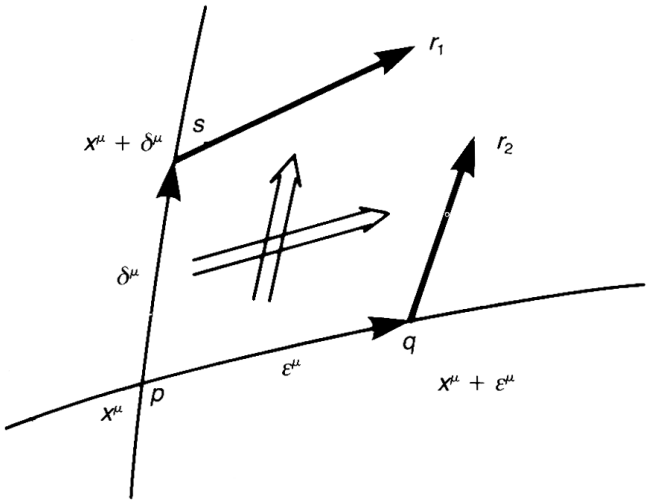
**Torsion tensor** measures the non-compatibility of the transformation.

$$S^\gamma_{\alpha\beta} = \frac{1}{2}\left(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}\right)$$

# Torsion: defect density

$$\nabla_\alpha \mathbf{e}_\beta \neq \nabla_\beta \mathbf{e}_\alpha \qquad S^\gamma_{\alpha\beta} = \frac{1}{2}\left(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}\right)$$

$$\text{Compatibility} \iff S^\gamma_{\alpha\beta} = 0$$




- **non-compatibility** of the transformation
- **non-holonomic** character of the map

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- **Burgers** vector **b**: displacement vector's slope (1939)

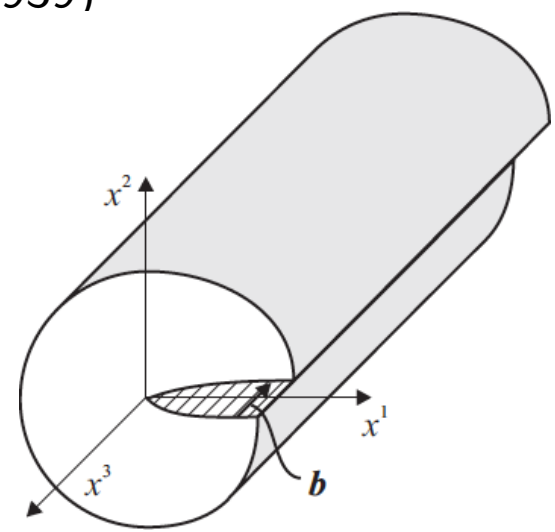
$$b^i = \oint_C \mathrm{d}U^i$$

- **Nye**'s tensor (1953)

$$b^i = \int_\Sigma \boldsymbol{\alpha}\mathbf{e}^{\mathbf{i}} \; \mathrm{d}x^\alpha \mathrm{d}x^\beta$$

- **Cartan**: density of structural defects

$$b^i = \int_\Sigma S^i_{\alpha\beta} \; \mathrm{d}x^\alpha \mathrm{d}x^\beta$$




**But torsion is a tensor with a full geometrical background**

# Connection, torsion and curvature

**Affine connection**

$$\begin{array}{rcl} \nabla & = & \overline{\nabla} + K \\ \Gamma^{\gamma}_{\alpha\beta} & = & \overline{\Gamma}^{\gamma}_{\alpha\beta} + K^{\gamma}_{\alpha\beta} \end{array}$$

$\Gamma^{\gamma}_{\alpha\beta}$ is **not** the Christoffel symbol $\overline{\Gamma}^{\gamma}_{\alpha\beta}$

- Christoffel symbol related to metric

$$\overline{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta})$$

- Contortion related to torsion

$$K^{\gamma}_{\alpha\beta} = \frac{1}{2}(S^{\gamma}_{\alpha\beta} + g^{\gamma\lambda} S^{\kappa}_{\lambda\beta} g_{\kappa\alpha} + g^{\gamma\lambda} S^{\kappa}_{\lambda\alpha} g_{\kappa\beta})$$

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- **Curvature tensor**

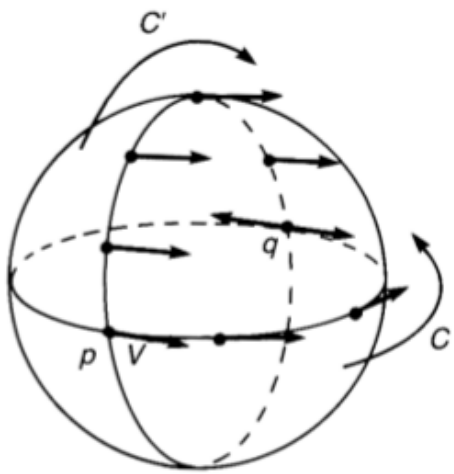
$$R^{\cdot\lambda}_{\alpha\cdot\beta\gamma} = \partial_\beta \Gamma^{\lambda}_{\gamma\alpha} - \partial_\gamma \Gamma^{\lambda}_{\beta\alpha} + \Gamma^{\lambda}_{\beta\kappa}\Gamma^{\kappa}_{\gamma\alpha} - \Gamma^{\lambda}_{\gamma\kappa}\Gamma^{\kappa}_{\beta\alpha}$$

- Defined by the metric and torsion

$$R^{\cdot\lambda}_{\alpha\cdot\beta\gamma} = \overline{R}^{\cdot\lambda}_{\alpha\cdot\beta\gamma} + \overline{\nabla}_\beta K^{\lambda}_{\gamma\alpha} - \overline{\nabla}_\gamma K^{\lambda}_{\beta\alpha} + K^{\lambda}_{\beta\kappa} K^{\kappa}_{\gamma\alpha} - K^{\lambda}_{\gamma\kappa} K^{\kappa}_{\beta\alpha}$$

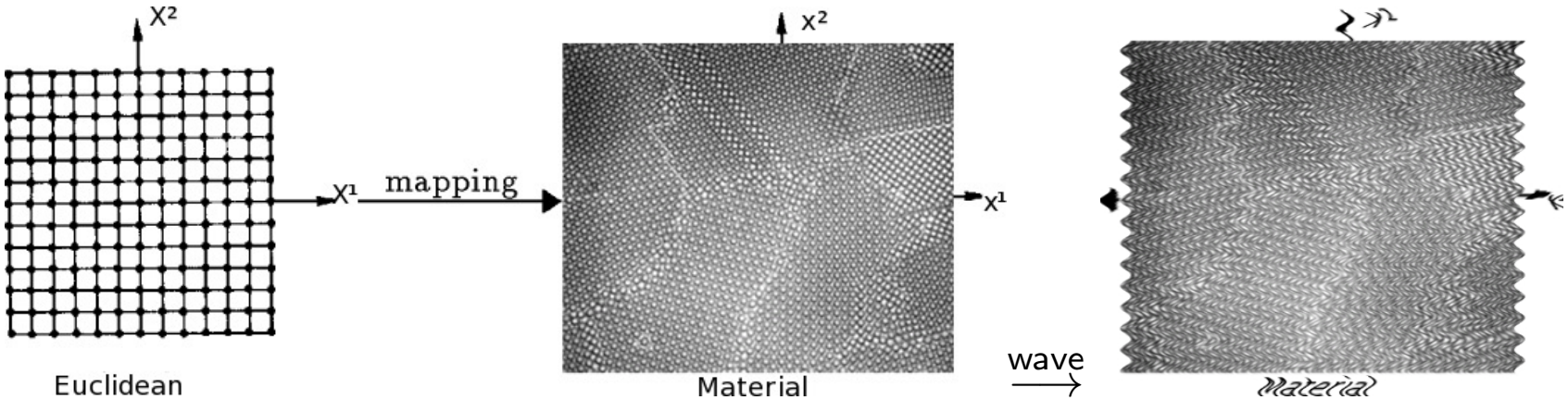
- For uniform torsion

$$R^{\cdot\lambda}_{\alpha\cdot\beta\gamma} = \overline{R}^{\cdot\lambda}_{\alpha\cdot\beta\gamma} + K^{\lambda}_{\beta\kappa} K^{\kappa}_{\gamma\alpha} - K^{\lambda}_{\gamma\kappa} K^{\kappa}_{\beta\alpha}$$



# Riemann-Cartan material manifold

- Reference state is supposed Euclidean.
- Initial state obtains by non-smooth transformations $\varphi$ and $\Psi$.




**Main ingredient**

- The metric tensor $\Leftarrow$ Riemann
- The torsion tensor $\Leftarrow$ Cartan

**Main tool**

Affine connection

$$\nabla = \overline{\nabla} + K$$

- Defect density varies continuously at macro-scale
- Torsion and metric are supposed smooth tensors

**Wave fields are**

- superimposed
- infinitesimal
- smooth

**motion.**

# Extended Navier equation

**Goal:** Analyze the wave in a media having structural defects.

**Media:** Defect density is *non-uniform* but *constant in time*.

**Method:** The conservation's law are stated under *linearized hypotheses*.

- Momentum conservation (no external force)

$$\rho_0 \, \partial_t^2 \mathbf{u} = \nabla \cdot \sigma$$

- Hooke's law: no energy associated to defect density

$$\sigma := \lambda \, \mathrm{Tr}(\varepsilon) \, \mathbf{G} + 2\mu \, \varepsilon$$

- Linearized strain tensor

$$\varepsilon := \frac{1}{2} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^t \right)$$

**... but with** $\boxed{\nabla = \overline{\nabla} + K}$

Futhazar G, LM L, Rakotomanana LR. *Covariant gradient continua applied to wave propagation within defective material*, Arch of Applied Mech, 2014

# Generalized Navier equation

$$\rho\, \partial_t^2 \mathbf{u} = (\lambda+\mu)\overline{\nabla}(\overline{\nabla}\cdot\mathbf{u}) + \mu\overline{\Delta}\mathbf{u} + \mathbf{F}$$

Configurational force:

$$\mathbf{F} = \lambda\Big((\overline{\nabla}\cdot\mathbf{u})(K\cdot\mathbf{G}) + \overline{\nabla}(K\cdot\mathbf{u}) + (K\cdot\mathbf{u})(K\cdot\mathbf{G})\Big) + \dots$$
$$\dots\mu\Big(K\cdot((\overline{\nabla}\mathbf{u}) + (\overline{\nabla}\mathbf{u})^t) + \overline{\nabla}\cdot(K\mathbf{u} + (K\mathbf{u})^t) + K\cdot(K\mathbf{u} + (K\mathbf{u})^t)\Big)$$

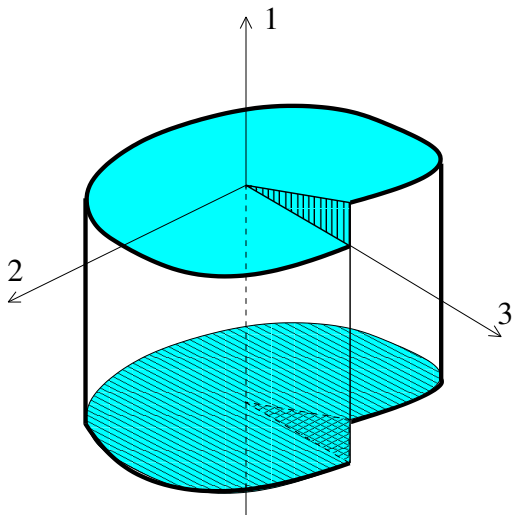
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**Hypotheses**

- Euclidean reference metric $\mathbf{G} = I$.
- Uniform density of defect
- Density of screw dislocation along $\mathbf{e_1}$

$\Rightarrow$ only $S := S^1_{23} \neq 0$ (uniform and constant)

$\Rightarrow$ *Simplified wave equation:*




$$\begin{aligned}
\partial_t^2 u_1 &= (c_l^2 - c_t^2)\partial_1(\overline{\nabla}\cdot\mathbf{u}) + c_t^2\overline{\Delta}u_1 + c_t^2 S\,(\partial_2 u_3 - \partial_3 u_2)\\
\partial_t^2 u_2 &= (c_l^2 - c_t^2)\partial_2(\overline{\nabla}\cdot\mathbf{u}) + c_t^2\overline{\Delta}u_2 + c_t^2 S\,(\partial_3 u_1 + 2\partial_1 u_3) - c_t^2 S^2 u_2\\
\partial_t^2 u_3 &= (c_l^2 - c_t^2)\partial_3(\overline{\nabla}\cdot\mathbf{u}) + c_t^2\overline{\Delta}u_3 - c_t^2 S\,(2\partial_1 u_2 + \partial_2 u_1) - c_t^2 S^2 u_3
\end{aligned}$$

$$c_l = \sqrt{\frac{\lambda+2\mu}{\rho}}$$
$$c_t = \sqrt{\frac{\mu}{\rho}}$$

# Breathing mode

$$
\begin{aligned}
\partial_t^2 u_1 &= (c_l^2 - c_t^2)\partial_1(\overline{\nabla} \cdot \mathbf{u}) + c_t^2 \overline{\Delta} u_1 + c_t^2 S\,(\partial_2 u_3 - \partial_3 u_2) \\
\partial_t^2 u_2 &= (c_l^2 - c_t^2)\partial_2(\overline{\nabla} \cdot \mathbf{u}) + c_t^2 \overline{\Delta} u_2 + c_t^2 S\,(\partial_3 u_1 + 2\partial_1 u_3) - c_t^2 S^2 u_2 \\
\partial_t^2 u_3 &= (c_l^2 - c_t^2)\partial_3(\overline{\nabla} \cdot \mathbf{u}) + c_t^2 \overline{\Delta} u_3 - c_t^2 S\,(2\partial_1 u_2 + \partial_2 u_1) - c_t^2 S^2 u_3
\end{aligned}
$$

---

# Breathing mode

$$
\begin{aligned}
\partial_t^2 u_1 &= (c_l^2 - c_t^2)\partial_1(\overline{\nabla} \cdot \mathbf{u}) + c_t^2 \overline{\Delta} u_1 + c_t^2 S\,(\partial_2 u_3 - \partial_3 u_2) \\
\partial_t^2 u_2 &= (c_l^2 - c_t^2)\partial_2(\overline{\nabla} \cdot \mathbf{u}) + c_t^2 \overline{\Delta} u_2 + c_t^2 S\,(\partial_3 u_1 + 2\partial_1 u_3) - c_t^2 S^2 u_2 \\
\partial_t^2 u_3 &= (c_l^2 - c_t^2)\partial_3(\overline{\nabla} \cdot \mathbf{u}) + c_t^2 \overline{\Delta} u_3 - c_t^2 S\,(2\partial_1 u_2 + \partial_2 u_1) - c_t^2 S^2 u_3
\end{aligned}
$$

---

**Perfect medium: $S = 0$**

Consider plane, irrotationnal, static problem in $(\mathbf{e_2}, \mathbf{e_3})$-direction

$$\underline{\mathbf{u}}(x_2, x_3) = \underline{u}_2 \mathbf{e_2} + \underline{u} \mathbf{e_3} \quad \text{solves} \quad (c_l^2 - c_t^2)\overline{\nabla}(\overline{\nabla} \cdot \underline{\mathbf{u}}) + c_t^2 \overline{\Delta} \underline{\mathbf{u}} = 0$$

# Breathing mode

$$
\begin{aligned}
\partial_t^2 u_1 &= (c_l^2 - c_t^2)\partial_1(\overline{\nabla}\cdot\mathbf{u}) + c_t^2\overline{\Delta}u_1 + c_t^2 S\,(\partial_2 u_3 - \partial_3 u_2)\\
\partial_t^2 u_2 &= (c_l^2 - c_t^2)\partial_2(\overline{\nabla}\cdot\mathbf{u}) + c_t^2\overline{\Delta}u_2 + c_t^2 S\,(\partial_3 u_1 + 2\partial_1 u_3) - c_t^2 S^2 u_2\\
\partial_t^2 u_3 &= (c_l^2 - c_t^2)\partial_3(\overline{\nabla}\cdot\mathbf{u}) + c_t^2\overline{\Delta}u_3 - c_t^2 S\,(2\partial_1 u_2 + \partial_2 u_1) - c_t^2 S^2 u_3
\end{aligned}
$$

---

## Perfect medium: $S = 0$

Consider plane, irrotationnal, static problem in $(\mathbf{e_2}, \mathbf{e_3})$-direction

$$\underline{\mathbf{u}}(x_2, x_3) = \underline{u}_2\mathbf{e_2} + \underline{u}\mathbf{e_3} \quad \text{solves} \quad (c_l^2 - c_t^2)\overline{\nabla}(\overline{\nabla}\cdot\underline{\mathbf{u}}) + c_t^2\overline{\Delta}\underline{\mathbf{u}} = 0$$

## With defects: $S \neq 0$

Consider plane, irrotationnal, dynamic problem in $(\mathbf{e_2}, \mathbf{e_3})$-direction

$\mathbf{u}(\mathbf{x}, t) = \underline{\mathbf{u}}(\mathbf{x})e^{i\omega t}$ is a solution of the new problem if $\boxed{\omega = c_t S}$

$\Longrightarrow$ $c_t S$ is the eigen-frequency of a breathing mode

$\Longrightarrow$ curvature induced

# Harmonic plane wave $\mathbf{u}(\mathbf{x}, t) = \mathbf{U} e^{i(k\mathbf{n}\cdot\mathbf{x} - \omega t)}$

- frequency $\omega$
- polarization $\mathbf{U}$
- direction of propagation $\mathbf{n}$, $\|\mathbf{n}\| = 1$
- wavenumber $k(\theta, \phi; \omega)$ must solves

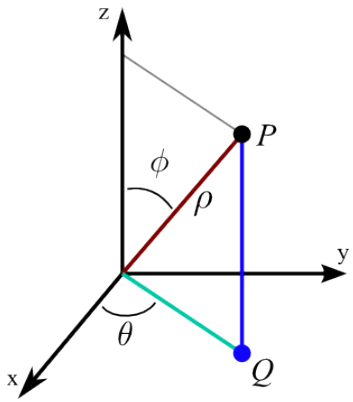
$\Rightarrow$ **anisotropic dispersion relation**

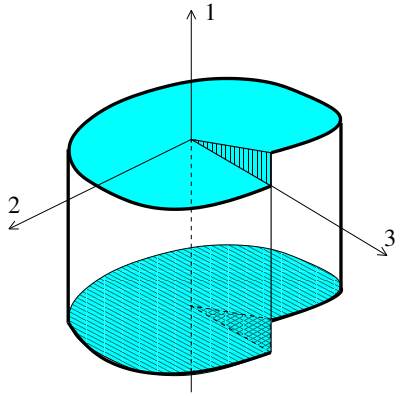
$$\det(\omega^2 \mathbb{I} - k^2 \mathbb{M} + ik\mathbb{N} + \mathbb{D}) = 0$$

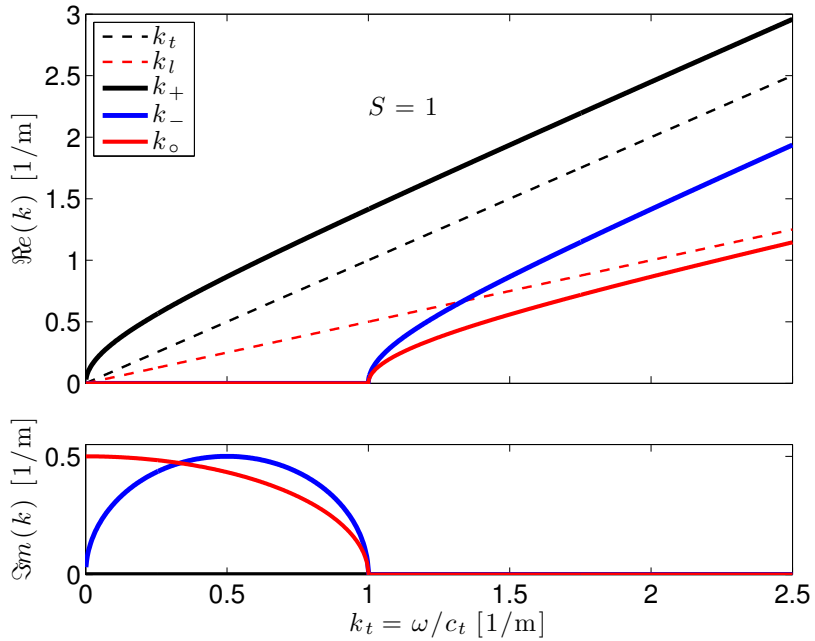
## Various phenomena !

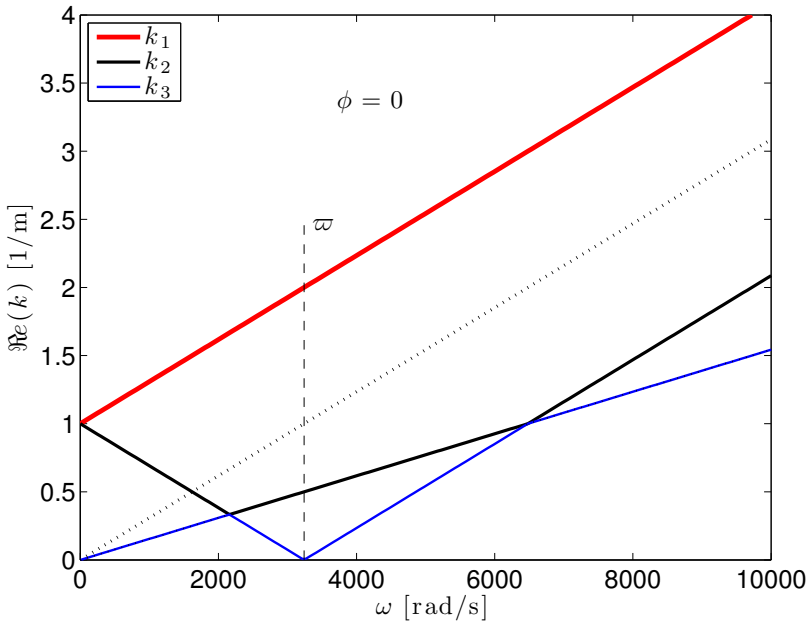
- Dispersive waves
- Zero group velocity (localisation)
- Attenuation only at low-frequency
- Possible assymptotic behavior

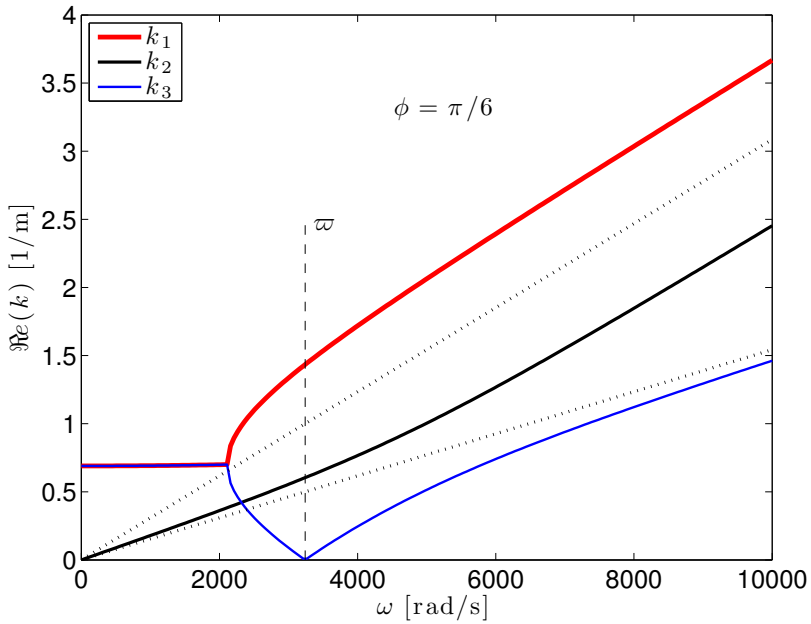
Propagation normal to defects ($\phi = \frac{\pi}{2}$) $\Rightarrow$

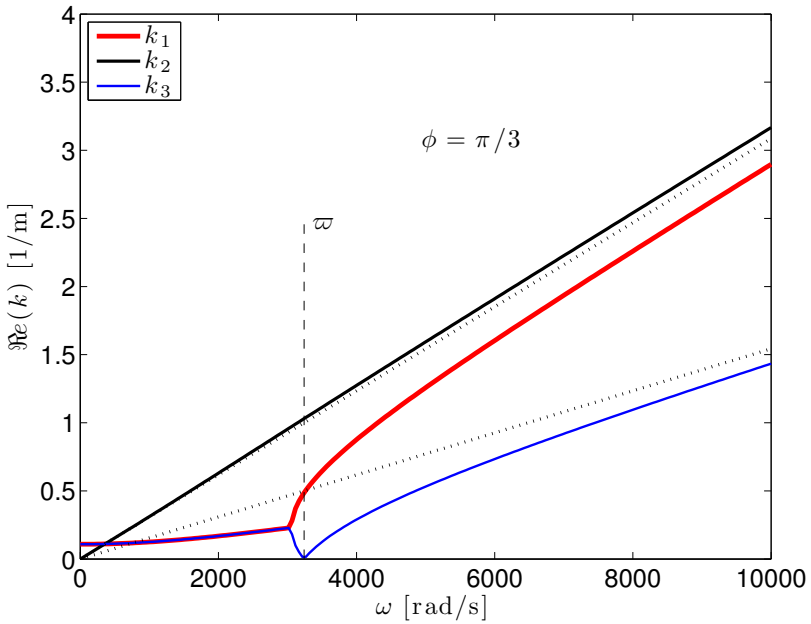


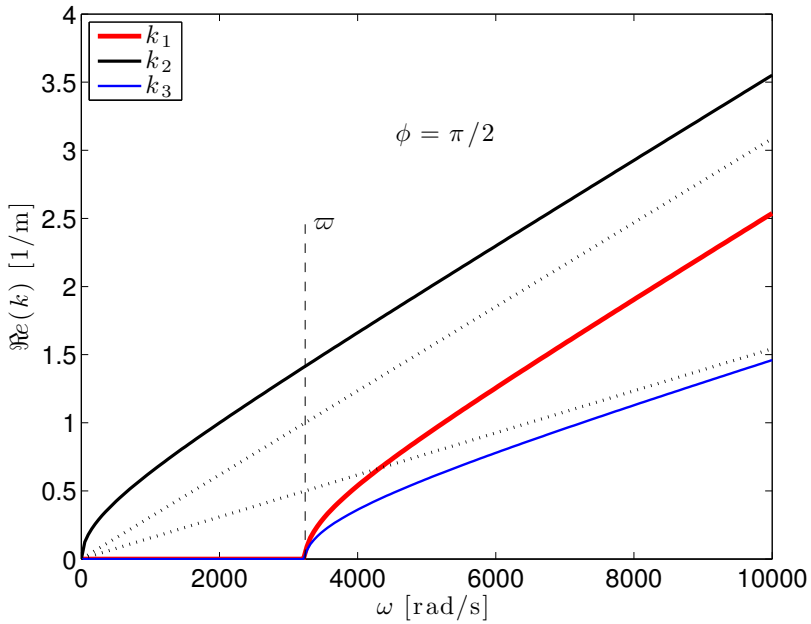



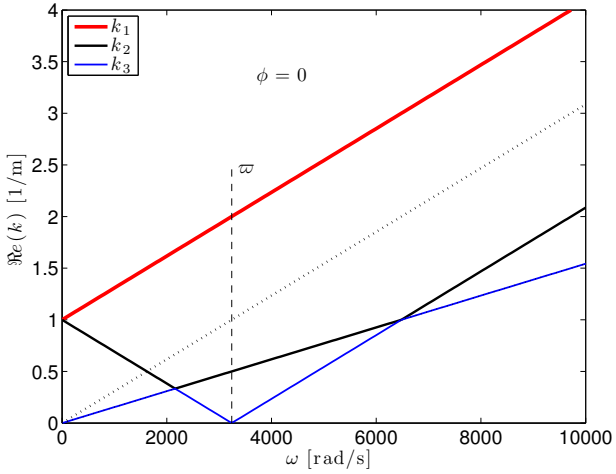

$k_1$
$k_2$
$k_3$
$\phi = 0$
$\varpi$
$\Re e(k)$ [1/m]
$\omega$ [rad/s]

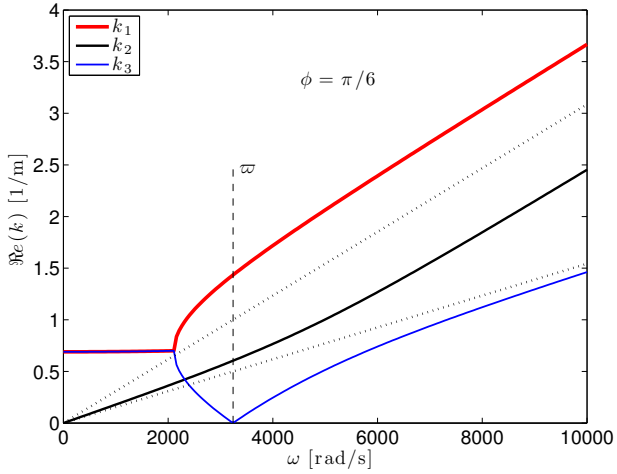

$k_1$
$k_2$
$k_3$
$\phi = \pi/6$
$\varpi$
$\Re e(k)$ [1/m]
$\omega$ [rad/s]


$k_1$
$k_2$
$k_3$
$\phi = \pi/3$
$\varpi$
$\Re e(k)$ [1/m]
$\omega$ [rad/s]


$k_1$
$k_2$
$k_3$
$\phi = \pi/2$
$\varpi$
$\Re e(k)$ [1/m]
$\omega$ [rad/s]

# Chirality in the high frequency regime



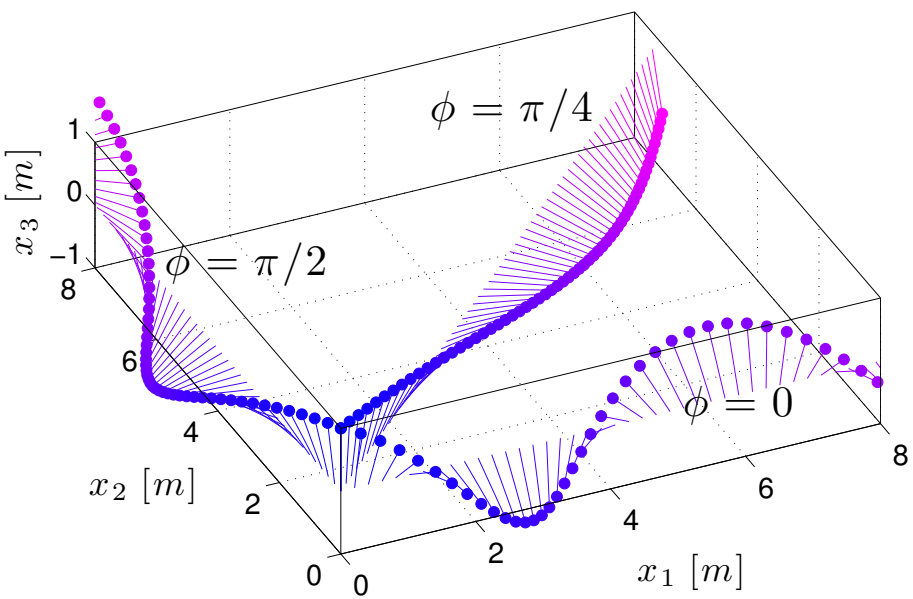




$$k_1 \sim k_t + \kappa$$
$$k_2 \sim k_t - \kappa \qquad k_i = \frac{\omega}{c_i}$$
$$k_3 \sim k_l$$

$$\kappa = S\frac{1 + 3\cos(2\phi)}{4}$$

Rotation of the polarization

$$u^1 = \left(\cos(\boldsymbol{\kappa}\cdot\mathbf{x})\mathbf{e}_\theta - \sin(\boldsymbol{\kappa}\cdot\mathbf{x})\mathbf{e}_\phi\right) e^{i(\mathbf{k_t}\cdot\mathbf{x}-\omega t}$$
$$u^2 = \left(\sin(\boldsymbol{\kappa}\cdot\mathbf{x})\mathbf{e}_\theta + \cos(\boldsymbol{\kappa}\cdot\mathbf{x})\mathbf{e}_\phi\right) e^{i(\mathbf{k_t}\cdot\mathbf{x}-\omega t}$$
$$u^3 = V_3\mathbf{e_r}\ e^{i(\mathbf{k_l}\cdot\mathbf{x}-\omega t)}$$



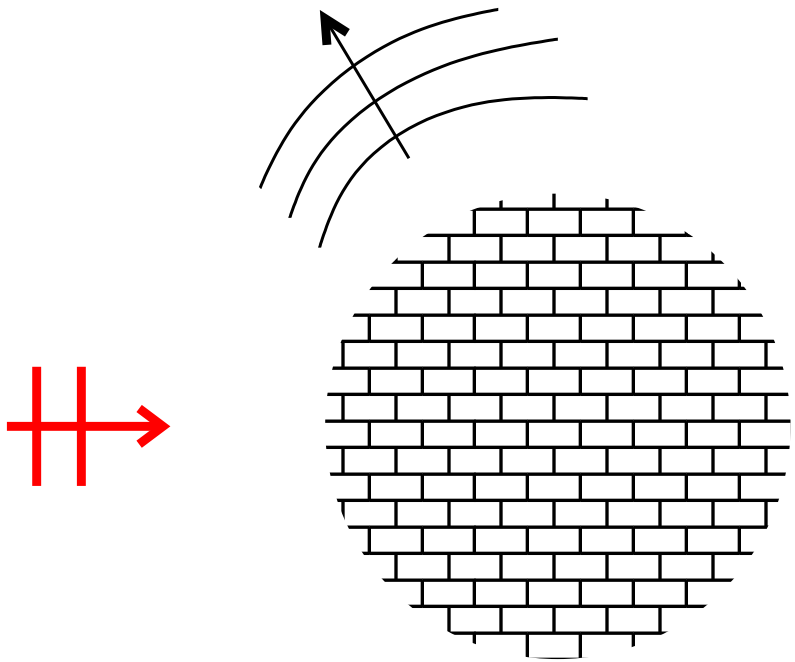

**Effect on long distances**

- cumulative effect
- experimental validation ?

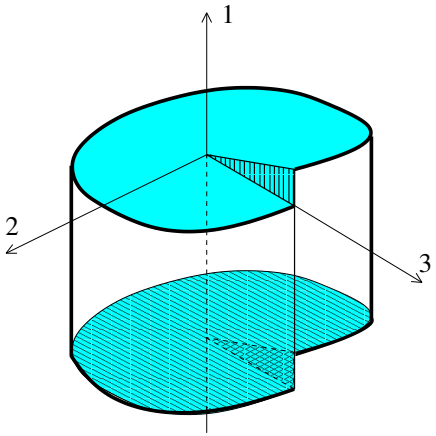
# Diffraction by a dislocation line

- A perfect ambiant medium
- A damaged sub-domain

- Known incoming harmonic wave
- Unknown scattered wave
- Unknown transmitted wave

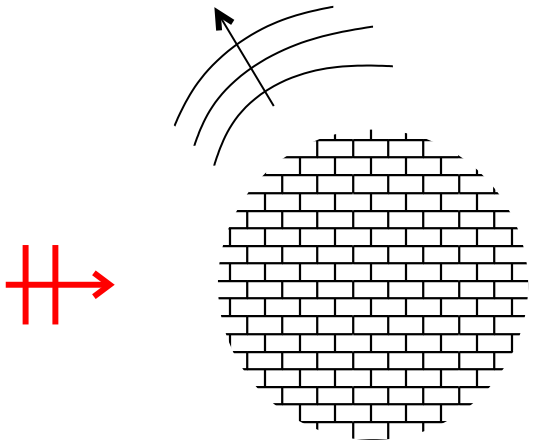


# A screw dislocation line




**Three typical lengths**

- $b$ Burgers vector
- $a$ core's size
- $L$ wavelength



- The Burgers vector $\mathbf{b} = b\mathbf{e_1}$ (fixed)

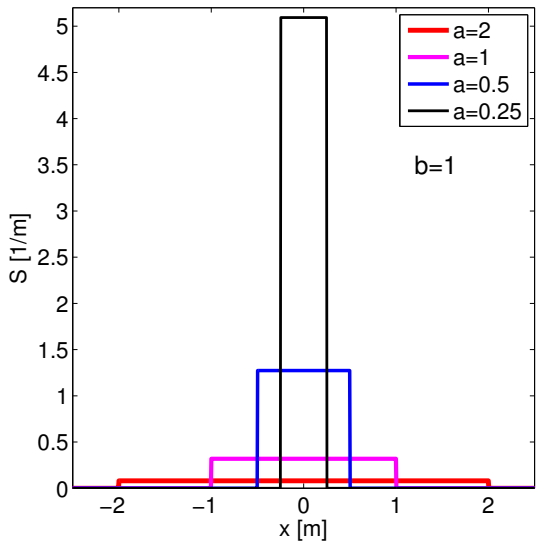
$$b = \oint_C \mathrm{d}U^1 = \int_\Sigma S^1_{23} \ \mathrm{d}x^2\,\mathrm{d}x^3$$

- Uniform density $S^1_{23}$ in the core

$$S(r) = \begin{cases} \dfrac{b}{\pi a^2}\,, & r \le a \\ 0\,, & r > a \end{cases}$$

- $a \ll L$ : the torsion is **almost** a $\delta$-function

$b \ll a \ll L$ **low** defect density
$b \sim a \ll L$ **high** defect density



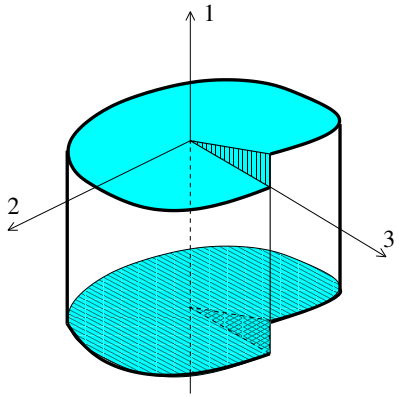

$$S^1_{23}(\mathbf{x}) \xrightarrow[a\to 0]{} b\ \delta(x_2 - 0, x_3 - 0)$$

# Wavefields in the defected area

- The Peach-Koehler force

$$\mathbf{F} = \mu S\, \overline{\nabla} \times \mathbf{u} - \mu S^2 \mathbf{w} \quad \text{with} \quad \mathbf{u} = \mathbf{w} + \mathbf{u_z}$$

  - $\mathbf{w}$ in-plane displacement
  - $\mathbf{u_z}$ vertical displacement




**Remarks:** close to Eshelby (1953) configurational force: $\mathbf{F} = \mu b\, \overline{\nabla} \times \mathbf{u_z}$

---

- Helmholtz decomposition

$$\mathbf{w} = \overline{\nabla}\varphi + \overline{\nabla} \times (\psi \mathbf{e_z})$$

  - $\varphi$ longitudinal potential
  - $\psi$ shear potential

$$\begin{aligned} (k_l^2 - \frac{c_t^2}{c_l^2} S^2)\varphi + \overline{\Delta}\varphi &= 0 \\ (k_t^2 - S^2)\psi + \overline{\Delta}\psi &= -S u_z \\ k_t^2 u_z + \overline{\Delta} u_z &= S \overline{\Delta}\psi \end{aligned}$$

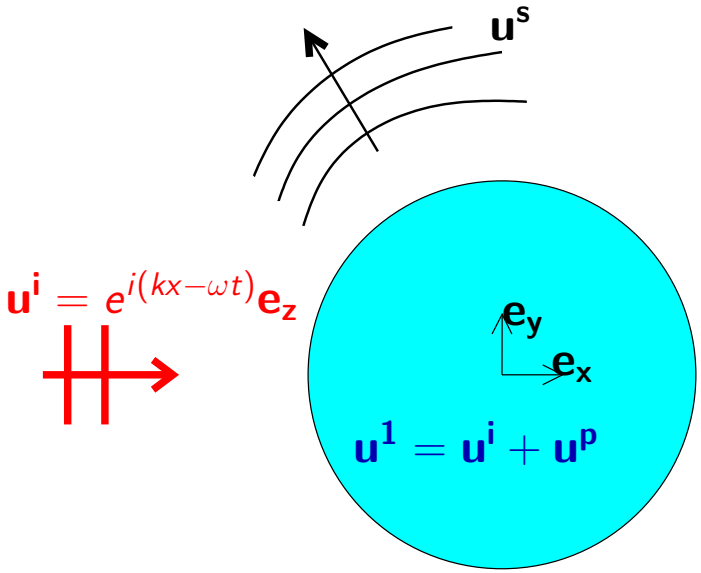
$\Rightarrow$ Shear potential $\psi$ and transverse displacement $u_z$ are **coupled**

$\Rightarrow$ Coupling of the P-SV and SH problems

# Scattering problem by a shear verticaly-polarized plane-wave

**Problem**: An unitary plane verticaly-polarized shear wave probes the dislocation

$$u_z^i = k_t \sum_n c_n^z J_n(k_t r) e^{in\theta}$$




- Scattering and transmission
  - in-plane : longitudinal and shear wave
  - out-of-plane : shear wave

Scattering wave $$u_z^s = k_t \sum_n a_n^z H_n(k_t r) e^{in\theta}$$

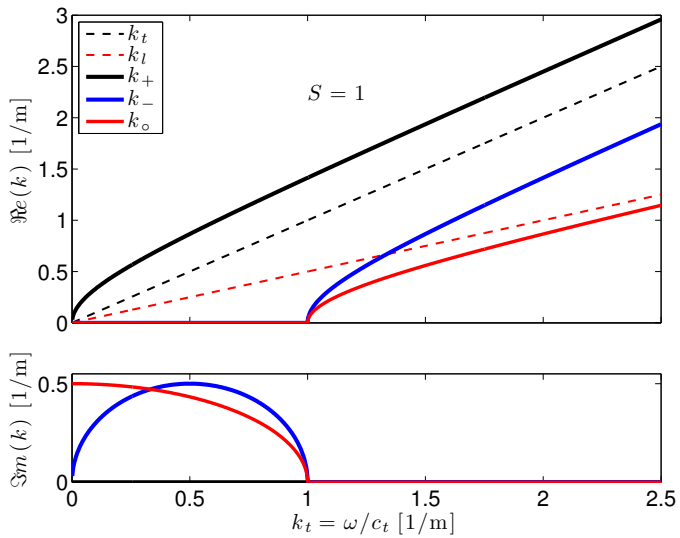
Transmitted wave $$\psi^+ = \sum_n b_n^+ J_n(k^+ r) e^{in\theta}$$

- At the interface, continuity of
  - *superimposed* displacement
  - *superimposed* stress

- Solve $\mathbb{L}\mathbf{x} = \mathbf{y}$
  - *unknown* modal amplitude **x**
  - *known* source term **y**: incoming wave

$\Rightarrow$ $b \ll a \ll L$ **low** defect density

$\Rightarrow$ $b \sim a \ll L$ **high** defect density



# Low defect density $b \ll a$ : *core displacement*

$$\mathbf{u^1} = \mathbf{u^i} + i\left(\gamma + i\frac{\pi}{2} + \ln\epsilon\right)\frac{c_t^2}{c_l^2}\frac{b}{L}\mathbf{e_y} + \mathcal{O}(\epsilon^3)$$

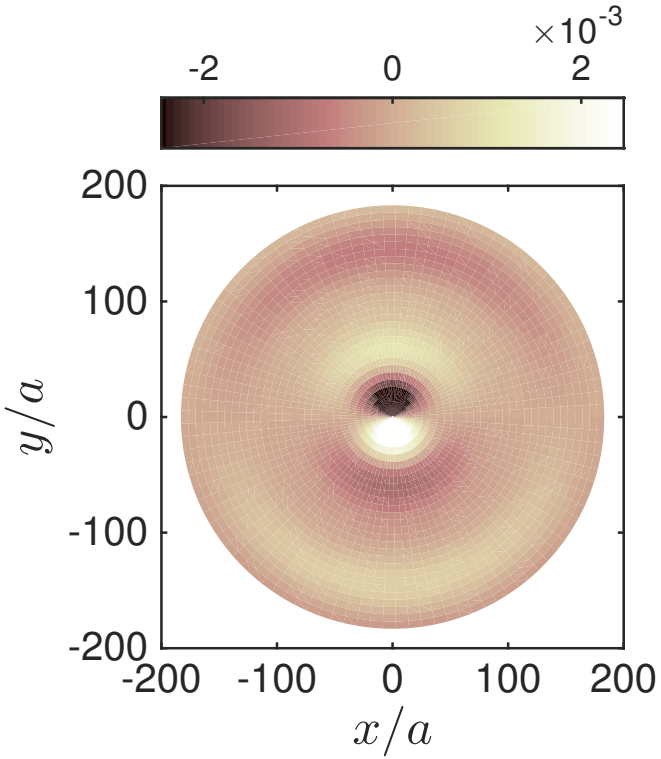
!! Independent of the core-size $a$

!!! $u_z^1 = u_z^i$ the obstacle is invisible... **vertically**

# Low defect density $b \ll a$ : *far-field radiation patern*

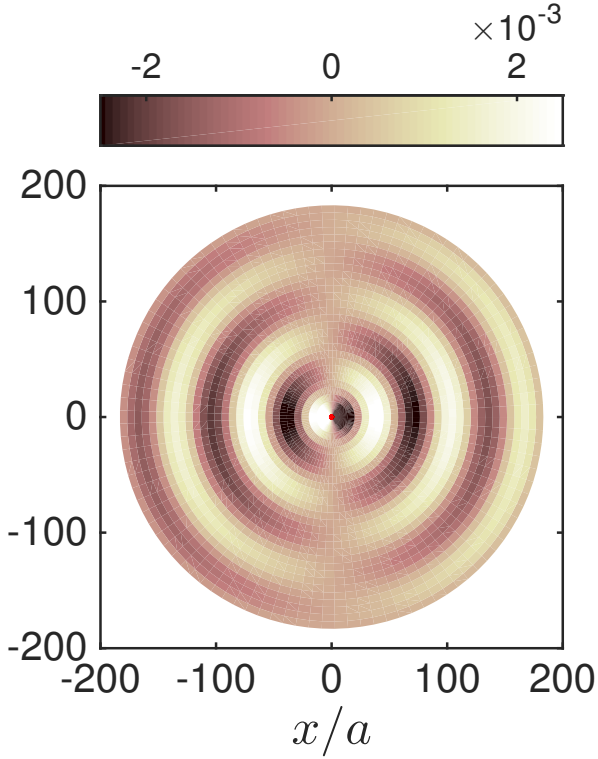
in-plane longitudinal dipole

$$f_{zl} = \frac{1-i}{2}\sqrt{\pi}\frac{c_t^2}{c_l^2}\frac{b}{L}\sin\theta$$




in-plane shear dipole

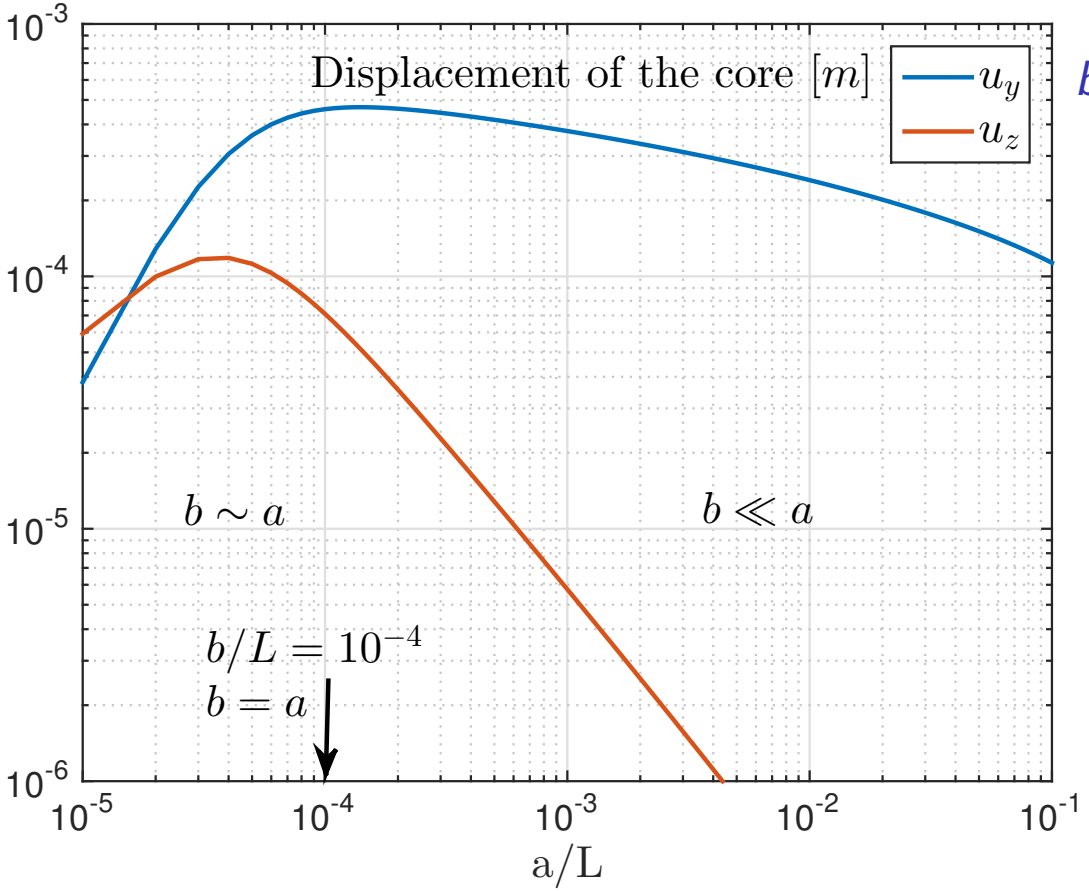
$$f_{zt} = \frac{1-i}{2}\sqrt{\pi}\frac{b}{L}\cos\theta$$




!! Independent of the core-size $a$

!!! pure SH→PSV conversion

# Effect of the core size, of the defects density




$b \ll a \Rightarrow u_z = u_z^1 - u_z^i$ negligible in the core

$b \sim a \Rightarrow u_z$ same order of magnitude than $u_y^1$

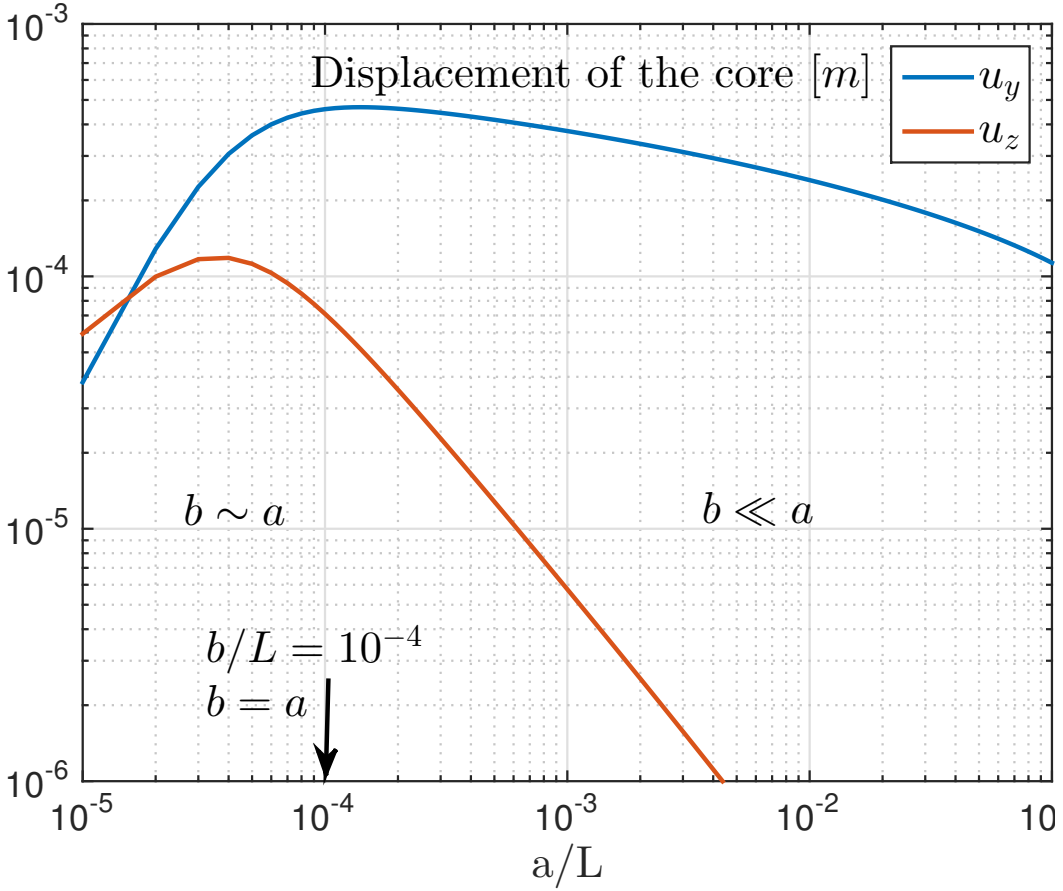
$b \sim a \Rightarrow u_z \sim \dfrac{b}{L}$

$\Rightarrow$ no pure PSV$\rightarrow$ SH conversion

$\Rightarrow$ shear-vertical scattering

**Two dynamical regimes**

# Effect of the core size, of the defects density




$b \ll a \Rightarrow u_z = u_z^1 - u_z^i$ negligible in the core

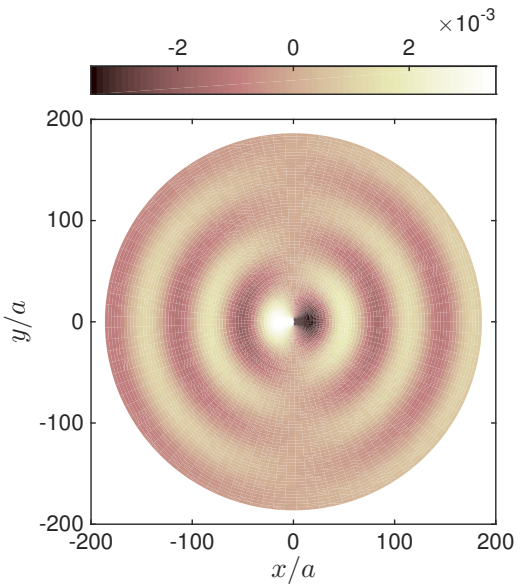
$b \sim a \Rightarrow u_z$ same order of magnitude than $u_y^1$

$b \sim a \Rightarrow u_z \sim \dfrac{b}{L}$

$\Rightarrow$ no pure PSV$\rightarrow$ SH conversion

$\Rightarrow$ shear-vertical scattering

**Two dynamical regimes**




Vertical-shear scattering

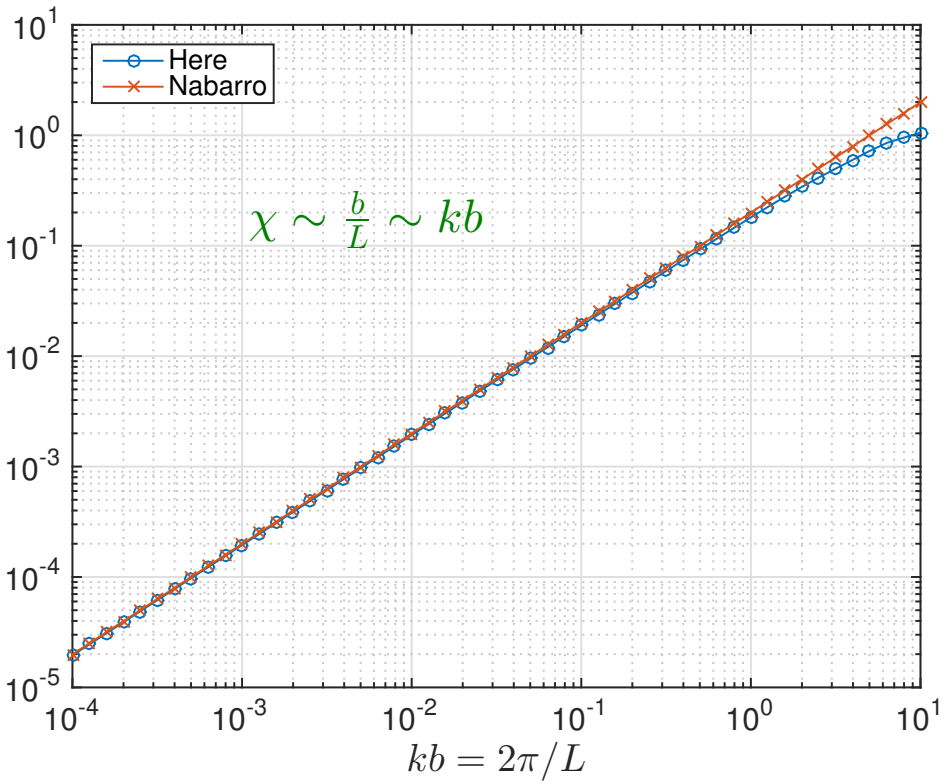
# Comparison with literature: scattering behavior

| The core | **Here** | **Nabarro et al** |
|---|---|---|
| • moves in the plane | • yes | • yes |
| • produces vertical-shear scattering | • only if $b \sim a$ | • yes |
| • produces in-plane scattering | • yes | • no |
| | ! coupled effect | ! out-of-plane pb |

- Dynamical models for interaction between wave and screw dislocation:
  FRN **Nabarro**, Proc R Soc A, 1951
  K Lücke, AV Granato, Physical Review B, 1981
  Murat 1963, Lund 1988, Maurel 2004
- Effect of the core size
  PY Pellegrini, J. of the Mech. and Physics of Solids, 60(2), 2012

# Scattering strength

$$a = b = 1\,nm \qquad 10^{-4} < kb < 10 \qquad \|\mathbf{u}^{\mathbf{i}}\| = b \qquad c_t = 3177 m/s$$
$$50\,\mu m > L > 0.5\,nm$$




Vertical-shear scattering

$$u_z^s = A\ u_z^i \cos\theta \frac{e^{i(k_t r - \omega t)}}{\sqrt{k_t r}}$$

Transverse core displacement

$$u_y^1 = B\ u_z^i$$

Scattering strength

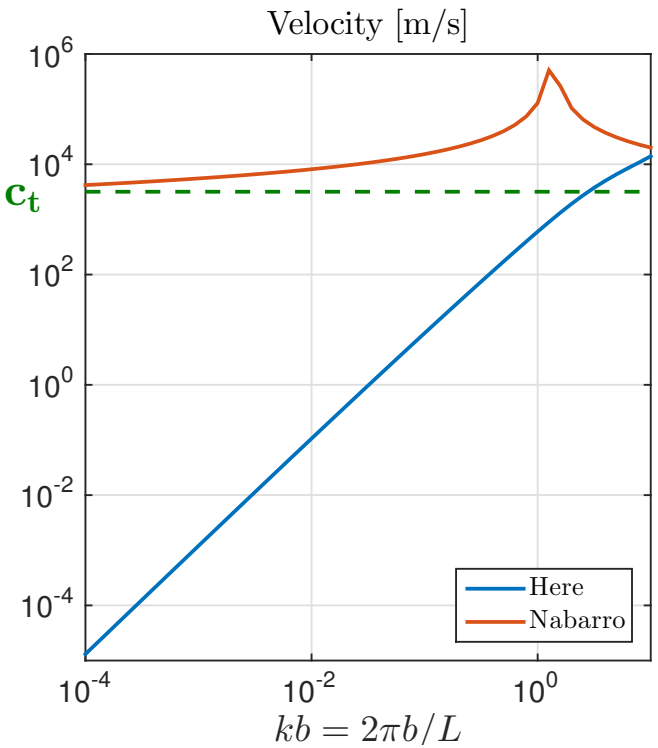
$$\chi = \frac{A}{B}$$

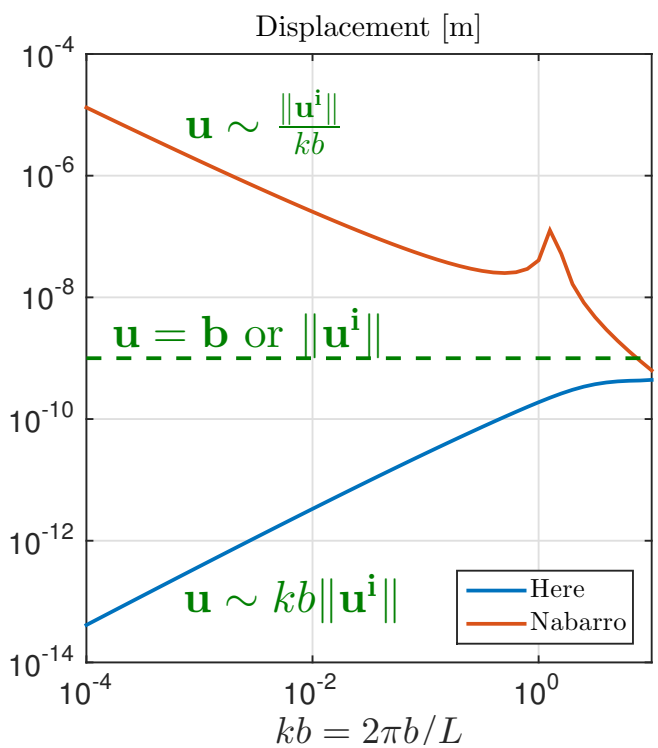
$\Rightarrow$ $A \ll B$ a large displacement produces a small scattering

$\Rightarrow$ Same qualitative phenomena

# In-plane displacement and velocity of the core

$$a = b = 1\,nm \qquad 10^{-4} < kb < 10 \qquad \|\mathbf{u}^{\mathbf{i}}\| = b \qquad c_t = 3177 m/s$$
$$50\,\mu m > L > 0.5\,nm$$




- D. Shilo, E. Zolotoyabko, Phys Rev Lett, 2003:
  dynamic regime, $v \sim 0.8 c_t = 2\,500\,m/s$, $(f = 500\,MHz)$
- T. Kruml *et al*, J. of Physics: Cond Matter, 2002:
  quasi-static regime, $v = 10^{-7} m/s$, $(10 - 50 MPa)$

# Comparison with literature

Dynamical models for interaction between wave and screw dislocation:

[1] FRN **Nabarro**, Proc R Soc A, 1951, K Lücke, AV Granato, Physical Review B, 1981, Murat 1963, Lund 1988, Maurel 2004

[1] The core **moves** in the material background

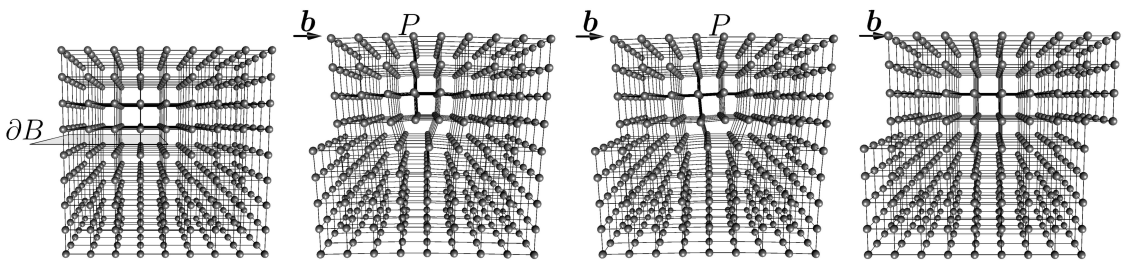
$\Rightarrow$ The wave energy is able to cut lattice

$\Rightarrow$ **High Energy**

*Here* the core **is fixed** in the material background

$\Rightarrow$ The wave is **not** able to cut lattice

$\Rightarrow$ **Low Energy**




$\Rightarrow$ Two dynamical regimes according to **defect density**

$\Rightarrow$ Two dynamical regimes according to **wave energy**

$\Rightarrow$ needs of **large transformations** and **geometrically exact** models !

# Conclusion

- A single ingredient have been introduced:

  **torsion of the material manifold**

- A large variety of phenomena have been observed in the dynamical problem :
  - Dirac-$\delta$-distribution looks not possible
  - Core size effect : two-regimes of propagation
  - PSV-SH coupling
  - Partial invisibility of the damaged zone.
- Analytical expansion are possible for any type of incoming wave.
- Some experimental validations would be welcome

LM L, Rakotomanana LR. *A Continuous Model for the Wave Scattering by a Bounded Defective Domain*, Mathematical Modelling in Solid Mechanics, 2017

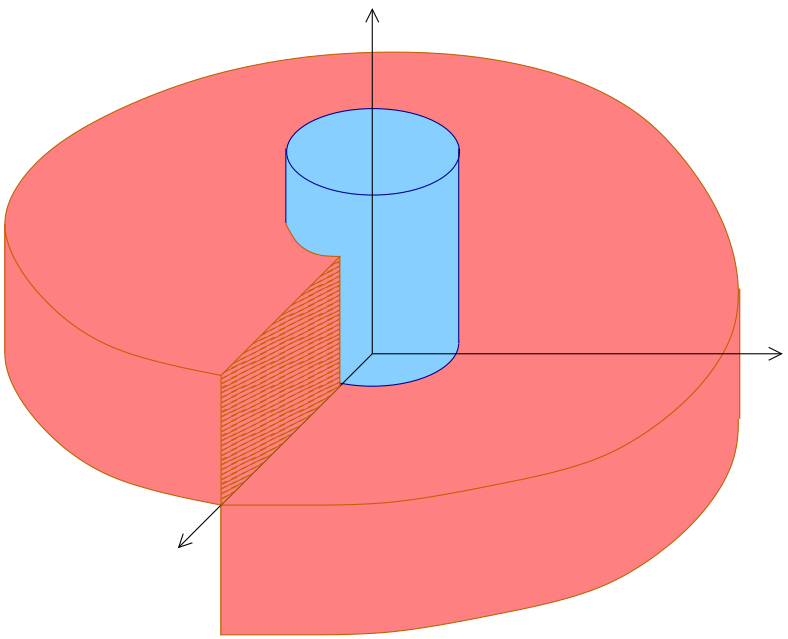
Futhazar G, LM L, Rakotomanana LR. *Covariant gradient continua applied to wave propagation within defective material*, Arch of Applied Mech, 2014

Thanks !

# Discussion

- In our work the dislocation core is fixed in the material background
- In Nabarro *et al* the dislocation core moves in the material background

However to see such effect we need a nanometric space-resolution with a GigaHertz time-resolution. . . what is quiet difficult !



- In our work the wave is a smooth field in the material background

$$[\mathbf{u}] = \mathbf{0}$$

- In Nabarro *et al* the full displacement supports the defect

$$[\mathbf{u}] = \mathbf{b}, \quad \left[\mathbf{u}^{\mathbf{s}} + \mathbf{u}^{\mathbf{i}}\right] = \mathbf{b}$$

but $\mathbf{u}^{\mathbf{i}}$ is a sinus-like field, then $\left[\mathbf{u}^{i}\right] = \mathbf{0}$. This implies $[\mathbf{u}^{\mathbf{s}}] = \mathbf{b}$

The dynamic perturbation, $\mathbf{u}^{\mathbf{s}}$, is higher if we consider wave-field lying in the Euclidian space.