Quelques malentendus et autres bévues tirés de la controverse autour du principe d'indifférence matérielle et de la théorie cinétique des gaz

Stories from the controversy on material frame indifference and the kinetic theory of gases

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THE MATHEMATICAL THEORY OF NON-UNIFORM GASES

AN ACCOUNT OF THE KINETIC THEORY OF VISCOSITY, THERMAL CONDUCTION AND DIFFUSION IN GASES

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[Chapman and Cowling, 1970]

CORRECTION OF TWO ERRORS IN THE KINETIC THEORY OF GASES WHICH HAVE BEEN USED TO CAST UNFOUNDED DOUBT UPON THE PRINCIPLE OF MATERIAL FRAME-INDIFFERENCE *

Clifford Truesdell**

He who regards the kinetic theory as providing the one and only right approach to gas flows should discard all of continuum mechanics, not just one or another part of it. Many physicists do so.

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MECCANICA [Truesdell, 1976]

CENTER FOR THE APPLICATION OF MATHEMATICS

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RED HERRINGS AND SUNDRY UNIDENTIFIED FISH IN NON-LINEAR CONTINUUM MECHANICS

by

R. S. RIVLIN

1. Introduction

The title of this talk is evidently provocative and represents a radical departure from other papers I have written. The modern resurgence of interest in the fundamentals of non-linear continuum theories and in the study of their implications dates very largely from the end of the Second

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[Rivlin, 1969]

Chapman and Cowling's notations

$\overline{\boldsymbol{w}}$ is the *conjugate* / transpose of second order tensor $\boldsymbol{w}$ p. 15

$\overline{\overline{\boldsymbol{w}}} = \frac{1}{2}(\boldsymbol{w} + \overline{\boldsymbol{w}})$ is the symmetric part of $\boldsymbol{w}$ p. 15

$\overset{\circ}{\boldsymbol{w}} = \boldsymbol{w} - \frac{1}{3}(w_{xx} + w_{yy} + w_{zz})\mathbf{U}$ is *non–divergent* / deviatoric p. 15

The gradient of a vector field is[1] $\boldsymbol{\nabla c} = \dfrac{\partial c_j}{\partial x_i}\mathbf{e}_i \otimes \mathbf{e}_j$ p. 18

The *rate-of-strain* and *rate-of-shear* tensors are

$$\boldsymbol{e} = \overline{\overline{\boldsymbol{\nabla c_0}}}, \quad \overset{\circ}{\boldsymbol{e}} = \overline{\overline{\boldsymbol{\nabla c_0}}}^{\circ}$$

where $\boldsymbol{c_0}$ is the velocity vector p. 19

One of the vectors in a dyadic may be a vector differential operator such as $\partial/\partial \boldsymbol{r}$ ($\equiv \nabla$). If, for example, $\boldsymbol{c}$ is a vector with components u, v, w, the components of $\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{c}$ ($\equiv \nabla \boldsymbol{c}$) are

$$\left.\begin{matrix} \frac{\partial u}{\partial x}, & \frac{\partial v}{\partial x}, & \frac{\partial w}{\partial x}, \\ \frac{\partial u}{\partial y}, & \frac{\partial v}{\partial y}, & \frac{\partial w}{\partial y}, \\ \frac{\partial u}{\partial z}, & \frac{\partial v}{\partial z}, & \frac{\partial w}{\partial z}, \end{matrix}\right\} \quad (1.33,\ 1) \quad \begin{matrix} w_{xx}, & w_{xy}, & w_{xz}, \\ w_{yx}, & w_{yy}, & w_{yz}, \\ w_{zx}, & w_{zy}, & w_{zz}, \end{matrix}$$

[1] Note that the transpose of this definition is frequently used.

Skew tensors

$\overset{\times}{\boldsymbol{w}} = \frac{1}{2}(\boldsymbol{w} - \overline{\boldsymbol{w}})$ is the skew–symmetric part of $\boldsymbol{w}$ p. 23

so that $\boldsymbol{w} = \overline{\overline{\boldsymbol{w}}} + \overset{\times}{\boldsymbol{w}}$ p. 23

The tensor $\overset{\times}{\boldsymbol{w}}$ has components[2]

$$\begin{array}{lll} 0, & \frac{1}{2}(w_{xy} - w_{yx}), & \frac{1}{2}(w_{xz} - w_{zx}), \\ \frac{1}{2}(w_{yx} - w_{xy}), & 0, & \frac{1}{2}(w_{yz} - w_{zy}), \\ \frac{1}{2}(w_{zx} - w_{xz}), & \frac{1}{2}(w_{zy} - w_{yz}), & 0. \end{array}$$

The vector $\overset{\times}{\omega}$ is associated with tensor $\overset{\times}{\boldsymbol{w}}$ with components[3]

$$\frac{1}{2}(w_{yz} - w_{zy}), \quad \frac{1}{2}(w_{zx} - w_{xz}), \quad \frac{1}{2}(w_{xy} - w_{yx})$$

and such that $\overset{\times}{\boldsymbol{w}} \cdot \boldsymbol{a} = \overline{\overline{\boldsymbol{w}}} \cdot \boldsymbol{a} + \boldsymbol{a} \wedge \overset{\times}{\omega}$ p. 24

[2] Note that this is the usual definition.

[3] The opposite vector is often used.

The method of solution of Boltzmann's equation

- Expansion of the *pressure tensor* and heat flux vector

$$\mathsf{p} = \sum_0^\infty \mathsf{p}^{(r)}, \quad \boldsymbol{q} = \sum_0^\infty \boldsymbol{q}^{(r)},$$

$$\mathsf{p}^{(r)} = \int m\boldsymbol{C}\boldsymbol{C}f^{(r)}\,d\boldsymbol{c}, \quad \boldsymbol{q}^{(r)} = \int E\boldsymbol{C}f^{(r)}\,d\boldsymbol{c};$$

p. 115

$\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{c}_0$ is the *peculiar velocity* p. 27

- Zero$^{\text{th}}$ order solution (first approximation)

$$\mathsf{p}^{(0)} = knT\mathsf{U} = \mathsf{U}p,$$

$\boldsymbol{q}^{(0)} = 0$ p. 115

p is the hydrostatic pressure

- First order solution (second approximation)

$\boldsymbol{p}^{(1)} = -2\mu\,\mathring{\boldsymbol{e}} = -2\mu\,\mathring{\overline{\overline{\boldsymbol{\nabla}\boldsymbol{c}_0}}}$ p. 126

$\boldsymbol{q}^{(1)} = -\lambda\boldsymbol{\nabla}T$ p. 125

the existence of *volume viscosity* can also be derived p. 408

- Viscosity and thermal conduction in a simple gas

$\lambda = \frac{5}{2}C_v\mu$, mean free path time $\tau = \mu/p$ p. 160

The third approximation (Burnett)

$$\mathbf{p}^{(2)} = \varpi_1 \frac{\mu^2}{p} \Delta \overset{0}{\mathbf{e}} + \varpi_2 \frac{\mu^2}{p} \left(\frac{D_0}{Dt} (\overset{0}{\mathbf{e}}) - 2 \overset{\circ}{\overline{\nabla \mathbf{c}_0 . \overset{0}{\mathbf{e}}}} \right) + \varpi_3 \frac{\mu^2}{\rho T} \overset{\circ}{\overline{\nabla \nabla T}}$$

$$+ \varpi_4 \frac{\mu^2}{\rho p T} \overset{\circ}{\overline{\nabla p \nabla T}} + \varpi_5 \frac{\mu^2}{\rho T^2} \overset{\circ}{\overline{\nabla T \nabla T}} + \varpi_6 \frac{\mu^2}{p} \overset{\circ}{\overline{\overset{0}{\mathbf{e}} . \overset{0}{\mathbf{e}}}},$$

p. 286

$\boldsymbol{p}^{(2)}$ is *both symmetrical and non–divergent*[4].

The ϖ_i are pure numbers with approximate values

$\varpi_1 = \frac{4}{3} \left(\frac{7}{2} - \frac{T}{\mu} \frac{d\mu}{dT} \right), \quad \varpi_2 = 2, \quad \varpi_3 = 3, \quad \varpi_4 = 0$

$\varpi_5 = \frac{3T}{\mu} \frac{d\mu}{dT}, \quad \varpi_6 = 8$

D/Dt is the *mobile operator, time–derivative following the motion* p. 48

$\frac{D_0}{Dt} = \frac{\partial_0}{\partial t} + \boldsymbol{c}_0 \cdot \frac{\partial}{\partial \boldsymbol{r}}$ denotes a first approximation to D/Dt p. 116

$\boldsymbol{c}_0$ is the *mean molecular velocity* p. 27

[Burnett, 1935]

[4] The notation $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{c}_0 = \mathrm{Tr}\, \boldsymbol{e}$ is introduced on p.284

The third approximation (Burnett)

$$q^{(2)} = \theta_1 \frac{\mu^2}{\rho T} \Delta \nabla T + \theta_2 \frac{\mu^2}{\rho T} \left\{ \frac{D_0}{Dt} (\nabla T) - (\nabla c_0) . \nabla T \right\}$$

$$+ \theta_3 \frac{\mu^2}{\rho p} \nabla p . \hat{\mathbf{e}} + \theta_4 \frac{\mu^2}{\rho} \nabla . (\hat{\mathbf{e}}) + \theta_5 \frac{3\mu^2}{\rho T} \nabla T . \hat{\mathbf{e}},$$

p. 285

The θ_i are pure numbers with approximate values

$$\theta_1 = \frac{15}{4} \left(\frac{7}{2} - \frac{T}{\mu} \frac{d\mu}{dT} \right), \quad \theta_2 = \frac{45}{8}, \quad \theta_3 = -3, \quad \theta_4 = 3$$

$$\theta_5 = \frac{35}{4} + \frac{T}{\mu} \frac{d\mu}{dT}$$

The fourth approximation (super–Burnett)

No exhaustive form of the super–Burnett terms is provided in the literature, apparently and unfortunately.

[Struchtrup, 2005]

Examen de la loi de comportement mécanique

Avec les notations désormais classiques[5],

$$\begin{aligned}\boldsymbol{\sigma} &= -p\mathbf{1} + 2\mu \boldsymbol{D}^{\mathrm{dev}} + \boldsymbol{\sigma}_2^{\mathrm{dev}} \\ \boldsymbol{\sigma}_2 &= -\varpi_1 \frac{\mu^2}{p} (\mathrm{trace}\, \boldsymbol{D})\, \boldsymbol{D} - 8\frac{\mu^2}{p} \boldsymbol{D}^{\mathrm{dev}} \cdot \boldsymbol{D}^{\mathrm{dev}} \\ &- 2\frac{\mu^2}{p} \left(\dot{\boldsymbol{D}} - \boldsymbol{L}^T \cdot \boldsymbol{D}^{\mathrm{dev}} - \boldsymbol{D}^{\mathrm{dev}} \cdot \boldsymbol{L} \right)\end{aligned}$$

où $L_{ij} = v_{i,j} = D_{ij} + W_{ij}$
La dernière contribution se développe de la manière suivante

$$\dot{\boldsymbol{D}} - \boldsymbol{L}^T \cdot \boldsymbol{D}^{\mathrm{dev}} - \boldsymbol{D}^{\mathrm{dev}} \cdot \boldsymbol{L} = \dot{\boldsymbol{D}} - \boldsymbol{L}^T \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{L} + \frac{2}{3} (\mathrm{trace}\, \boldsymbol{D})\, \boldsymbol{D}$$

Noter la dérivée temporelle similaire intervenant dans l'expression du flux :

$$\dot{\boldsymbol{g}} - \boldsymbol{L}^T \cdot \boldsymbol{g}, \quad \text{avec} \quad \boldsymbol{g} = \boldsymbol{\nabla} T$$

[5] *C&C term $\boldsymbol{L}^T$ the velocity gradient* [Murdoch, 1983], note 5, p. 188

Examen de la loi de comportement mécanique

La contribution $\dot{\boldsymbol{D}} - \boldsymbol{L}^T \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{L}$ peut être comparée avec

- les *dérivées convectives*

$$
\begin{aligned}
\overset{\triangledown}{\boldsymbol{A}}_{..} &= \dot{\boldsymbol{A}}_{..} + \boldsymbol{L}^T \boldsymbol{A}_{..} + \boldsymbol{A}_{..} \boldsymbol{L} \quad \text{Cotter} - \text{Rivlin} \\
\overset{\triangledown}{\boldsymbol{A}}_{.}^{.} &= \dot{\boldsymbol{A}}_{.}^{.} + \boldsymbol{L}^T \boldsymbol{A}_{.}^{.} - \boldsymbol{A}_{.}^{.} \boldsymbol{L}^T \\
\overset{\triangledown}{\boldsymbol{A}}_{.}^{.} &= \dot{\boldsymbol{A}}_{.}^{.} - \boldsymbol{L} \boldsymbol{A}_{.}^{.} + \boldsymbol{A}_{.}^{.} \boldsymbol{L} \\
\overset{\triangledown}{\boldsymbol{A}}^{..} &= \dot{\boldsymbol{A}}^{..} - \boldsymbol{L} \boldsymbol{A}^{..} - \boldsymbol{A}^{..} \boldsymbol{L}^T \quad \text{Oldroyd ou Lie}
\end{aligned}
$$

  Les dérivées convectives d'un champ de vecteur sont

$$
\overset{\triangledown}{\boldsymbol{g}}_{.} = \dot{\boldsymbol{g}}_{.} + \boldsymbol{L}^T \boldsymbol{g}_{.}, \quad \overset{\triangledown}{\boldsymbol{g}}^{.} = \dot{\boldsymbol{g}}^{.} - \boldsymbol{L} \boldsymbol{g}^{.}
$$

- Dérivée de Truesdell $\overset{\triangledown}{\boldsymbol{A}}^{..} = \dot{\boldsymbol{A}}^{..} - \boldsymbol{L} \boldsymbol{A}^{..} - \boldsymbol{A}^{..} \boldsymbol{L}^T + (\operatorname{trace} \boldsymbol{L}) \boldsymbol{A}^{..}$
- les *tenseurs de Rivlin–Ericksen* [Truesdell, 1991] p. 123

$$
\boldsymbol{A}_1 = 2\boldsymbol{D}, \quad \boldsymbol{A}_{n+1} = \dot{\boldsymbol{A}}_n + \boldsymbol{A}_n \boldsymbol{L} + (\boldsymbol{A}_n \boldsymbol{L})^T
$$

  En particulier, $\boldsymbol{A}_2/2 = \dot{\boldsymbol{D}} + \boldsymbol{L}^T \boldsymbol{D} + \boldsymbol{D}\boldsymbol{L}$

Examen de la loi de comportement mécanique

Le terme du second ordre de la théorie cinétique des gaz peut s'écrire sous la forme :

$$
\begin{aligned}
-\frac{p}{2\mu^2} &= \frac{\varpi_1}{2}(\mathrm{trace}\,\boldsymbol{D})\boldsymbol{D} + \dot{\boldsymbol{D}} - \boldsymbol{L}^T\boldsymbol{D}^{\mathrm{dev}} - \boldsymbol{D}^{\mathrm{dev}}\boldsymbol{L} + 4\boldsymbol{D}^{\mathrm{dev}}\boldsymbol{D}^{\mathrm{dev}} \\
&= (\frac{\varpi_1}{2} + \frac{2}{3} - \frac{8}{3})(\mathrm{trace}\,\boldsymbol{D})\boldsymbol{D} + 4\boldsymbol{D}\boldsymbol{D} + \dot{\boldsymbol{D}} - \boldsymbol{L}^T\boldsymbol{D} - \boldsymbol{D}\boldsymbol{L} \\
&= (\frac{7}{3} - \frac{6}{3} - \frac{2}{3}s)(\mathrm{trace}\,\boldsymbol{D})\boldsymbol{D} + 4\boldsymbol{D}\boldsymbol{D} \\
&+ \overset{\triangledown}{\boldsymbol{D}}^{Truesdell} + \boldsymbol{L}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{L}^T - \boldsymbol{L}^T\boldsymbol{D} - \boldsymbol{D}\boldsymbol{L} - (\mathrm{trace}\,\boldsymbol{D})\boldsymbol{D} \\
&= (\frac{1}{3} - \frac{2}{3}s - 1)(\mathrm{trace}\,\boldsymbol{D})\boldsymbol{D} + 4\boldsymbol{D}\boldsymbol{D} + \overset{\triangledown}{\boldsymbol{D}}^{Truesdell} + 2(\boldsymbol{W}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W}) \\
&= \frac{2}{3}(1-s)(\mathrm{trace}\,\boldsymbol{D})\boldsymbol{D} + 4\boldsymbol{D}\boldsymbol{D} + \overset{\triangledown}{\boldsymbol{D}}^{Truesdell} + 2(\boldsymbol{W}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W})
\end{aligned}
$$

Examen de la loi de comportement mécanique

La loi de comportement issue de la théorie cinétique des gaz

- satisfait au *principe d'invariance galiléenne*, i.e. elle est invariante dans la transformation

$$\boldsymbol{\sigma} \longrightarrow \boldsymbol{Q}_0 \boldsymbol{\sigma} \boldsymbol{Q}^{\boldsymbol{T}}{}_0, \quad \boldsymbol{L} \longrightarrow \boldsymbol{Q}_0 \boldsymbol{L} \boldsymbol{Q}_0^T, \quad \boldsymbol{D} \longrightarrow \boldsymbol{Q}_0 \boldsymbol{D} \boldsymbol{Q}_0^T$$

- ne satisfait pas au *principe d'indifférence matérielle*, i.e. elle n'est pas invariante dans la transformation euclienne

$$\boldsymbol{\sigma} \longrightarrow \boldsymbol{Q}(t) \boldsymbol{\sigma} \boldsymbol{Q}^{\boldsymbol{T}}(t), \quad \boldsymbol{L} \longrightarrow \boldsymbol{Q}(t) \boldsymbol{L} \boldsymbol{Q}^{\boldsymbol{T}}(t) + \dot{\boldsymbol{Q}} \boldsymbol{Q}^{\boldsymbol{T}}, \quad \boldsymbol{D} \longrightarrow \boldsymbol{Q}(t) \boldsymbol{D} \boldsymbol{Q}^{\boldsymbol{T}}(t)$$

à cause de la contribution $\dot{\boldsymbol{D}} - \boldsymbol{L}^{\boldsymbol{T}} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{L}$ qui se transforme en

$$\boldsymbol{Q}(\dot{\boldsymbol{D}} - \boldsymbol{L}^{\boldsymbol{T}} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{L} + 2\boldsymbol{Q}^{\boldsymbol{T}} \dot{\boldsymbol{Q}} \boldsymbol{D} + 2\boldsymbol{D} \dot{\boldsymbol{Q}}^{\boldsymbol{T}} \boldsymbol{Q}) \boldsymbol{Q}^{\boldsymbol{T}}$$

The bogus axioms of continuum mechanics by [Woods, 1981]

"The **axiom of frame–indifference (material objectivity)** *claims that the response of a thermodynamic system to imposed thermodynamic forces is independent of the motion of the observer O, and* **so far as Galilean transformations of O is concerned**, *this is certainly true. But the axiom is applied confidently to rotating frames, where some physical considerations would surely provide a caution."*
$\longrightarrow$ Müller's argument and impact of Coriolis forces on particles [Müller, 1972]

Woods ne dit explicitement à aucun moment que la loi de comportement issue de la théorie cinétique des gaz n'a de sens que par rapport au référentiel galiléen[6]

[6] C&C, *The translational motions of the individual molecules may be specified by their 'actual' velocities* $\boldsymbol{c}$ *(i.e. their velocities relative to* **some standard frame of reference**). p. 27.

A note on "Axioms of continuum mechanics" by [Green, 1982]

Si, comme A.E. Green, on méconnaît le fait que la définition du gradient d'un champ de vecteur diffère chez Chapman, Cowling et Woods,

$$(\boldsymbol{\nabla} \boldsymbol{c})_{ij} = c_{j,i}$$

et chez Green[7] et Ogden[8] $\qquad L_{ij} = v_{i,j}$

alors, on est enclin à s'étonner du débat faisant rage dans les années 1970 et 1980 en constatant que *"We now see that the equation for the heat flux* **is** *frame indifferent! (as is also the expression of the pressure tensor)."*

C'est une bévue de Green, qui fait jubiler Woods dans

More on the bogus axioms of continuum mechanics by [Woods, 1982]

[7] Voir par exemple la relation (1.3.1) dans [Green and Atkins, 1970], Green utilisant dans son œuvre exclusivement la notation indicielle.

[8] Voir la relation p. 123 dans [Ogden, 1997], et (2.3) p. 4 dans [Fu and Ogden, 2000].

Frame–indifferent kinetic theory by [Woods, 1983]

L'affaire ne s'arrête pas là et, de manière surprenante, Woods se convertit au principe d'indifférence matérielle, après l'avoir pourfendu, et sombre en affirmant :

The second–order transport terms in a monatomic gaz have the curious property of being dependent on the observer's reference frame. However, frame–indifference has long been accepted as an essential property of constitutive equations in continuum mechanics.

The generally accepted Chapman–Enskog theory has the defect of being frame–dependent, with the result that the expressions it gives for pressure $\boldsymbol{p}$ *and the heat–flux vector* $\boldsymbol{q}$ *depend on the choice of the reference frame in which molecular velocities are specified.*

Under normal conditions the second–order terms $\boldsymbol{p}^{(2)}$ *and* $\boldsymbol{q}^{(2)}$ *are only small corrections to the frame–indifferent contributions* $\boldsymbol{q}^{(1)}$ *and* $\boldsymbol{q}^{(1)}$*, so that their frame–dependence appears to be no more than a curious anomaly, without pratical significance.*

This is in contradiction with the fact that the kinetic theory is intrinsically formulated with respect to a Galilean observer.

Frame–indifferent kinetic theory by [Woods, 1983]

Woods proposes to consider the *peculiar velocity* with respect to a frame that is *convected in both position and orientation with the fluid particle*. He arrives at the following modified Burnett contribution :

$$\begin{aligned}\boldsymbol{\sigma}_2^{Woods} &= -p\tau^2\left(\varpi_1^W(\operatorname{trace}\boldsymbol{D})\boldsymbol{D} + 4\boldsymbol{D}^{\mathrm{dev}}\boldsymbol{D}^{\mathrm{dev}} + \dot{\boldsymbol{D}} + \boldsymbol{D}^{\mathrm{dev}}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{D}^{\mathrm{dev}}\right)\\ &= -p\tau^2\left(\varpi_1^W(\operatorname{trace}\boldsymbol{D})\boldsymbol{D} + 4\boldsymbol{D}^{\mathrm{dev}}\boldsymbol{D}^{\mathrm{dev}} + \dot{\boldsymbol{D}} + \boldsymbol{D}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{D}\right)\end{aligned}$$

avec $\varpi_1^W = \frac{4}{3}(3-s)$ contre $\varpi_1 = \frac{4}{3}(\frac{7}{2}-s)$ chez C&C.
La dernière contribution fait intervenir la **dérivée objective de Jaumann** et change donc complètement de nature[9] par rapport à la théorie de C&C[10].
La tentative précédente se double d'une erreur bénigne de calcul que révèle la différence

$$\frac{p}{\mu^2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2^{Woods}) = (\varpi_1^W - \varpi_1 + \frac{4}{3}) + 2(\boldsymbol{D}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{D})$$

où le premier terme s'annule[11] si $\varpi_1^W = \varpi_1 - \frac{4}{3} = \frac{4}{3}(\frac{5}{2}-s)$.

[9] La conversion : *Professor A.E. Green and I recently debated several axioms of continuum mechanics, an important one being MFI. I am pleased to acknowledge that for this axiom his convictions were sound, whereas my confidence in the opposing evidence from kinetic theory etc. was misplaced.*

[10] En bref, Woods substitue $\dot{\boldsymbol{D}} + \boldsymbol{D}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{D}$ à $\dot{\boldsymbol{D}} + \boldsymbol{W}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W}$ présent dans $\boldsymbol{\sigma}_2$.

[11] L'erreur de Woods se situe dans l'équation (3.11), ce qui précède étant exact.

Intrinsic spin as a constitutive variable by [Murdoch, 1983]

The appearance in the previous relations of the spin $\boldsymbol{W}$ *has led a number of authors to conclude that these, regarded as constitutive equations, violate the principle of material frame–indifference. However, the above equations pertain to the stress and heat flux only in* **inertial frames**. *This becomes clear when comparison is made with the analysis of [Müller, 1972], who derived approximative expressions for* $\boldsymbol{\sigma}$ *and* $\boldsymbol{q}$ *in a general frame.*

Considérons la loi de comportement $\boldsymbol{\sigma}^{gal} \propto \dot{\boldsymbol{D}}^{gal} + \boldsymbol{W}^{gal}\boldsymbol{D}^{gal} - \boldsymbol{D}^{gal}\boldsymbol{W}^{gal}$, formulée dans un référentiel *gal*iléen ϕ^{gal} *(and hence any)*.
Soit $\boldsymbol{Q}$ la transformation orthogonale permettant de passer de ϕ^{gal} à un référentiel quelconque ϕ, et $\boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T$, le taux de rotation associé :

$$\boldsymbol{D} = \boldsymbol{Q}\boldsymbol{D}^{gal}\boldsymbol{Q}^T, \quad \boldsymbol{W} = \boldsymbol{Q}\boldsymbol{W}^{gal}\boldsymbol{Q}^T + \boldsymbol{\Omega}$$

Loi de comportement dans ϕ s'écrit :

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{Q}\boldsymbol{\sigma}^{gal}\boldsymbol{Q}^T \propto \dot{\boldsymbol{D}} + \boldsymbol{W}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W} + 2\boldsymbol{\Omega}\boldsymbol{D} - 2\boldsymbol{D}\boldsymbol{\Omega} \\ &\propto \dot{\boldsymbol{D}} - \boldsymbol{W}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{W} + 2\boldsymbol{W}^{int}\boldsymbol{D} - 2\boldsymbol{D}\boldsymbol{W}^{int} \end{aligned}$$

où $\boldsymbol{W}^{int} = \boldsymbol{W} - \boldsymbol{\Omega}$ *is the intrinsic spin*, which is nothing but the Galilean spin tensor: $\boldsymbol{W}^{int} = \boldsymbol{Q}\boldsymbol{W}^{gal}\boldsymbol{Q}^T$.

Intrinsic spin as a constitutive variable by [Murdoch, 1983]

La quantité $\boldsymbol{W}^{int}$ est *objective*.
En effet, si l'on passe de ϕ^{gal} à un autre référentiel quelconque ϕ' par $\boldsymbol{Q'}$, et de ϕ à ϕ' par $\boldsymbol{Q''} = \boldsymbol{Q'}\boldsymbol{Q}^T$,

$$\begin{aligned} \boldsymbol{W}^{int\prime} &= \boldsymbol{W}' - \dot{\boldsymbol{Q}}'\boldsymbol{Q}'^T \\ &= \boldsymbol{Q}''\boldsymbol{W}\boldsymbol{Q}''^T + \dot{\boldsymbol{Q}}''\boldsymbol{Q}''^T - \dot{\boldsymbol{Q}}'\boldsymbol{Q}'^T \\ &= \boldsymbol{Q}''(\boldsymbol{W} - \dot{\boldsymbol{Q}}\boldsymbol{Q}^T)\boldsymbol{Q}''^T = \boldsymbol{Q}''\boldsymbol{W}^{int}\boldsymbol{Q}''^T \end{aligned}$$

Objectivity vs. frame–indifference

- *On n'engendre pas de contraintes dans un corps en tournant autour!*[12]
  Cela va de soi mais la loi ci–dessous ne satisfait pas à cette exigence élémentaire :
  $$\boldsymbol{\sigma} = 2\mu \boldsymbol{D} + \alpha(\boldsymbol{W}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W})$$
  si elle est stipulée dans un référentiel quelconque.
  This relation is not objective.
- La loi précédente, formulée par rapport à un référentiel galiléen
  $$\boldsymbol{\sigma}^{gal} = 2\mu \boldsymbol{D} + \alpha(\boldsymbol{W}^{gal}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W}^{gal})$$
  ne souffre pas du travers précédent.
  Elle prend la forme $\boldsymbol{\sigma} = 2\mu \boldsymbol{D} + \alpha((\boldsymbol{W} - \boldsymbol{\Omega})\boldsymbol{D} - \boldsymbol{D}(\boldsymbol{W} - \boldsymbol{\Omega}))$ par rapport à un référentiel quelconque
  This relation is objective but not frame–indifferent[13] dans la mesure où elle fait appel au référentiel galiléen.
  Un telle loi est exclue par MFI.

[12] C'est à cette exigence que correspondent les conditions (1.5) dans [Woods, 1983], affirmant **l'objectivité des contraintes et du flux de chaleur**, propriétés admises (mais approchées!).

[13] not Euclidean invariant

Comments on the "material frame–indifference" controversy by [Speziale, 1987]

The debate over the validity of the principle of material frame indiference of modern continuum mechanics is examined from a more rigorous theoretical standpoint based on statistical mechanics.
Constitutive equations can only represent special solutions of the microscopic dynamics and, as such, can have a larger invariance group than the Galilean group. While there are reasons to question the general validity of material frame indifference, **these doubts arise exclusively from the presence of unbalanced Coriolis forces** *whose effect becomes vanishingly small in the classical continuum limit.*

Turbulence et loi incluant le taux de rotation

motivation de [Razafindralandy and Hamdouni, 2006]

Lund and Novikov include the filtered vorticity tensor $\overline{\mathbf{W}} = (\nabla\overline{\boldsymbol{u}} - {}^T\nabla\overline{\boldsymbol{u}})$ in the expression of the subgrid model. Cayley–Hamilton theorem gives then the *Lund–Novikov model* (see [22]):

$$
\begin{aligned}
-\mathbf{T}_s^d = {} & C_1\overline{\delta}^2|\overline{\mathbf{S}}|\overline{\mathbf{S}} + C_2\overline{\delta}^2(\overline{\mathbf{S}}^2)^d + C_3\overline{\delta}^2(\overline{\mathbf{W}}^2)^d \\
& + C_4\overline{\delta}^2(\overline{\mathbf{S}}\,\overline{\mathbf{W}} - \overline{\mathbf{W}}\,\overline{\mathbf{S}}) + C_5\overline{\delta}^2\frac{1}{|\overline{\mathbf{S}}|}(\overline{\mathbf{S}}^2\overline{\mathbf{W}} - \overline{\mathbf{S}}\,\overline{\mathbf{W}}^2),
\end{aligned}
\tag{9}
$$

where the coefficients C_i depend on the invariants obtained from $\overline{\mathbf{S}}$ and $\overline{\mathbf{W}}$. The expression of these coefficients are so complex that they are considered as constants and evaluated with statistic techniques.

Approche phénoménologique

formulation de la loi de comportement[14] en partant du **fluide homogène de degré** n selon [Truesdell and Euvrard, 1974], p. 217 (fluide de Rivlin–Ericksen), et en ajoutant $\boldsymbol{W}$.
application à $n = 2$

$$\boldsymbol{\sigma} = \lambda(\operatorname{trace} \boldsymbol{D})\mathbf{1} + 2\mu\boldsymbol{D} + \alpha_1\boldsymbol{D}^2 + \alpha_2(\boldsymbol{D}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{D}) + \alpha_3\boldsymbol{W}^2 + \alpha_4\dot{\boldsymbol{D}}$$

Les termes en rouge sont présents dans la loi d'ordre 2 de la théorie cinétique des gaz. Cette dernière a le mérite de fournir les expressions des paramètres $\mu, \alpha_1, \alpha_2, \alpha_4$.

[14] par rapport au référentiel galiléen.

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La positivité de la dissipation $\boldsymbol{\sigma} : \boldsymbol{D} \geq 0$ exige que $\alpha_3 = 0$.

[14] par rapport au référentiel galiléen.

Approche phénoménologique

La formulation des lois de comportement doit respecter les 3 principes suivants :

- Covariance (tensorialité) des relations à exploiter grâce aux théorèmes disponibles, notamment [Boehler, 1987, Zheng, 1994]
- Invariance galiléenne (faire fi de MFI!) (groupe galiléen étendu, au sens du principe d'équivalence d'Einstein)
- Second principe de la thermodynamique

Les lois satisfaisant à MFI constituent une classe particulière de comportement de la matière, en général suffisante pour l'ingénieur, mais pas toujours (turbulence en particulier).


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