Dynamique de Dirac pour les problemes mecaniques

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Generalized geometry for modelling in mechanics

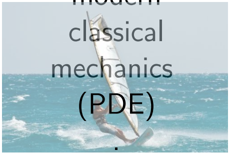
- Some history and motivation
  arXiv 2007.11081
- Dirac dynamics
  - Reminder about Dirac structures
  - Lie algebroids and cohomology
  - Variational approach to Dirac dynamics
    arXiv 2109.00313
- Applications
  - Constraint systems (slightly) revisited
  - Impicit Lagrangian systems
  - Structure preserving numerics

(some in progress, j/w Oscar Cosserat, Aziz Hamdouni, Camille Laurent-Gengoux, Daria Loziienko, Alexei Kotov, Leonid Ryvkin)

In the previous episodes...

Philosophy:

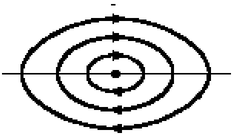
Geometry encodes the physics of the system

	Mechanical property	Geometric description
 classical classical mechanics (ODE)	conservation of energy	Poisson / symplectic
	symmetries	Lie groups/algebras, Cartan moving frames
	dissipation / interaction power balance; constraints	(almost) Dirac
	control	(singular) foliations
 modern classical mechanics (PDE) .	conservation of energy	multisymplectic
	symmetries	Cartan moving frames
	dissipation / interaction	Stokes–Dirac
	$rot(grad) = 0$, $div(rot) = 0$	$d^2 = 0$ – DEC
	control	foliations

Preserving this geometry in computations is fruitful

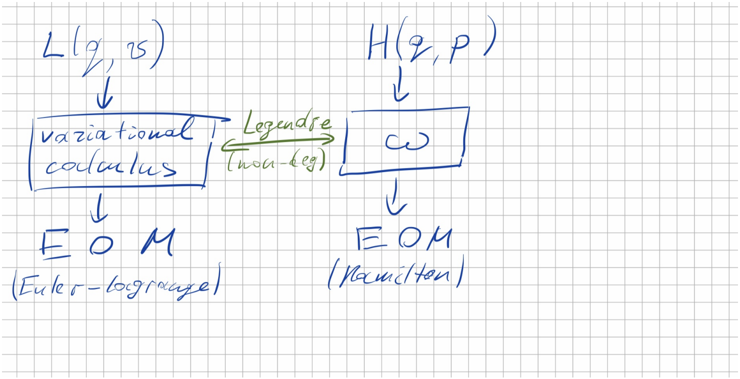
Very classical story

Verlet '67
Yoshida '80s

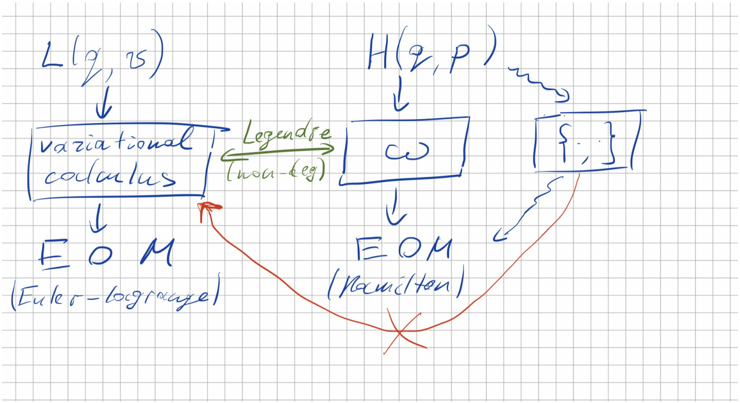
Canonical case: given $H\colon T^*Q \to \mathbb{R}$ $\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$	Symplectic geometry $\omega = \sum_i dp_i \wedge dq^i$ $\iota_{X_H}\omega = \mathrm{d}H$	
More general case: given $H\colon M \to \mathbb{R}$ and an antisymmetric $J(\mathbf{x})$ $\dot{\mathbf{x}} = J(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}}$	Poisson geometry $\{\cdot,\cdot\}$ on : $C^\infty(M)$ $X_H = \{H, \cdot\}$ $\dot{\mathbf{x}} = \{H, \mathbf{x}\}$	

Oscar Cosserat, 2022

Classical story in modern language

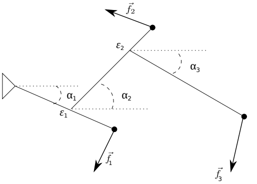


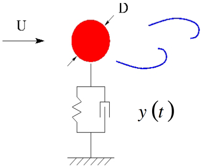
Classical story in modern language

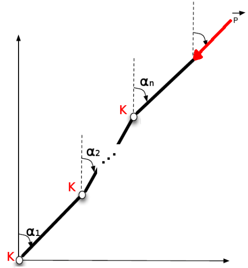


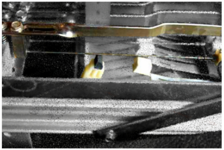
Beyond: port-Hamiltonian, constraint Lagrangian

Traditional references: J. Marsden, H. Yoshimura (ILS);
A. van der Schaft, B. Maschke (PHS)



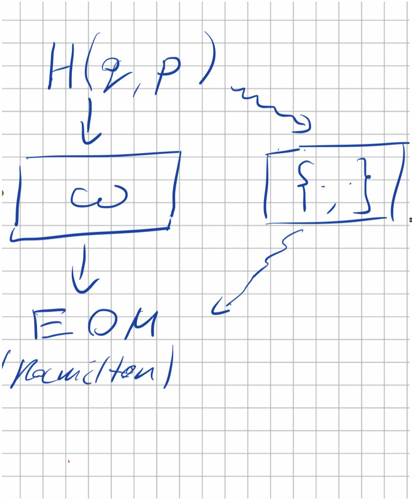






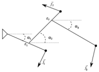
Conjecture (VS): Everything is port-Hamiltonian.
Geometry: (almost) Dirac structures for both classes

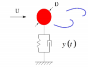
What is the conceptual difference?

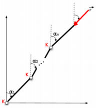


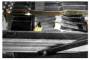
Beyond: port-Hamiltonian systems; constraints

A lot of examples
(ask Antoine Falaize, David Roze) :







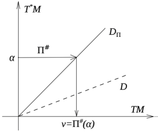


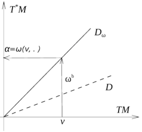
Conjecture (VS): Everything is port-Hamiltonian.

Geometry behind: Courant algebroids, Dirac structures

On $E = TM \oplus T^*M$ (or more generally $F \oplus F^*$)
Symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
Dorfman bracket: $[v \oplus \eta, v' \oplus \eta']_D = [v, v']_{\mathrm{Lie}} \oplus (\mathcal{L}_v \eta' - \mathrm{d}\eta(v'))$.

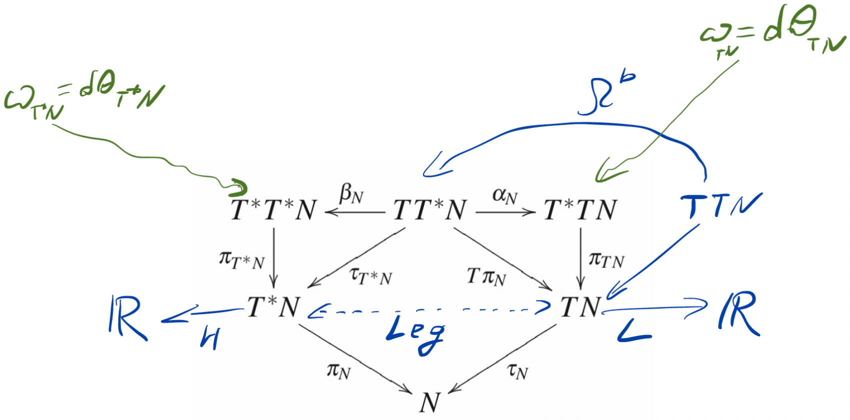
A *Dirac structure* $\mathcal{D}$ is a maximally isotropic (Lagrangian) subbundle of E closed w.r.t. $[\cdot,\cdot]_D$





$\mathcal{D}_\Pi = graph(\Pi^\sharp) = \{(\Pi^\sharp\alpha, \alpha)\}$ $\quad$ $\mathcal{D}_\omega = graph(\omega^\flat) = \{(v, \iota_v\omega)\}$

Classical story revisited (Tulczyjew)



Classical story revisited (Tulczyjew)

4.3.5 Theorem (W.M. Tulczyjew)**.** *With the notations specified above (4.3.4), let* $X_H : T^*N \to TT^*N$ *be the Hamiltonian vector field on the symplectic manifold* $(T^*N, \mathrm{d}\theta_N)$ *associated to the Hamiltonian* $H : T^*N \to \mathbb{R}$*, defined by* $\mathrm{i}(X_H)\mathrm{d}\theta_N = -\mathrm{d}H$*. Then*

$$X_H(T^*N) = \beta_N^{-1}\big(\mathrm{d}H(T^*N)\big) .$$

Moreover, the equality

$$\alpha_N^{-1}\big(\mathrm{d}L(TN)\big) = \beta_N^{-1}\big(\mathrm{d}H(T^*N)\big)$$

holds if and only if the Lagrangian L is hyper-regular and such that

$$\mathrm{d}H = \mathrm{d}\big(E_L \circ \mathcal{L}_L^{-1}\big) ,$$

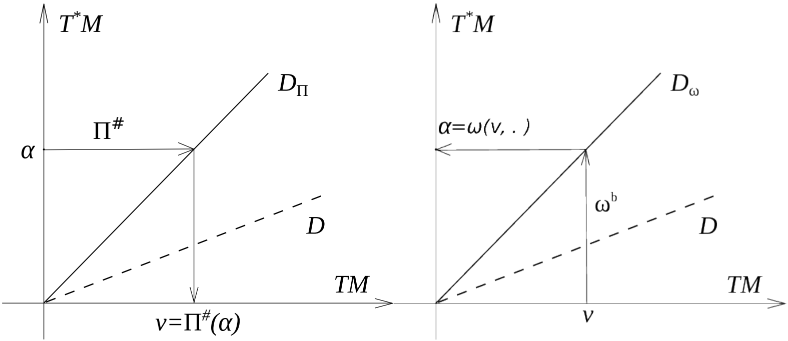
where $\mathcal{L}_L : TN \to T^*N$ *is the Legendre map and* $E_L : TN \to \mathbb{R}$ *the energy associated to the Lagrangian L.*

Season 2, Episode 1: What is Dirac dynamics? Main ingredients.

Geometry behind: Courant algebroids, Dirac structures

On $\mathbb{T}M = TM \oplus T^*M$ (or more generally $E \oplus E^*$)
Symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
Dorfman bracket: $[v \oplus \eta, v' \oplus \eta']_D = [v, v']_{\mathrm{Lie}} \oplus (\mathcal{L}_v \eta' - \mathrm{d}\eta(v'))$.

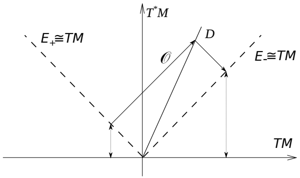
A *Dirac structure* $\mathcal{D}$ is a maximally isotropic (Lagrangian) subbundle of $\mathbb{T}M$ closed w.r.t. $[\cdot, \cdot]_D$



$\mathcal{D}_\Pi = graph(\Pi^\sharp) = \{(\Pi^\sharp \alpha, \alpha)\}$ $\qquad$ $\mathcal{D}_\omega = graph(\omega^\flat) = \{(v, \iota_v \omega)\}$

Dirac structures: general

Choose a metric on $M \Rightarrow TM \oplus T^*M \cong TM \oplus TM$,
Introduce the eigenvalue subbundles $E_\pm = \{v \oplus \pm v\}$
of the involution $(v, \alpha) \mapsto (\alpha, v)$. Clearly, $E_+ \cong E_- \cong TM$.



<u>(Almost) Dirac structure</u> – graph of an orthogonal operator $\mathcal{O} \in \Gamma(\mathrm{End}(TM))$:
$(v, \alpha) = ((\mathrm{id} - \mathcal{O})w, g((\mathrm{id} + \mathcal{O})w, \cdot))$
<u>Dirac structure</u> = almost Dirac + (Jacobi-type) integrability condition:

$$g\left(\mathcal{O}^{-1}\nabla_{(\mathrm{id}-\mathcal{O})\xi_1}(\mathcal{O})\xi_2, \xi_3\right) + cycl(1,2,3) = 0$$

Remark. If the operator $(\mathrm{id} + \mathcal{O})$ is invertible, one recovers D_Π with $\Pi = \frac{\mathrm{id} - \mathcal{O}}{\mathrm{id} + \mathcal{O}}$ (Cayley transform), integrability $\Leftrightarrow [\Pi, \Pi]_{SN} = 0$.

Remark. Lie algebroid structure:
$\rho = (\mathrm{id} - \mathcal{O})$, $C_{ij}^k = (\mathrm{id} - \mathcal{O})_i^m \Gamma_{mj}^k - (i \leftrightarrow j) + \mathcal{O}_j^{m;k}\mathcal{O}_{mi}$

Remark. The same, using degree 1 DG-manifolds

Reminder about Lie algebroids

Definition

Let M be a smooth manifold. A Lie algebroid $(A, \rho, [\cdot,\cdot])$ is given by a finite-dimensional vector bundle A, a vector bundle morphism $\rho : A \to TM$, called anchor and a ($\mathbb{R}$-bilinear) Lie bracket on the sections of A

$$[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

satisfying for all $f \in C^\infty(M)$, $s, s' \in \Gamma(A)$:

$$[s, fs'] = f[s, s'] + \rho(s)(f) \cdot s'.$$

Fun facts about Lie algebroids

• Lie algebroids can be alternatively defined as differential graded manifolds of degree 1. In particular, there is a degree 1 (Lichnerowicz) differential $d_A : \Gamma(\Lambda^\bullet A^*) \to \Gamma(\Lambda^\bullet A^*)$:

$$(d_A\eta)(\xi_1, ..., \xi_{n+1}) = \sum_i (-1)^{i+1}\rho(\xi_i)(\eta(\xi_1, ..., \hat{\xi}_i, .., \xi_{n+1})) + \sum_{i<j}(-1)^{i+j}\eta([\xi_i, \xi_j], \xi_1, ..., \hat{\xi}_i, ..., \hat{\xi}_j, .., \xi_{n+1})$$

It satisfies $d_A^2 = 0$ and induces the so-called Lie algebroid cohomology and $H^\bullet(A)$.
The anchor ρ induces a morphism $H^\bullet_{dR}(M) \to H^\bullet(A)$.
• A Lie algebroid always induces a singular foliation on M: $\rho(A) \subset TM$ is involutive, hence $M = \bigsqcup_\alpha N_\alpha$ such that $TN_\alpha = \rho(A)|_{N_\alpha}$ for all N_α (immersed connected submanifolds). Moreover, the bracket on A restricts to well-defined brackets on $A|_{N_\alpha}$, turning $A|_{N_\alpha} \to N_\alpha$ into Lie algebroids.

Basic cohomology of Lie algebroids

Definition Let $A \xrightarrow{\rho} TM$ be a Lie algebroid over the smooth manifold M. We define:

- The subspace of ρ-horizontal forms at $m \in M$ as:

$$(\Lambda^\bullet A_m^*)^{hor} := \{\alpha \in \Lambda^\bullet A_m^* \mid \iota_v \alpha = 0 \ \forall v \in \ker(\rho_m : A_m \to T_m M)\}$$

- The subspaces of ρ-basic forms:

$$\Gamma(\Lambda^\bullet A^*)^{bas} = \{\alpha \in \Gamma(\Lambda^\bullet A^*) \mid \alpha_m \text{ and } (d_A \alpha)_m \text{ are horizontal for all } m\}$$

- The basic cohomology of A as the quotient

$$\mathcal{H}^\bullet_{bas}(A) = \frac{\ker(d_A : \Gamma(\Lambda^\bullet A^*)^{bas} \to \Gamma(\Lambda^\bullet A^*)^{bas})}{\mathrm{Image}(d_A : \Gamma(\Lambda^\bullet A^*)^{bas} \to \Gamma(\Lambda^\bullet A^*)^{bas})}$$

Basic forms – fun facts

Lemma

Let A be an algebroid, $N \subset M$ a leaf of A and $\eta \in \Gamma((\Lambda^k A^*)^{hor})$ a ρ-horizontal form.

- $\eta|_N$ is a horizontal k-form on the restricted Lie algebroid $A|_N \to N$, i.e. it induces a unique k-form $\eta_N \in \Omega^k(N)$.
- η is completely determined by the collection $\{\eta_N \mid N \text{ leaf of } A\}$.
- When η is basic, we have $(d_A\eta)_N = d\eta_N$.
- Let $[\eta] = 0 \in \mathcal{H}^k_{bas}(A)$, then $[\eta_N] = 0 \in H^k_{dR}(N)$ for all leaves N of the algebroid A.

Season 2, Episode 2: Variational approach

The natural basic two-cocycle of a Dirac structure

Let $D \subset \mathbb{T}M$ be a Dirac structure. We define $\omega_D \in \Gamma(\Lambda^2 D^*)$ by

$$\omega_D((v,\alpha),(w,\beta)) = \alpha(w) - \beta(v).$$

Since D is isotropic, $\omega_D((v,\alpha),(w,\beta)) = 2\alpha(w) = -2\beta(v)$, i.e. ω_D is horizontal.

Since D is involutive ω_D is closed in Dirac cohomology, i.e. $d_D\omega_D = 0$, and hence ω_D is basic. It thus yields a natural class in $\mathcal{H}^2_{bas}(D)$. Hence, in view of Lemma on basic forms:

Lemma Let $D \subset \mathbb{T}M$ be a Dirac structure.

- There is a naturally induced basic cocycle $\omega_D \in \Gamma(\Lambda^2 D^*)^{bas}$ associated to any Dirac structure D.
- If $[\omega_D] = 0 \in \mathcal{H}^2_{bas}(D)$, then for any leaf N of D, $[(\omega_D)_N] = 0 \in H^2_{dR}(N)$.

Dirac paths

Theorem 1. Let $D \subset \mathbb{T}M$ be a Dirac structure over M, $H \in C^\infty(M)$ be a Hamiltonian function and γ a path on M.

Assume that the basic 2-class $[\omega_D]$ vanishes, and let $\theta \in \Gamma(D^*)^{hor}$ be such that $d_D\theta = \omega_D$, then the following statements are equivalent:

(i) The path γ is a Hamiltonian curve, i.e. $(\dot\gamma(t), dH_{\gamma(t)}) \in D$ for all t.

(ii) All Dirac paths $\zeta : I \to D$ over γ (i.e. $\rho(\zeta) = \dot\gamma$) are critical points among the Dirac paths with the same end points of the following functional:

$$\zeta \mapsto \int_I \big(\theta_{\gamma(t)}(\zeta(t)) + H(\gamma(t))\big)\, dt \tag{1}$$

Inspiration from Tulczyjew's business

Definition. Let $L\colon TQ \to \mathbb{R}$ a (possibly degenerate) Lagrangian.

a) Tulczyjew's differential – map $u \mapsto \mathcal{D}_u L := \kappa(d_u L)$, where $\kappa : T^*TQ \to T^*T^*Q$ is the Tulczyjew isomorphism. Its image is a submanifold of T^*T^*Q.

b) Legendre – map from TQ to T^*Q: $\mathbb{F}L(v)$ for every $v \in T_qQ$:

$$\frac{\partial}{\partial t}\bigg|_{t=0} L(v+tw) = \langle \mathbb{F}L(v), w\rangle$$

c) We denote by $\mathfrak{Leg} = \mathbb{F}L(TQ) \subset T^*Q$ the image of $\mathbb{F}L$.

d) We call partial vector fields on $\mathfrak{Leg}$ sections[1] of $\Gamma(TT^*Q)|_{\mathfrak{Leg}}$.

c) An integral curve of a partial vector field X on $\mathfrak{Leg}$ is a path $t \mapsto u_t \in TQ$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{F}L(u_t) = X_{\mathbb{F}L(u_t)}$.

d) An implicit Lagrangian system for an almost Dirac structure $\mathbb{D} \subset \mathbb{T}T^*Q$ is a pair (X, L), with X a partially defined vector field on $\mathfrak{Leg}$, such that $(X(\mathbb{F}L(u)), \mathcal{D}_u L) \in \mathbb{D}$ for all u in TQ.

[1] For E a vector bundle over a manifold X and $Y \subset X$ an arbitrary subset (not necessarily a manifold), we denote by $\Gamma(E)|_Y$ restrictions to Y of smooth sections of E in a neighborhood of Y in X.

Operations with Dirac structures

Definition Let $D \subset \mathbb{T}M$ be a subbundle.

- For all $\phi\colon M' \to M$, we denote by $\phi^! D$ the set

$$\phi^! D_{m'} := \left\{(X, \phi^* \beta) \text{ with } X \in T_{m'}M', \beta \in T^*_{\phi(m')}M | (\phi_*(X), \beta) \in D_{\phi(m')}\right\}$$

When D is an (almost-)Dirac structure call $\phi^! D$ the pullback of D.

- Let ω be a 2-form $\omega \in \Omega^2(M)$, we denote by $e^\omega D$ the set

$$e^\omega D = \{(v, \beta + \iota_v \omega) \mid (v, \beta) \in D\}$$

and call it the gauge transform of D.

Lemma Let $D \subset \mathbb{T}M$ be a Dirac structure and M' be a manifold.
- For any smooth map $\phi\colon M' \to M$, $\phi^! D$ is a Dirac structure on M'.
- For any closed 2-form $\omega \in \Omega^2(M)$, $e^\omega D$ is a Dirac structure on M.

Implicit Lagrangian systems with magnetic terms

Given $D \subset \mathbb{T}Q$ a Dirac structure on Q, consider
(i) its pull back $\pi^! D$ on T^*Q through the canonical base map $\pi\colon T^*Q \to Q$, then
(ii) the gauge transformation $e^{\Omega}\pi^! D$ of this pull-back with respect to the canonical symplectic 2-form Ω.

Definition Let $D \subset \mathbb{T}Q$ be a Dirac structure on Q. We call constrained magnetic Lagrangian system an implicit Lagrangian system for the Dirac structure $\mathbb{D} = e^{\Omega}\pi^! D \subset \mathbb{T}T^*Q$ as above.

Implicit Lagrangian systems with magnetic terms

Theorem 2. Let $D \subset \mathbb{T}Q$ be a Dirac structure and $L : TQ \to \mathbb{R}$ a Lagrangian. Assume that the 2-form $\omega_D \in \Gamma(\Lambda^2 D^*)^{hor}$ admits a basic primitive $\theta \in \Gamma(D^*)^{hor}$. Then for $q : I \to Q$ the following are equivalent:

a) There exists a Dirac path $\zeta : I \to D$ such that $\rho(\zeta) = \dot{q}$ which is the critical point among Dirac paths with the same end points of

$$\int_I (L(\rho(\zeta(t))) + \theta(\zeta(t)))dt. \tag{2}$$

b) For all $t \in I$, the following condition holds.

$$\left(\frac{\partial}{\partial t}\mathbb{F}L(\dot{q}(t)), \mathcal{D}_{\dot{q}(t)}L\right) \in \mathbb{D} = e^{\Omega}\pi^! D. \tag{3}$$

Season 2, Episode 3: Applications and examples

Example (Classical symplectic magnetic terms)

Let Q be any manifold, $\omega \in \Omega^2_{cl}(Q)$ and $D = \Gamma_\omega \subset \mathbb{T}Q$. Let $L : TQ \to \mathbb{R}$ be a Lagrangian. Then $e^{\Omega}\pi^! D = \Gamma_{\Omega + \pi^*\omega} \subset \mathbb{T}T^*Q$.

As $\mathcal{H}^\bullet_{basic}(D) = H^\bullet_{dR}(M)$, the 2-form on D admits a basic potential if and only if ω is de-Rham exact, i.e. $\omega = d\theta$, $\theta \in \Omega^1(Q)$.

Assume that the Legendre transform $\mathbb{F}L : TQ \to T^*Q$ is bijective, and denote the Legendre transfrom of the Lagrangian by H, i.e.

$$H(p) = \langle p, (\mathbb{F}L)^{-1} p \rangle - L \circ (\mathbb{F}L)^{-1}(p)$$

In this case $\mathcal{D}L$ is simply dH. Theorem 2 yields that the critical points of $L(q, \dot{q}) + \theta(\dot{q})$ correspond under the Legendre transform to integral curves of the Hamiltonian flow of H for the symplectic structure $\Omega + \pi^*\omega$.

Hence (X, dH) is an implicit Lagrangian system with respect to $e^{\Omega}\pi^! D$ if and only if the vector field X is the Hamiltonian vector field of H with respect to $\Omega + \pi^*\omega$.

Example (Holonomic constraints)

Let $F \subset TQ$ be a regular foliation, and $F^\circ \subset T^*Q$ its annihilator. The Dirac structure $D = F \oplus F^\circ$ always admits a basic potential, as the 2-form in $\Lambda^2 D^*$ is zero (there is no magnetic term). Then $\pi^! D$ is the Dirac structure associated to the pullback foliation $\pi^{-1}(F)$ and

$$e^\Omega \pi^! D = \{(w, \alpha) \in TT^*Q \oplus T^*T^*Q \mid \pi_*(w) \in F, \alpha - \Omega^\flat w \in \pi^{-1}(F)^\circ\}$$

Let $L : TQ \to \mathbb{R}$ be a Lagrangian. Then Theorem 2 yields that the integral curves of any implicit Lagrangian system $(X, \mathcal{D}L)$ for $e^\Omega \pi^! D$ are critical points of L among curves that are tangent to F. The condition (3) (belonging to $\mathbb{D}$) translates directly to the Euler-Lagrange equations for a system subject to holonomic constraints, which are classically spelled-out using the Lagrange multipliers.

Instead of conclusion: work in progress

Poisson.

Let $D \subset \mathbb{T}M$ be the graph of a Poisson structure π. Then the Lie algebroid D is isomorphic to T^*M and $H^\bullet(D) \cong H^\bullet_\pi(M)$ is known as the Poisson cohomology. The class of ω_D in $H^\bullet(D)$ corresponds to the class of π in $H^2_\pi(M)$. The class of ω in the finer cohomology $\mathcal{H}^\bullet_{bas}(D)$ is zero if and only if $\pi \in \mathfrak{X}^2(M)$ admits a primitive $E \in \mathfrak{X}(M)$ (a vector field E satisfying $L_E\pi = \pi$), which is tangent to the Poisson structure, i.e. is a section of $\rho(D) \subset TM$.

Question: reformulate the Theorem 2.

Other Dirac structure. Pick your favourite one.

Oscar

Generalizations.

– Interpret obstructions

– Almost Dirac? $\to$ "almost classes"

– ...

Demain

Discretization and numerics – very long story

Merci pour votre attention!