Problèmes différentiels causaux fractionnaires et irrationnels :

outils pour la simulation de systèmes linéaires ou faiblement non linéaires

Thomas Hélie, CNRS

Équipe S3AM
Laboratoire des Sciences et Technologies de la Musique et du Son
IRCAM – CNRS – Sorbonne Université – Ministère de la Culture
Paris, France

– CFM 2022 –
Congrés Frandcais de Mécanique
Mini-Symposium "Rencontres Mathématiques-Mécanique"
31 août 2022
Nantes, France

Outline

1. **Introduction**
2. **Linear fractional/irrational systems: integral representations and simulation** *(coll.: D. Matignon & R. Mignot)*
3. **Weakly nonlinear irrational systems and Volterra series** *(coll.: M. Hasler & V. Smet)*
4. **Conclusion**

Outline

1. **Introduction**
   - Linear Time Invariant causal operators and Laplace transform
   - Causal one-half integrator $I^{1/2}$
   - Zoology of fractional and irrational) operators(/systems)
   - Integral representations: basic ideas on $I^{1/2}$
   - Questions about generalizations
2. **Linear fractional/irrational systems: integral representations and simulation** *(coll.: D. Matignon & R. Mignot)*
3. **Weakly nonlinear irrational systems and Volterra series** *(coll.: M. Hasler & V. Smet)*
4. **Conclusion**

Outline

1. **Introduction**
   - Linear Time Invariant causal operators and Laplace transform
   - Causal one-half integrator $I^{1/2}$
   - Zoology of fractional and irrational) operators(/systems)
   - Integral representations: basic ideas on $I^{1/2}$
   - Questions about generalizations
2. **Linear fractional/irrational systems: integral representations and simulation** *(coll.: D. Matignon & R. Mignot)*
3. **Weakly nonlinear irrational systems and Volterra series** *(coll.: M. Hasler & V. Smet)*
4. **Conclusion**

Linear Time Invariant (LTI) causal operators & Laplace Transform

Set of signals: $\mathcal{E} = \{x : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$, defined almost everywhere s.t. (i) & (ii)$\}$

(i) causality: $\forall t < 0, \quad x(t) = 0,$

(ii) integrability: $\forall T > 0, \quad \int_0^T |x(t)| \, \mathrm{d}t$ is convergent.

Laplace transform at $s \in \mathbb{C}$: $L[x](s) = X(s) := \int_0^{\infty} \mathrm{e}^{-st} x(t) \, \mathrm{d}t,$

(iii) defined if $\int_0^{\infty} |\mathrm{e}^{-st} x(t)| \, \mathrm{d}t$ is convergent.

Linear Time Invariant (LTI) causal operators & Laplace Transform

Set of signals: $\mathcal{E} = \{x : \mathbb{R} \to \mathbb{R} \text{ or } \mathbb{C}, \text{ defined almost everywhere s.t. (i) \& (ii)}\}$

(i) causality: $\forall t < 0, \quad x(t) = 0,$

(ii) integrability: $\forall T > 0, \quad \int_0^T |x(t)| \, \mathrm{d}t$ is convergent.

Laplace transform at $s \in \mathbb{C}$: $L[x](s) = X(s) := \int_0^\infty \mathrm{e}^{-st} x(t) \, \mathrm{d}t,$

(iii) defined if $\int_0^\infty |\mathrm{e}^{-st} x(t)| \, \mathrm{d}t$ is convergent.

General theorems *(complementary results for L^1, L^2, distributions, etc.)*

Existence: $\exists! a \in \overline{\mathbb{R}}$ s.t. (iii) is false if $\Re e(s) < a$ and true if $\Re e(s) > a$.

Analyticity: for all $s \in \mathbb{C}_a^+ := \{s \in \mathbb{C} \,|\, \Re e(s) > a\}$ (Rk: $\mathbb{C}_{-\infty}^+ = \mathbb{C}$, $\mathbb{C}_{+\infty}^+ = \emptyset$).

Fourier transform: $F[x](f) := X(2i\pi f)$, if $a < 0$ ($x \equiv$ IR of a strictly stable system).

Linear Time Invariant (LTI) causal operators & Laplace Transform

Set of signals: $\mathcal{E} = \{x : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$, defined almost everywhere s.t. (i) & (ii)$\}$

(i) causality: $\forall t < 0, \quad x(t) = 0,$

(ii) integrability: $\forall T > 0, \quad \int_0^T |x(t)|\,\mathrm{d}t$ is convergent.

Laplace transform at $s \in \mathbb{C}$: $L[x](s) = X(s) := \int_0^\infty \mathrm{e}^{-st} x(t)\,\mathrm{d}t,$

(iii) defined if $\int_0^\infty |\mathrm{e}^{-st} x(t)|\,\mathrm{d}t$ is convergent.

General theorems *(complementary results for L^1, L^2, distributions, etc.)*

Existence: $\exists! a \in \overline{\mathbb{R}}$ s.t. (iii) is false if $\Re e(s) < a$ and true if $\Re e(s) > a$.

Analyticity: for all $s \in \mathbb{C}_a^+ := \{s \in \mathbb{C} \,|\, \Re e(s) > a\}$ (Rk: $\mathbb{C}_{-\infty}^+ = \mathbb{C}$, $\mathbb{C}_{+\infty}^+ = \emptyset$).

Fourier transform: $F[x](f) := X(2i\pi f)$, if $a < 0$ ($x \equiv$ IR of a strictly stable system).

Theorems on integral, differential and LTI operators

Integrator $[I^1 x](t) := \int_0^t x(\tau)\,\mathrm{d}\tau$: $L[I^1 x](s) = \dfrac{1}{s} X(s),$ if $s \in \mathbb{C}_{\max(0,a)}^+$

Derivative $[D^1 x](t) := x'(t)$: $L[D^1 x](s) = s\,X(s) - x(0^+),$ if $x|_{\mathbb{R}^+}$ is $\mathcal{C}^0$ and $\exists A_0, t_0 > 0, \forall t > t_0, |x(t)| \leq A_0 \mathrm{e}^{at}$ *(if x is $\mathcal{C}^0$ on $\mathbb{R}$, $x(0^+) = 0$ and $D^1 \equiv s\times$).*

Convolution operator $[h \star x](t) = \int_{\mathbb{R}} h(\tau)\,x(t-\tau)\mathrm{d}\tau$: $L[h \star x](s) = H(s)\,X(s).$

Outline

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$I^{1/2}\, I^{1/2} x \quad = \quad I^1 x$$

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s)$$

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s) \implies H(s)^2 X(s) = \frac{1}{s} X(s)$$

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s) \implies H(s)^2 X(s) = \frac{1}{s} X(s)$$

Solution analytic on $\mathbb{C}_0^+$: $H(s) = 1/\sqrt{s}$ (or its negative version $-1/\sqrt{s}$)

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s) \implies H(s)^2 X(s) = \frac{1}{s} X(s)$$

Solution analytic on $\mathbb{C}_0^+$: $H(s) = 1/\sqrt{s}$ (or its negative version $-1/\sqrt{s}$)

Specimen in physics: 1D heat equation in a semi-infinite rod ($z \geq 0$)

a) Heat flow: $q(z,t) = -\kappa\, \partial_z \theta(z,t)$, ($\theta$: temperature, $\kappa > 0$: thermal conductivity)

b) Heat equation: $\partial_t \theta(z,t) = -\partial_z q(z,t) = \kappa\, \partial_z^2 \theta(z,t)$, for all $(z,t) \in (0,+\infty)^2$,

c) Initial condition: $\theta(z, t=0) = 0$, for all $z > 0$,

d) Controlled boundary: $q(z=0,t) = x(t)$, for all $t > 0$, with $X \in \mathcal{E}$.

e) Bounded solution ($\theta(z,.)$ in $\mathcal{E}$ for all z)

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s) \implies H(s)^2 X(s) = \frac{1}{s} X(s)$$

Solution analytic on $\mathbb{C}_0^+$: $H(s) = 1/\sqrt{s}$ (or its negative version $-1/\sqrt{s}$)

Specimen in physics: 1D heat equation in a semi-infinite rod ($z \geq 0$)

a) Heat flow: $q(z,t) = -\kappa\, \partial_z \theta(z,t)$, ($\theta$: temperature, $\kappa > 0$: thermal conductivity)

b) Heat equation: $\partial_t \theta(z,t) = -\partial_z q(z,t) = \kappa\, \partial_z^2 \theta(z,t)$, for all $(z,t) \in (0,+\infty)^2$,

c) Initial condition: $\theta(z, t=0) = 0$, for all $z > 0$,

d) Controlled boundary: $q(z=0, t) = x(t)$, for all $t > 0$, with $X \in \mathcal{E}$.

e) Bounded solution ($\theta(z,.)$ in $\mathcal{E}$ for all z)

$$\text{(b-c)} \Rightarrow s\Theta(z,s) = \partial_z^2 \Theta(z,s) \implies \exists A, B \text{ s.t. } \Theta(z,s) = A(s)\mathrm{e}^{-\sqrt{s}\, z} + B(s)\mathrm{e}^{+\sqrt{s}\, z}$$

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s) \implies H(s)^2 X(s) = \frac{1}{s} X(s)$$

Solution analytic on $\mathbb{C}_0^+$: $H(s) = 1/\sqrt{s}$ (or its negative version $-1/\sqrt{s}$)

Specimen in physics: 1D heat equation in a semi-infinite rod ($z \geq 0$)

a) Heat flow: $q(z,t) = -\kappa\, \partial_z \theta(z,t)$, ($\theta$: temperature, $\kappa > 0$: thermal conductivity)

b) Heat equation: $\partial_t \theta(z,t) = -\partial_z q(z,t) = \kappa\, \partial_z^2 \theta(z,t)$, for all $(z,t) \in (0,+\infty)^2$,

c) Initial condition: $\theta(z, t=0) = 0$, for all $z > 0$,

d) Controlled boundary: $q(z=0,t) = x(t)$, for all $t > 0$, with $X \in \mathcal{E}$.

e) Bounded solution ($\theta(z,.)$ in $\mathcal{E}$ for all z)

(b-c) $\Rightarrow$ $s\Theta(z,s) = \partial_z^2 \Theta(z,s) \implies \exists A, B$ s.t. $\Theta(z,s) = A(s)\mathrm{e}^{-\sqrt{s}\,z} + B(s)\mathrm{e}^{+\sqrt{s}\,z}$

(e and $s \in \mathbb{C}_0^+$, e.g. positive s) $\implies \Theta(z,s) = A(s)\mathrm{e}^{-\sqrt{s}\,z}$ (and $B \equiv 0$)

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s) \implies H(s)^2 X(s) = \frac{1}{s} X(s)$$

Solution analytic on $\mathbb{C}_0^+$: $H(s) = 1/\sqrt{s}$ (or its negative version $-1/\sqrt{s}$)

Specimen in physics: 1D heat equation in a semi-infinite rod ($z \geq 0$)

a) Heat flow: $q(z,t) = -\kappa\, \partial_z \theta(z,t)$, ($\theta$: temperature, $\kappa > 0$: thermal conductivity)

b) Heat equation: $\partial_t \theta(z,t) = -\partial_z q(z,t) = \kappa\, \partial_z^2 \theta(z,t)$, for all $(z,t) \in (0,+\infty)^2$,

c) Initial condition: $\theta(z, t=0) = 0$, for all $z > 0$,

d) Controlled boundary: $q(z=0, t) = x(t)$, for all $t > 0$, with $X \in \mathcal{E}$.

e) Bounded solution ($\theta(z,.)$ in $\mathcal{E}$ for all z)

(b-c) $\Rightarrow$ $s\Theta(z,s) = \partial_z^2 \Theta(z,s) \implies \exists A, B$ s.t. $\Theta(z,s) = A(s)\mathrm{e}^{-\sqrt{s}\,z} + B(s)\mathrm{e}^{+\sqrt{s}\,z}$

(e and $s \in \mathbb{C}_0^+$, e.g. positive s) $\implies \Theta(z,s) = A(s)\mathrm{e}^{-\sqrt{s}\,z}$ (and $B \equiv 0$)

(a) $\Rightarrow q(z,t) = \kappa\, A(s)\, \sqrt{s}\mathrm{e}^{-\sqrt{s}\,z}$

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s) \implies H(s)^2 X(s) = \frac{1}{s} X(s)$$

Solution analytic on $\mathbb{C}_0^+$: $H(s) = 1/\sqrt{s}$ (or its negative version $-1/\sqrt{s}$)

Specimen in physics: 1D heat equation in a semi-infinite rod ($z \geq 0$)

a) Heat flow: $q(z,t) = -\kappa\, \partial_z \theta(z,t)$, ($\theta$: temperature, $\kappa > 0$: thermal conductivity)
b) Heat equation: $\partial_t \theta(z,t) = -\partial_z q(z,t) = \kappa\, \partial_z^2 \theta(z,t)$, for all $(z,t) \in (0,+\infty)^2$,
c) Initial condition: $\theta(z, t=0) = 0$, for all $z > 0$,
d) Controlled boundary: $q(z=0,t) = x(t)$, for all $t > 0$, with $X \in \mathcal{E}$.
e) Bounded solution ($\theta(z,.)$ in $\mathcal{E}$ for all z)

(b-c) $\Rightarrow$ $s\Theta(z,s) = \partial_z^2 \Theta(z,s) \implies \exists A, B$ s.t. $\Theta(z,s) = A(s) e^{-\sqrt{s}\, z} + B(s) e^{+\sqrt{s}\, z}$
(e and $s \in \mathbb{C}_0^+$, e.g. positive s) $\implies \Theta(z,s) = A(s) e^{-\sqrt{s}\, z}$ (and $B \equiv 0$)
(a) $\Rightarrow q(z,t) = \kappa\, A(s) \sqrt{s} e^{-\sqrt{s}\, z}$, so that at $z = 0$, (d) $\Rightarrow \kappa\, A(s) \sqrt{s} = X(s)$

Causal one-half integrator $I^{1/2}$ & specimen in classical physics

For all $s \in \mathbb{C}_0^+$ and $x \in \mathcal{E}$ s.t. $s \mapsto X(s)$ is defined in $\mathbb{C}_a^+$ with $a \leq 0$

Laplace transfer function H of $I^{1/2}$

$$L[I^{1/2}\, I^{1/2} x](s) = L[I^1 x](s) \implies H(s)^2 X(s) = \frac{1}{s} X(s)$$

Solution analytic on $\mathbb{C}_0^+$: $H(s) = 1/\sqrt{s}$ (or its negative version $-1/\sqrt{s}$)

Specimen in physics: 1D heat equation in a semi-infinite rod ($z \geq 0$)

a) Heat flow: $q(z,t) = -\kappa\, \partial_z \theta(z,t)$, ($\theta$: temperature, $\kappa > 0$: thermal conductivity)

b) Heat equation: $\partial_t \theta(z,t) = -\partial_z q(z,t) = \kappa\, \partial_z^2 \theta(z,t)$, for all $(z,t) \in (0,+\infty)^2$,

c) Initial condition: $\theta(z, t=0) = 0$, for all $z > 0$,

d) Controlled boundary: $q(z=0,t) = x(t)$, for all $t > 0$, with $X \in \mathcal{E}$.

e) Bounded solution ($\theta(z,.)$ in $\mathcal{E}$ for all z)

(b-c) $\Rightarrow$ $s\Theta(z,s) = \partial_z^2 \Theta(z,s) \implies \exists A, B$ s.t. $\Theta(z,s) = A(s)\mathrm{e}^{-\sqrt{s}\,z} + B(s)\mathrm{e}^{+\sqrt{s}\,z}$

(e and $s \in \mathbb{C}_0^+$, e.g. positive s) $\implies \Theta(z,s) = A(s)\mathrm{e}^{-\sqrt{s}\,z}$ (and $B \equiv 0$)

(a) $\Rightarrow q(z,t) = \kappa\, A(s)\sqrt{s}\mathrm{e}^{-\sqrt{s}\,z}$, so that at $z=0$, (d) $\Rightarrow \kappa\, A(s)\sqrt{s} = X(s)$

Result: $\Theta(z,s) = \frac{\mathrm{e}^{-\sqrt{s}\,z}}{\kappa\sqrt{s}} X(s)$ and $\Theta(z=0,s) = \frac{1}{\kappa\sqrt{s}} X(s)$

At $z=0$, the temperature $\theta(z=0,t)$ evolves as $\frac{1}{\kappa} I^{1/2}$ of the heat flow $x(t)$

Outline

1. **Introduction**
   - Linear Time Invariant causal operators and Laplace transform
   - Causal one-half integrator $I^{1/2}$
   - Zoology of fractional and irrational) operators(/systems)
   - Integral representations: basic ideas on $I^{1/2}$
   - Questions about generalizations
2. **Linear fractional/irrational systems: integral representations and simulation** *(coll.: D. Matignon & R. Mignot)*
3. **Weakly nonlinear irrational systems and Volterra series** *(coll.: M. Hasler & V. Smet)*
4. **Conclusion**

Zoology of $\xrightarrow{x(t)}$ Fractional(/Irrational) Syst. $\xrightarrow{y(t)}$

Fractional/Irrational syst.	**Transfer fct.** (analytic in $\Re e(s) > 0$)
Integrator $I^{1/2}$	$H_1(s) = 1/\sqrt{s}$ $(\rightarrow H(s)^2 = 1/s)$
Derivative $\partial_t^{1/2}$	$H_2(s) = \sqrt{s}$ $(\rightarrow H(s)^2 = s)$
Frac. Diff. Eq. $(0 < \alpha < 1)$ $\sum_{p=0}^{P} \partial_t^{p\alpha} y = \sum_{q=0}^{Q} \partial_t^{q\alpha} x$	$H_3(s) = \sum_{q=0}^{Q} s^{q\alpha} / \sum_{p=0}^{P} s^{p\alpha}$

Zoology of $x(t) \rightarrow$ Fractional(/Irrational) Syst. $\rightarrow y(t)$

Fractional/Irrational syst.	**Transfer fct.** (analytic in $\Re e(s) > 0$)
Integrator $I^{1/2}$	$H_1(s) = 1/\sqrt{s}$ $(\rightarrow H(s)^2 = 1/s)$
Derivative $\partial_t^{1/2}$	$H_2(s) = \sqrt{s}$ $(\rightarrow H(s)^2 = s)$
Frac. Diff. Eq. $(0<\alpha<1)$ $\sum_{p=0}^{P} \partial_t^{p\alpha} y = \sum_{q=0}^{Q} \partial_t^{q\alpha} x$	$H_3(s) = \sum_{q=0}^{Q} s^{q\alpha} / \sum_{p=0}^{P} s^{p\alpha}$
Bessel: $y(t) = [J_0 \star x](t)$	$H_4(s) = 1/\sqrt{s^2+1}$
Fract. PDE : $(\partial_z + \partial_t^{1/2})w = 0$ $y(t) = w(z,t)$, $\partial_z w(0,t) = -x(t)$	$H_5(s) = \mathrm{e}^{-\sqrt{s}z}/\sqrt{s}$
Flared lossy acoustic pipe	$H_6(s) = 2\Gamma(s)\mathrm{e}^{s-\Gamma(s)}/[s+\Gamma(s)]$ with $\Gamma(s) = \sqrt{s^2+\varepsilon s^{3/2}+1}$

Zoology of $\xrightarrow{x(t)}$ Fractional(/Irrational) Syst. $\xrightarrow{y(t)}$

Fractional/Irrational syst.	**Transfer fct.** (analytic in $\Re e(s)>0$)
Integrator $I^{1/2}$	$H_1(s) = 1/\sqrt{s}$ $(\to H(s)^2 = 1/s)$
Derivative $\partial_t^{1/2}$	$H_2(s) = \sqrt{s}$ $(\to H(s)^2 = s)$
Frac. Diff. Eq. $(0<\alpha<1)$ $\sum_{p=0}^{P} \partial_t^{p\alpha} y = \sum_{q=0}^{Q} \partial_t^{q\alpha} x$	$H_3(s) = \sum_{q=0}^{Q} s^{q\alpha} / \sum_{p=0}^{P} s^{p\alpha}$
Bessel: $y(t) = \left[J_0 \star x\right](t)$	$H_4(s) = 1/\sqrt{s^2+1}$
Fract. PDE : $(\partial_z + \partial_t^{1/2})w = 0$ $y(t) = w(z,t)$, $\partial_z w(0,t) = -x(t)$	$H_5(s) = \mathrm{e}^{-\sqrt{s}z}/\sqrt{s}$
Flared lossy acoustic pipe	$H_6(s) = 2\Gamma(s)\mathrm{e}^{s-\Gamma(s)}/[s+\Gamma(s)]$ with $\Gamma(s) = \sqrt{s^2+\varepsilon s^{3/2}+1}$

$\to$ **long memory:** $\forall t>0,\ h_1(t) = 1/\sqrt{\pi t},\ h_5(t) \underset{\infty}{\sim} \sqrt{2/(\pi t)}\cos(t-\pi/4)$

Zoology of $x(t) \to$ Fractional(/Irrational) Syst. $\to y(t)$

Fractional/Irrational syst.	**Transfer fct.** (analytic in $\Re e(s)>0$)
Integrator $I^{1/2}$	$H_1(s) = 1/\sqrt{s}$ $(\to H(s)^2 = 1/s)$
Derivative $\partial_t^{1/2}$	$H_2(s) = \sqrt{s}$ $(\to H(s)^2 = s)$
Frac. Diff. Eq. $(0<\alpha<1)$ $\sum_{p=0}^{P} \partial_t^{p\alpha} y = \sum_{q=0}^{Q} \partial_t^{q\alpha} x$	$H_3(s) = \sum_{q=0}^{Q} s^{q\alpha} / \sum_{p=0}^{P} s^{p\alpha}$
Bessel: $y(t) = [J_0 \star x](t)$	$H_4(s) = 1/\sqrt{s^2+1}$
Fract. PDE : $(\partial_z + \partial_t^{1/2}) w = 0$ $y(t) = w(z,t)$, $\partial_z w(0,t) = -x(t)$	$H_5(s) = \mathrm{e}^{-\sqrt{s}z}/\sqrt{s}$
Flared lossy acoustic pipe	$H_6(s) = 2\Gamma(s)\mathrm{e}^{s-\Gamma(s)}/[s+\Gamma(s)]$ with $\Gamma(s) = \sqrt{s^2+\varepsilon s^{3/2}+1}$

$\to$ **long memory:** $\forall t>0$, $h_1(t) = 1/\sqrt{\pi t}$, $h_5(t) \underset{\infty}{\sim} \sqrt{2/(\pi t)} \cos(t-\pi/4)$

$\to$ **singularities of $H_k(s)$:** poles and **cuts** in $\Re e(s) < 0$

Outline

1. **Introduction**
   - Linear Time Invariant causal operators and Laplace transform
   - Causal one-half integrator $I^{1/2}$
   - Zoology of fractional and irrational) operators(/systems)
   - Integral representations: basic ideas on $I^{1/2}$
   - Questions about generalizations
2. **Linear fractional/irrational systems: integral representations and simulation** *(coll.: D. Matignon & R. Mignot)*
3. **Weakly nonlinear irrational systems and Volterra series** *(coll.: M. Hasler & V. Smet)*
4. **Conclusion**

Case of the fractional integrator $I^{1/2}$ $(\mathbf{H_1(s) = 1/\sqrt{s}})$

- Consider $s = \rho e^{i\theta} \in \mathbb{C}$ with $\rho > 0$ and $\theta \in]-\pi, \pi]$

Case of the fractional integrator $I^{1/2}$ $\quad (\mathbf{H_1(s) = 1/\sqrt{s}})$

- Consider $s = \rho e^{i\theta} \in \mathbb{C}$ with $\rho > 0$ and $\theta \in]-\pi, \pi]$
- $\sqrt{s} = \sqrt{\rho} e^{i\theta/2}$ generalizes the square-root which is positive on $s \in \mathbb{R}^+$

Case of the fractional integrator $I^{1/2}$ $\quad (\mathbf{H_1(s)=1/\sqrt{s}})$

- Consider $s = \rho e^{i\theta} \in \mathbb{C}$ with $\rho > 0$ and $\theta \in]-\pi, \pi]$
- $\sqrt{s} = \sqrt{\rho} e^{i\theta/2}$ generalizes the square-root which is positive on $s \in \mathbb{R}^+$
- For these choices, $\arg\sqrt{s} = \frac{\theta}{2} \in]-\frac{\pi}{2}, \frac{\pi}{2}]$ and there is a jump of $H_1(s) = 1/\sqrt{s}$ when s crosses $\mathbb{R}^-$

Case of the fractional integrator $I^{1/2}$ $(\mathbf{H_1(s)=1/\sqrt{s}})$

- Consider $s = \rho \mathrm{e}^{i\theta} \in \mathbb{C}$ with $\rho > 0$ and $\theta \in]-\pi, \pi]$
- $\sqrt{s} = \sqrt{\rho}\mathrm{e}^{i\theta/2}$ generalizes the square-root which is positive on $s \in \mathbb{R}^+$
- For these choices, $\arg\sqrt{s} = \frac{\theta}{2} \in]-\frac{\pi}{2}, \frac{\pi}{2}]$ and there is a jump of $H_1(s) = 1/\sqrt{s}$ when s crosses $\mathbb{R}^-$

$\mathbb{R}^-$ is called a cut of $H_1(s)$ and the jump at $-\xi \in \mathbb{R}^-$ is

$$H_1(-\xi + i0^-) - H_1(-\xi + i0^+) = \frac{i}{\sqrt{\xi}} - \frac{-i}{\sqrt{\xi}} = \frac{2i}{\sqrt{\xi}}$$

Case of the fractional integrator $I^{1/2}$ $(\mathbf{H_1(s) = 1/\sqrt{s}})$

- Consider $s = \rho e^{i\theta} \in \mathbb{C}$ with $\rho > 0$ and $\theta \in]-\pi, \pi]$
- $\sqrt{s} = \sqrt{\rho} e^{i\theta/2}$ generalizes the square-root which is positive on $s \in \mathbb{R}^+$
- For these choices, $\arg\sqrt{s} = \frac{\theta}{2} \in]-\frac{\pi}{2}, \frac{\pi}{2}]$ and there is a jump of $H_1(s) = 1/\sqrt{s}$ when s crosses $\mathbb{R}^-$

$\mathbb{R}^-$ is called a cut of $H_1(s)$ and the jump at $-\xi \in \mathbb{R}^-$ is

$$H_1(-\xi + i0^-) - H_1(-\xi + i0^+) = \frac{i}{\sqrt{\xi}} - \frac{-i}{\sqrt{\xi}} = \frac{2i}{\sqrt{\xi}}$$

- **Why choosing the cut $\mathbb{R}^-$ (that is $\theta \in]-\pi, \pi]$) ?**

Case of the fractional integrator $I^{1/2}$ $(\mathbf{H_1(s)=1/\sqrt{s}})$

- Consider $s = \rho e^{i\theta} \in \mathbb{C}$ with $\rho > 0$ and $\theta \in]-\pi, \pi]$
- $\sqrt{s} = \sqrt{\rho}e^{i\theta/2}$ generalizes the square-root which is positive on $s \in \mathbb{R}^+$
- For these choices, $\arg\sqrt{s} = \frac{\theta}{2} \in]-\frac{\pi}{2}, \frac{\pi}{2}]$ and there is a jump of $H_1(s) = 1/\sqrt{s}$ when s crosses $\mathbb{R}^-$

$\mathbb{R}^-$ is called a cut of $H_1(s)$ and the jump at $-\xi \in \mathbb{R}^-$ is

$$H_1(-\xi + i0^-) - H_1(-\xi + i0^+) = \frac{i}{\sqrt{\xi}} - \frac{-i}{\sqrt{\xi}} = \frac{2i}{\sqrt{\xi}}$$

- **Why choosing the cut $\mathbb{R}^-$ (that is $\theta \in]-\pi, \pi]$) ?**
  (i) Causal stable system $\Rightarrow$ H analytic in $\Re e(s) > 0$

Case of the fractional integrator $I^{1/2}$ $\quad (\mathbf{H_1(s) = 1/\sqrt{s}})$

- Consider $s = \rho e^{i\theta} \in \mathbb{C}$ with $\rho > 0$ and $\theta \in]-\pi, \pi]$
- $\sqrt{s} = \sqrt{\rho} e^{i\theta/2}$ generalizes the square-root which is positive on $s \in \mathbb{R}^+$
- For these choices, $\arg\sqrt{s} = \frac{\theta}{2} \in]-\frac{\pi}{2}, \frac{\pi}{2}]$ and there is a jump of $H_1(s) = 1/\sqrt{s}$ when s crosses $\mathbb{R}^-$

> $\mathbb{R}^-$ is called a cut of $H_1(s)$ and the jump at $-\xi \in \mathbb{R}^-$ is
> $$H_1(-\xi + i0^-) - H_1(-\xi + i0^+) = \frac{i}{\sqrt{\xi}} - \frac{-i}{\sqrt{\xi}} = \frac{2i}{\sqrt{\xi}}$$

- **Why choosing the cut $\mathbb{R}^-$ (that is $\theta \in]-\pi, \pi]$) ?**
  - (i) Causal stable system $\Rightarrow$ H analytic in $\Re e(s) > 0$
  - (ii) It is "natural" to preserve the Hermitian symmetry since a real system $\Rightarrow$ $H_1(\bar{s}) = \overline{H_1(s)}$ in $\Re e(s) > 0$

Basic idea: Laplace inverse transform and adapted Bromwich contour

Let $\mathrm{e}_+^t = \mathrm{e}^t \, \mathbf{1}_{\mathbb{R}^+}(t)$ be the causal exponential.

- Causal convolution kernel: $h_1(t) = \lim_{\epsilon \to 0^+} \int_{\epsilon - i\infty}^{\epsilon + i\infty} H_1(s) \mathrm{e}_+^{st} \mathrm{d}s$

Basic idea: Laplace inverse transform and adapted Bromwich contour

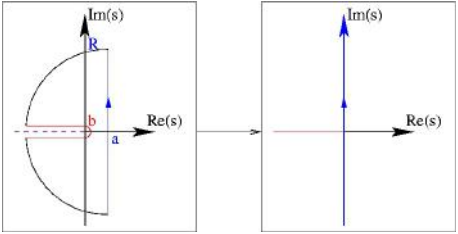
Let $\mathrm{e}_+^t = \mathrm{e}^t\,\mathbf{1}_{\mathbb{R}^+}(t)$ be the causal exponential.

- Causal convolution kernel: $h_1(t) = \lim_{\epsilon\to 0^+} \int_{\epsilon - i\infty}^{\epsilon + i\infty} H_1(s)\mathrm{e}_+^{st}\mathrm{d}s$
- Residue theorem: $\frac{1}{2i\pi}\oint_{\mathcal{C}} H_1(s)\mathrm{e}_+^{st}\mathrm{d}s = \sum_{\gamma\in\mathcal{P}} \mathrm{Res}_{H_1,\gamma}\,\mathrm{e}_+^{\gamma t}$

Basic idea: Laplace inverse transform and adapted Bromwich contour

Let $\mathrm{e}_+^t = \mathrm{e}^t \,\mathbf{1}_{\mathbb{R}^+}(t)$ be the causal exponential.

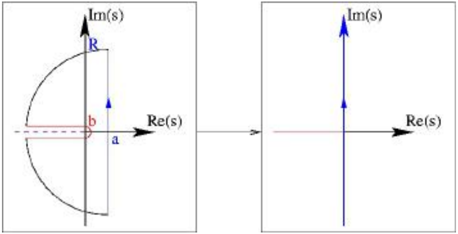
- Causal convolution kernel: $h_1(t) = \lim_{\epsilon \to 0^+} \int_{\epsilon - i\infty}^{\epsilon + i\infty} H_1(s) \mathrm{e}_+^{st} \mathrm{d}s$
- Residue theorem: $\frac{1}{2i\pi} \oint_{\mathcal{C}} H_1(s) \mathrm{e}_+^{st} \mathrm{d}s = \sum_{\gamma \in \mathcal{P}} \mathrm{Res}_{H_1,\gamma} \, \mathrm{e}_+^{\gamma t}$
- Bromwich contour $\mathcal{C}_{R,a,b}$ with $(R, a, b) \to (+\infty, 0^+, 0^+)$



Basic idea: Laplace inverse transform and adapted Bromwich contour

Let $e_+^t = e^t \mathbf{1}_{\mathbb{R}^+}(t)$ be the causal exponential.

- Causal convolution kernel: $h_1(t) = \lim_{\epsilon \to 0^+} \int_{\epsilon - i\infty}^{\epsilon + i\infty} H_1(s) e_+^{st} ds$
- Residue theorem: $\frac{1}{2i\pi} \oint_{\mathcal{C}} H_1(s) e_+^{st} ds = \sum_{\gamma \in \mathcal{P}} \text{Res}_{H_1,\gamma}\, e_+^{\gamma t}$
- Bromwich contour $\mathcal{C}_{R,a,b}$ with $(R, a, b) \to (+\infty, 0^+, 0^+)$



- $\boxed{h(t) + 0 - \int_0^{+\infty} \mu(-\xi) e_+^{-\xi t} d\xi + 0 = 0}$ with

$\mu(-\xi) = \frac{H_1(-\xi + i0^-) - H_1(-\xi + i0^+)}{2i\pi} = \frac{1}{\pi\sqrt{\xi}}$

Basic idea: Integral representations

- Kernel: $\boxed{h_1(t) = \int_0^{+\infty} \mu(-\xi) \mathrm{e}_+^{-\xi t} \mathrm{d}\xi}$ with $\mu(-\xi) = \frac{1}{\pi\sqrt{\xi}}$

Basic idea: Integral representations

- Kernel: $\boxed{h_1(t) = \int_0^{+\infty} \mu(-\xi) \mathrm{e}_+^{-\xi t} \mathrm{d}\xi}$ with $\mu(-\xi) = \frac{1}{\pi\sqrt{\xi}}$
- Input/Output system: a continuous aggregation of convolutions with damped exponential

$$y(t) = [h_1 \star x](t) = \int_0^{\infty} \mu(-\xi) \underbrace{[\mathrm{e}_+^{-\xi t} \star_t x(t)]}_{=\phi(-\xi, t)} \mathrm{d}\xi$$

Basic idea: Integral representations

- Kernel: $\boxed{h_1(t) = \int_0^{+\infty} \mu(-\xi) \mathrm{e}_+^{-\xi t} \mathrm{d}\xi}$ with $\mu(-\xi) = \frac{1}{\pi\sqrt{\xi}}$
- Input/Output system: a continuous aggregation of convolutions with damped exponential

$$y(t) = [h_1 \star x](t) = \int_0^{\infty} \mu(-\xi) \underbrace{[\mathrm{e}_+^{-\xi t} \star_t x(t)]}_{=\phi(-\xi,t)} \mathrm{d}\xi$$

- Time-realization:

$$\begin{cases} \partial_t \phi(-\xi, t) = -\xi \phi(-\xi, t) + x(t), \quad \phi(-\xi, 0) = 0, \quad \forall \xi > 0 \\ y(t) = \int_0^{+\infty} \mu(-\xi) \phi(-\xi, t) \mathrm{d}\xi \end{cases}$$

Basic idea: Integral representations

- Kernel: $\boxed{h_1(t) = \int_0^{+\infty} \mu(-\xi) \mathrm{e}_+^{-\xi t} \mathrm{d}\xi}$ with $\mu(-\xi) = \frac{1}{\pi\sqrt{\xi}}$
- Input/Output system: a continuous aggregation of convolutions with damped exponential

$$y(t) = [h_1 \star x](t) = \int_0^{\infty} \mu(-\xi) \underbrace{[\mathrm{e}_+^{-\xi t} \star_t x(t)]}_{=\phi(-\xi,t)} \mathrm{d}\xi$$

- Time-realization:

$$\begin{cases} \partial_t \phi(-\xi, t) = -\xi \phi(-\xi, t) + x(t), \quad \phi(-\xi, 0) = 0, \quad \forall \xi > 0 \\ y(t) = \int_0^{+\infty} \mu(-\xi) \phi(-\xi, t) \mathrm{d}\xi \end{cases}$$

- Transfer function: aggregation of first order systems

$F(-\xi, s) = \frac{\Phi(-\xi, s)}{X(s)} = \frac{1}{s+\xi}, \quad \forall \xi > 0$

$$H_1(s) = \frac{Y(s)}{X(s)} = \frac{\int_0^{+\infty} \mu(-\xi) \Phi(-\xi, s) \mathrm{d}\xi}{E(s)} = \int_0^{+\infty} \mu(-\xi) F(-\xi, s) \mathrm{d}\xi$$

$$= \int_0^{+\infty} \frac{\mu(-\xi)}{s+\xi} \mathrm{d}\xi \left(= \frac{1}{\sqrt{s}} \right), \quad \text{for } \Re e(s) > 0$$

Outline

Questions about generalizations

Summary:

- Determine the singularities (poles and cuts) of H(s).
- Compute their associated residues and jumps
- Derive an integral representation from an adapted Bromwich contour and the residue theorem
- long memory (damping slower than any exponential) $\leftrightarrow$ infinite continuous aggregation of exponentials

Questions:

- Are such integral representations always well-posed ?
- How to perform accurate approximations and simulations in the time domain ?

Outline

Outline

Definitions

- Many transfer functions can be decomposed as follows, in some right-half complex plane $\mathbb{C}_a^+ := \{\Re e(s) > a\}$,

$$H(s) = \sum_{k=1}^{K} \sum_{l=1}^{L_k} \frac{r_{k,l}}{(s - s_k)^l} + \int_{\mathcal{C}} \frac{M(\mathrm{d}\gamma)}{s - \gamma} \,,$$

Definitions

- Many transfer functions can be decomposed as follows, in some right-half complex plane $\mathbb{C}_a^+ := \{\Re e(s) > a\}$,

$$H(s) = \sum_{k=1}^{K} \sum_{l=1}^{L_k} \frac{r_{k,l}}{(s - s_k)^l} + \int_{\mathcal{C}} \frac{M(\mathrm{d}\gamma)}{s - \gamma},$$

- which translates in the time domain into the following decomposition of the impulse response:

$$h(t) = \sum_{k=1}^{K} \sum_{l=1}^{L_k} r_{k,l} \frac{1}{l!} t^{l-1} e^{s_k t} + \int_{\mathcal{C}} e^{\gamma t} M(\mathrm{d}\gamma), \qquad \text{for } t > 0.$$

Definitions

- Many transfer functions can be decomposed as follows, in some right-half complex plane $\mathbb{C}_a^+ := \{\Re e(s) > a\}$,

$$H(s) = \sum_{k=1}^{K} \sum_{l=1}^{L_k} \frac{r_{k,l}}{(s - s_k)^l} + \int_{\mathcal{C}} \frac{M(\mathrm{d}\gamma)}{s - \gamma} \, ,$$

- which translates in the time domain into the following decomposition of the impulse response:

$$h(t) = \sum_{k=1}^{K} \sum_{l=1}^{L_k} r_{k,l} \frac{1}{l!} \, t^{l-1} \, e^{s_k \, t} + \int_{\mathcal{C}} e^{\gamma \, t} \, M(\mathrm{d}\gamma), \qquad \text{for } t > 0.$$

- The integral part can be realized by a dynamical system:

$$\begin{aligned} \partial_t \phi(\gamma, t) &= \gamma \, \phi(\gamma, t) + u(t), \quad \phi(\gamma, 0) = 0, \qquad \forall \gamma \in \mathcal{C} \\ y(t) &= \int_{\mathcal{C}} \phi(\gamma, t) \, M(\mathrm{d}\gamma) \, , \end{aligned}$$

Technical conditions

- A well-posedness condition must be fulfilled:

$$\int_{\mathcal{C}} \left| \frac{M(\mathrm{d}\gamma)}{a+1-\gamma} \right| < \infty \,.$$

- When measure M has a density μ, and the curve $\mathcal{C}$ admits a $\mathcal{C}^1$-regular parametrization $\xi \mapsto \gamma(\xi)$ which is non-degenerate ($\gamma'(\xi) \neq 0$), we have:

$$\mu\big(\gamma\big) = \lim_{\epsilon \to 0^+} \frac{H\big(\gamma + i\gamma'\epsilon\big) - H\big(\gamma - i\gamma'\epsilon\big)}{2i\pi} .$$

Outline

Method M1: approximation by interpolation of the state

- Approximation of the state $\phi(\gamma, t)$, for $\{\gamma_p\}_{0 \leq p \leq P+1} \subset \mathcal{C}$
  $\widetilde{\phi}(\gamma, t) = \sum_{p=1}^{P} \phi_p(t)\, \Lambda_p(\gamma)$, where $\phi_p(t) = \phi(\gamma_p, t)$.
- $\{\Lambda_p\}_{1 \leq p \leq P}$ are *cont. piecewise lin. interpolating functions.*

Method M1: approximation by interpolation of the state

- Approximation of the state $\phi(\gamma, t)$, for $\{\gamma_p\}_{0\leq p\leq P+1} \subset \mathcal{C}$
  $\widetilde{\phi}(\gamma, t) = \sum_{p=1}^{P} \phi_p(t)\,\Lambda_p(\gamma)$, where $\phi_p(t) = \phi(\gamma_p, t)$.
- $\{\Lambda_p\}_{1\leq p\leq P}$ are *cont. piecewise lin. interpolating functions.*
- The corresponding realization reads:

$$\begin{aligned}
\partial_t \phi_p(t) &= \gamma_p\,\phi_p(t) + u(t),\ 1 \leq p \leq P, \\
\widetilde{y}(t) &= \Re\mathrm{e} \sum_{p=1}^{P} \mu_p\,\phi_p(t) \quad \text{with } \mu_p = \int_{[\gamma_{p-1},\gamma_{p+1}]_{\mathcal{C}}} \mu(\gamma)\Lambda_p(\gamma)\mathrm{d}\gamma.
\end{aligned}$$

- The corresponding transfer function has the structure:

$$\widetilde{H}_\mu(s) = \frac{1}{2}\sum_{p=1}^{P}\left[\frac{\mu_p}{s-\gamma_p} + \frac{\overline{\mu_p}}{s-\overline{\gamma_p}}\right]$$

- Convergence results can be proved, as dim. $P \longrightarrow \infty$.

Method M2: optimization
Step 1: re-interpreting Sobolev spaces

- Optimization in the frequency domain, stemming from

$$\widehat{h}(f) = \lim_{\epsilon \to 0^+} H(\epsilon + 2i\pi f)$$

Method M2: optimization
Step 1: re-interpreting Sobolev spaces

- Optimization in the frequency domain, stemming from

$$\widehat{h}(f) = \lim_{\epsilon \to 0^+} H(\epsilon + 2i\pi f)$$

- Norms in L^2, or Sobolev spaces H^s, are defined as:

$$\|h\|^2_{H^s(\mathbb{R}_t)} = \int_{\mathbb{R}_f} w_s(f) \, |H(2i\pi f)|^2 \, \mathrm{d}f \, , \text{ with } w_s(f) = (1 + 4\pi^2 f^2)^s \, .$$

where $s \in \mathbb{R}$ tunes the balance between low and high frequencies.

Method M2: optimization
Step 1: re-interpreting Sobolev spaces

- Optimization in the frequency domain, stemming from

$$\widehat{h}(f) = \lim_{\epsilon \to 0^+} H(\epsilon + 2i\pi f)$$

- Norms in L^2, or Sobolev spaces H^s, are defined as:

$$\|h\|^2_{H^s(\mathbb{R}_t)} = \int_{\mathbb{R}_f} w_s(f)\,|H(2i\pi f)|^2\,\mathrm{d}f\,, \text{ with } w_s(f) = (1 + 4\pi^2 f^2)^s\,.$$

  where $s \in \mathbb{R}$ tunes the balance between low and high frequencies.

- For specific applications, more general frequency dependent weights can be used: bounded frequency range, logarithmic scale, relative error measurement, bounded dynamics ...

Method M2: optimization
Step 2: building up specific weights for audio applications

For audio applications, $w(f)$ can be adapted and modified according to the following requirements:

1. a bounded frequency range $f \in [f^-, f^+]$: $w(f)\,1_{[f^-,f^+]}(f)$;
2. a frequency log-scale: $w(f)/f$;
3. a relative error measurement: $w(f)/|H(2i\pi f)|^2$
4. a relative error on a bounded dynamics: $w(f)/\left(\mathrm{Sat}_{H,\Theta}(f)\right)^2$ where the saturation function $\mathrm{Sat}_{H,\Theta}$ with threshold Θ is defined by

$$\mathrm{Sat}_{H,\Theta}(f) = \begin{cases} |H(2i\pi f)| & \text{if } |H(2i\pi f)| \geq \Theta_H \\ \Theta_H & \text{otherwise} \end{cases}$$

Note: normalization of the samples is desirable in most audio applications, before the sequence is sent to DAC audio converters.

Method M2: optimization
Step 3: Regularized criterion with equality constraints

- The regularized criterion reads:

$$\mathcal{C}_R(\mu) = \int_{\mathbb{R}^+} \left| \widetilde{H_\mu}(2i\pi f) - H(2i\pi f) \right|^2 w(f) \mathrm{d}f + \sum_{p=1}^{P} \epsilon_p |\mu_p|^2,$$

- Equality constraints for $\widetilde{H_\mu}^{(d_j)}$ at prescribed frequency points η_j, $1 \leq j \leq J$ are taken into account thanks to a Lagrangian $\mathcal{C}_{R,L}$ by adding to $\mathcal{C}_R$:

$$\Re\mathrm{e} \left(\boldsymbol{\ell}^* \begin{bmatrix} H^{(d_1)}(2i\pi\eta_1) - \widetilde{H_\mu}^{(d_1)}(2i\pi\eta_1) \\ \vdots \\ H^{(d_J)}(2i\pi\eta_J) - \widetilde{H_\mu}^{(d_J)}(2i\pi\eta_J) \end{bmatrix} \right),$$

Method M2: optimization
Step 4: Discrete criterion

- Discrete version of the criterion for frequencies increasing from $f_1 = f_-$ to $f_{N+1} = f_+$ is, with $s_n = 2i\pi f_n$:

$$\mathcal{C}(\mu) \approx \sum_{n=1}^{N} w_n \left| \widetilde{H_\mu}(s_n) - H(s_n) \right|^2 \text{ with } w_n = \int_{f_n}^{f_{n+1}} w(f)\mathrm{d}f.$$

- In matrix notations, this rewrites

$$\mathcal{C}_{R,L}(\boldsymbol{\mu}) = \left(\boldsymbol{M\mu} - \boldsymbol{h}\right)^* \boldsymbol{W}\left(\boldsymbol{M\mu} - \boldsymbol{h}\right) + \boldsymbol{\mu}^t \boldsymbol{E\mu} + \Re\mathrm{e}\left(\ell^* \left[\boldsymbol{k} - \boldsymbol{N\mu}\right]\right),$$

with

$\boldsymbol{M}$:	model	$N \times (P + P_2)$
$\boldsymbol{N}$:	constraint model	$J \times (P + P_2)$
$\boldsymbol{E}$:	regularization	$(P + P_2) \times (P + P_2)$
$\boldsymbol{W}$:	weights	$N \times N$
$\boldsymbol{h}$:	data	$N \times 1$
$\boldsymbol{k}$:	constaints	$J \times 1$

Method M2: optimization
Step 5: Closed-form solution

- If $J = 0$ (no constraint), the solution reduces to

$$\boldsymbol{\mu} = \mathcal{M}^{-1}\mathcal{H}\,,$$

where $\mathcal{M} = \Re\mathrm{e}\left(\boldsymbol{M}^*\boldsymbol{W}\boldsymbol{M} + \boldsymbol{E}\right)$ and $\mathcal{H} = \Re\mathrm{e}\left(\boldsymbol{M}^*\boldsymbol{W}\boldsymbol{h}\right)$.

- For $J \geq 1$, the solution reads:

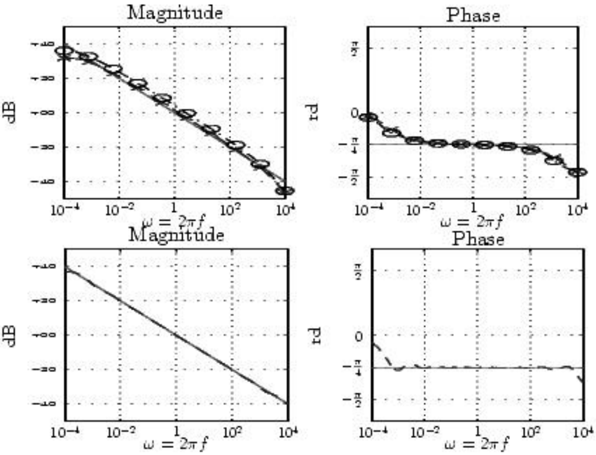
$$\boldsymbol{\mu} = \mathcal{M}^{-1}\left[\mathcal{H} + \underline{\boldsymbol{N}}^t\mathcal{N}^{-1}\left(\underline{\boldsymbol{k}} - \underline{\boldsymbol{N}}\mathcal{M}^{-1}\mathcal{H}\right)\right],$$

where $\mathcal{N} = \underline{\boldsymbol{N}}\mathcal{M}^{-1}\underline{\boldsymbol{N}}^t$ is invertible for non-redundant constraints, and

$$\begin{cases} \underline{\boldsymbol{N}}^t & \text{denotes} \quad [\Re\mathrm{e}(\boldsymbol{N}^t), \Im\mathrm{m}(\boldsymbol{N}^t)] \\ \underline{\boldsymbol{k}}^t & \text{denotes} \quad [\Re\mathrm{e}(\boldsymbol{k}^t), \Im\mathrm{m}(\boldsymbol{k}^t)] \end{cases}.$$

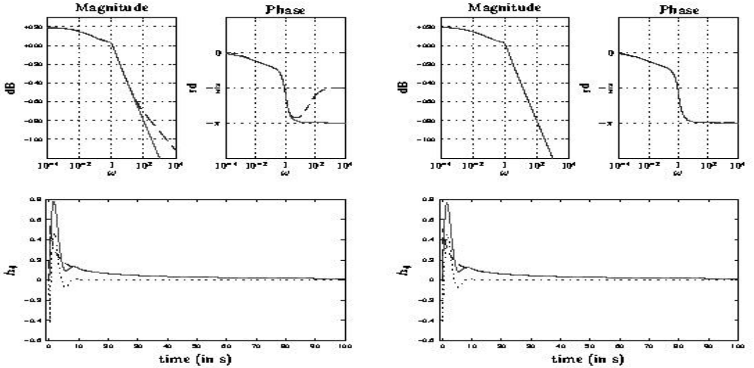
Outline

Academic example: $H_1(s) = 1/\sqrt{s},\ \mu_1(-\xi) = 1/(\pi\sqrt{\xi})$



Top: Interpolation, $P = 16$. Bottom: Optimization, $P = 10$.

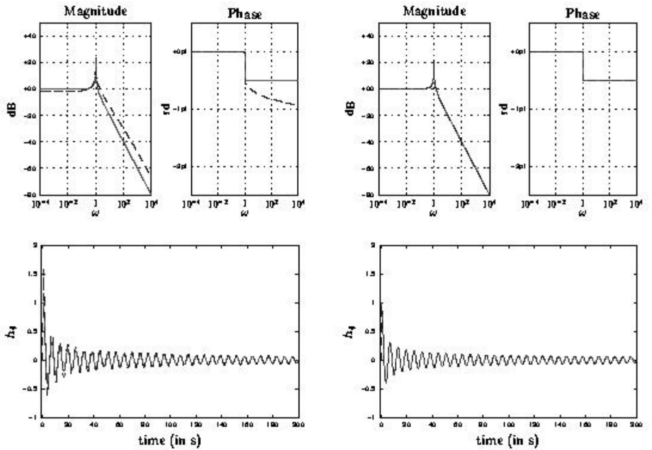
Fractional auto-regressive system: $H_3(s) = 1/(s^2 + 0.1s^{3/2} + s^{1/2} + 0.1)$ (poles and $\mathbb{R}^-$)



Left: Interpolation, $P = 18$. Right: Optimization, $P = 18$.
$(\ldots)$: poles only. $(--)$: cut only. $(-)$: poles and cut.

Bessel kernel: 2 cuts $\pm i + \mathbb{R}^-$

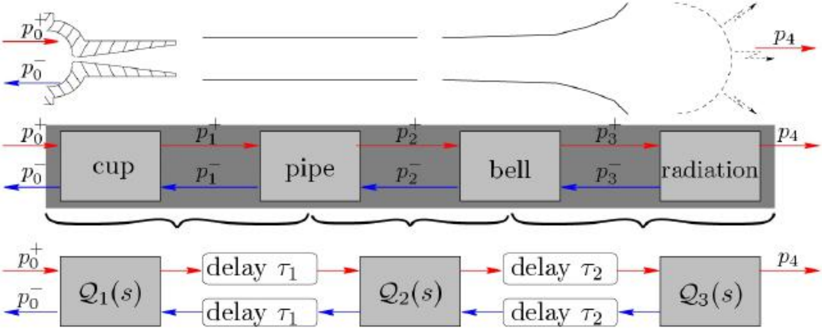
$H_4(s) = 1/\sqrt{s^2+1}$, $\mu_4^{\pm}(-\xi) = 1\,/\,(\pi\sqrt{\xi(\pm 2i - \xi)})$



Left: Interpolation, $P = 10$. Right: Optimization, $P = 10$.

Trumpet-like instrument (I)

Decomposition into elementary subsystems.

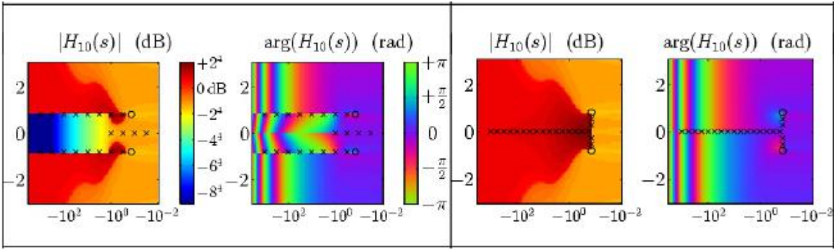


Transfer functions of interest:

- Reflection between p_0^+ and p_0^-.
- Transmission between p_0^+ and p_4.

Trumpet-like instrument (II): various choices of the cuts

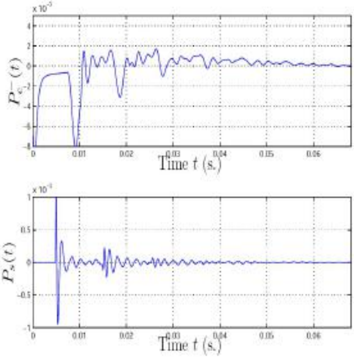
- with 3 Horizontal cuts, with a Cross cut



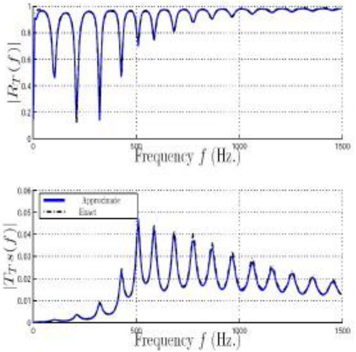
- Remark: the values of $H(s)$ in $\mathbb{C}_0^+$ do not depend on the choice of the cut!

Trumpet-like instrument (III)

Time-domain representation



Frequency-domain rep.



Real-time simulations in Pure-Data environment on optimized models with $P \leq 10$ for each quadripole $\mathcal{Q}_k$: bounded freq. range, log-scale & relat. error.

Outline

Outline

Purpose (audio effects and sound synthesis)

- Simulate the realistic propagation of a progressive plane wave in a pipe

- Include the **nonlinearity** responsible for the **brightness of « brass sounds »** at **fortissimo** nuances (|p|<160 dB spl)
- Low-cost input/output relation
  Choice: Volterra series

1. Nonlinear acoustic model
(planar progressive wave)

- [Mak97] adimensional version:

For x>0, t>0, $\partial_x p + \partial_t p + A(p) = \frac{\beta}{2}\partial_t p^2$

Boundary Cnd.: $p(x=0,t) = p_0(t)$ (input)

- Damping models A(p):

Simplest: $A_0(p) = \alpha_0\, p$

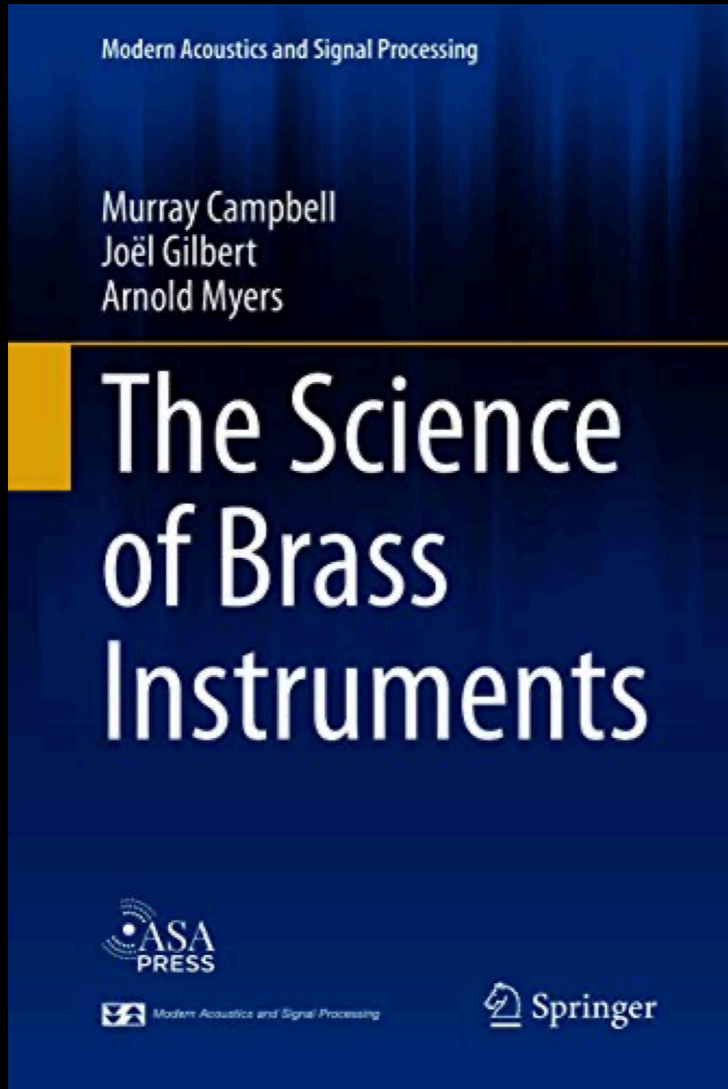
Realistic (brass instr., **[MJ00]**): $A_1(p) = \alpha_1\, \partial^{1/2} p$

Tribute to Joël Gilbert

Joël Gilbert (1963-2022)

Research director, CNRS
Laboratory of Acoustics, Le Mans University

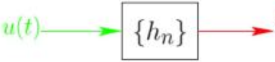
Medal of the French Acoustical Society, 2022



Outline

Definition and properties

- Volterra series with kernels $\{h_n\}_{n\in\mathbb{N}^*}$



$$y(t) = \underbrace{\sum_{n=1}^{+\infty}}_{\text{sum}} \underbrace{\int \cdots \int_{-\infty}^{+\infty} h_n(t_1, \ldots, t_n)\, u(t-t_1) \ldots u(t-t_n)\, dt_1 \ldots dt_n}_{\text{of} \quad \text{multi-convolutions}}$$

- Convergence: *(not studied here)* $|u(t)| < \rho$ radius of the series $\sum_{n=1}^{+\infty} \|h_n\|_1 \, x^n$

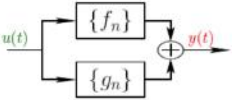
- Laplace transform: transfer kernels $H_n(s_1, \ldots, s_n)$
(analytic for stable causal system on $\Re e(s_k) > 0$)

Interconnexion laws

Denoting $s_{1:n} = s_1, s_2, \dots s_n$:

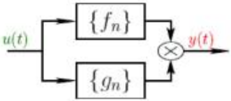
- **Sum :** $\rho_h \geq \min(\rho_f, \rho_g)$

$$H_n(s_{1:n}) = F_n(s_{1:n}) + G_n(s_{1:n})$$



- **Product :** $\rho_h \geq \min(\rho_f, \rho_g)$

$$H_n(s_{1:n}) = \sum_{p=1}^{n-1} F_p(s_{1:p}) G_{n-p}(s_{p+1:n})$$



- **Cascade :** $\rho_h \geq \rho_f$

$$H_n(s_{1:n}) = F_n(s_{1:n}) G_1(\widehat{s_{1:n}})$$

where $\widehat{s_{1:n}} = s_1 + s_2 + \ldots + s_n$

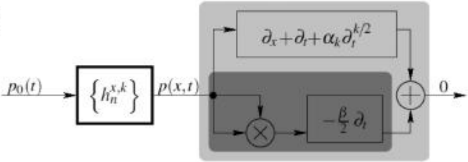
Outline

Kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$ and cancelling system

- For x>0, t>0, $\partial_x p + \partial_t p + \alpha_k \partial_t^{k/2} p = \frac{\beta}{2}\partial_t p^2$
- x-parameterized kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$

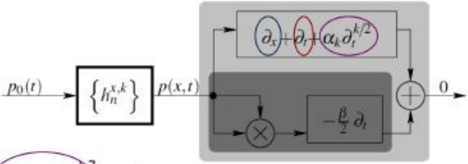
Kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$ and cancelling system

- For x>0, t>0, $\partial_x p + \partial_t p + \alpha_k \partial_t^{k/2} p = \frac{\beta}{2}\partial_t p^2$
- x-parameterized kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$
- Zero system:



Kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$ and cancelling system

- For x>0, t>0, $\partial_x p + \partial_t p + \alpha_k \partial_t^{k/2} p = \frac{\beta}{2}\partial_t p^2$
- x-parameterized kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$
- Zero system:

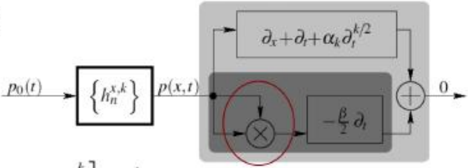


- From laws:

$$\partial_x H_n^{x,k}(s_{1:n}) + \left[\widehat{s_{1:n}} + \alpha_k \left(\widehat{s_{1:n}}\right)^{\frac{k}{2}}\right] H_n^{x,k}(s_{1:n})$$

Kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$ and cancelling system

- For x>0, t>0, $\partial_x p + \partial_t p + \alpha_k \partial_t^{k/2} p = \frac{\beta}{2}\partial_t p^2$
- x-parameterized kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$
- Zero system:



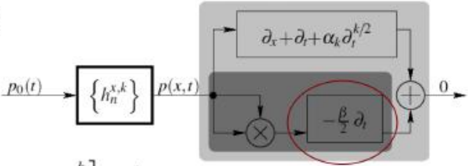
- From laws:

$$\partial_x H_n^{x,k}(s_{1:n}) + \left[\widehat{s_{1:n}} + \alpha_k \left(\widehat{s_{1:n}}\right)^{\frac{k}{2}}\right] H_n^{x,k}(s_{1:n})$$

$$\sum_{p=1}^{n-1} H_p^{x,k}(s_{1:p}) H_{n-p}^{x,k}(s_{p+1:n})$$

Kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$ and cancelling system

- For x>0, t>0, $\partial_x p + \partial_t p + \alpha_k \partial_t^{k/2} p = \frac{\beta}{2}\partial_t p^2$
- x-parameterized kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$
- Zero system:

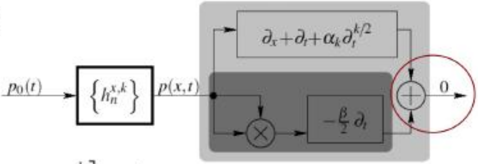


- From laws:

$$\partial_x H_n^{x,k}(s_{1:n}) + \left[\widehat{s_{1:n}} + \alpha_k \left(\widehat{s_{1:n}}\right)^{\frac{k}{2}}\right] H_n^{x,k}(s_{1:n})$$
$$-\frac{\beta}{2}\widehat{s_{1:n}} \sum_{p=1}^{n-1} H_p^{x,k}(s_{1:p}) H_{n-p}^{x,k}(s_{p+1:n})$$

Kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$ and cancelling system

- For x>0, t>0, $\partial_x p + \partial_t p + \alpha_k \partial_t^{k/2} p = \frac{\beta}{2}\partial_t p^2$
- x-parameterized kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$
- Zero system:

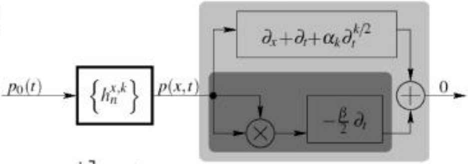


- From laws:

$$\partial_x H_n^{x,k}(s_{1:n}) + \left[\widehat{s_{1:n}} + \alpha_k \left(\widehat{s_{1:n}}\right)^{\frac{k}{2}}\right] H_n^{x,k}(s_{1:n})$$
$$-\frac{\beta}{2}\widehat{s_{1:n}} \sum_{p=1}^{n-1} H_p^{x,k}(s_{1:p}) H_{n-p}^{x,k}(s_{p+1:n}) = 0$$

Kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$ and cancelling system

- For x>0, t>0, $\partial_x p + \partial_t p + \alpha_k \partial_t^{k/2} p = \frac{\beta}{2}\partial_t p^2$
- x-parameterized kernels $\{h_n^{x,k}\}_{n\in\mathbb{N}^*}$
- Zero system:



- From laws:

$$\partial_x H_n^{x,k}(s_{1:n}) + \left[\widehat{s_{1:n}} + \alpha_k \left(\widehat{s_{1:n}}\right)^{\frac{k}{2}}\right] H_n^{x,k}(s_{1:n})$$
$$-\frac{\beta}{2}\widehat{s_{1:n}} \sum_{p=1}^{n-1} H_p^{x,k}(s_{1:p}) H_{n-p}^{x,k}(s_{p+1:n}) \quad = \quad 0$$

Linear ODEs

Boundary cnd. and solution

- If $x=0$, then $p(x=0,t)=p_0(t)$ (Identity system)
  $H_1^{x=0,k}(s_1) = 1$ and $H_n^{x=0,k}(s_{1:n}) = 0$ if $n \geq 2$

Boundary cnd. and solution

- If $x=0$, then $p(x=0,t)=p_0(t)$ (Identity system)

  $H_1^{x=0,k}(s_1) = 1$ and $H_n^{x=0,k}(s_{1:n}) = 0$ if $n \geq 2$

- Solution: $H_n^{x,k}(s_{1:n}) = G_n^{x,k}(s_{1:n}) \mathrm{e}^{-x\widehat{s_{1:n}}}$ with

  wave delay

$$G_1^{x,k}(s_{1:n}) = \mathrm{e}^{-\alpha_k s_1^{\frac{k}{2}}}$$

$$G_n^{x,k}(s_{1:n}) = \frac{\beta}{2}\widehat{s_{1:n}} \sum_{p=1}^{n-1} \int_0^x \mathrm{e}^{-\alpha_k (\widehat{s_{1:n}})^{\frac{k}{2}}(x-\xi)} G_p^{\xi,k}(s_{1:p}) G_{n-p}^{\xi,k}(s_{p+1:n}) \, \mathrm{d}\xi$$

Boundary cnd. and solution

- If $x=0$, then $p(x=0,t)=p_0(t)$ (Identity system)
  $H_1^{x=0,k}(s_1) = 1$ and $H_n^{x=0,k}(s_{1:n}) = 0$ if $n \geq 2$
- Solution: $H_n^{x,k}(s_{1:n}) = G_n^{x,k}(s_{1:n})\,\mathrm{e}^{-x\widehat{s_{1:n}}}$ with

$$
\begin{aligned}
G_1^{x,k}(s_{1:n}) &= \mathrm{e}^{-\alpha_k s_1^{\frac{k}{2}}} \\
G_n^{x,k}(s_{1:n}) &= \frac{\beta}{2}\widehat{s_{1:n}} \sum_{p=1}^{n-1} \int_0^x \mathrm{e}^{-\alpha_k (\widehat{s_{1:n}})^{\frac{k}{2}}(x-\xi)}\, G_p^{\xi,k}(s_{1:p}) G_{n-p}^{\xi,k}(s_{p+1:n})\,\mathrm{d}\xi
\end{aligned}
$$

- First kernels (k=0)

$$
\begin{aligned}
G_1^{x,0}(s_1) &= \mathrm{e}^{-\alpha_0 x} \\
G_2^{x,0}(s_{1:2}) &= \frac{\beta\,\widehat{s_{1:2}}}{2\alpha_0}\left(1-\mathrm{e}^{-\alpha_0 x}\right)
\end{aligned}
$$

Boundary cnd. and solution

- If $x=0$, then $p(x=0,t)=p_0(t)$ (Identity system)

  $H_1^{x=0,k}(s_1) = 1$ and $H_n^{x=0,k}(s_{1:n}) = 0$ if $n \geq 2$

- Solution: $H_n^{x,k}(s_{1:n}) = G_n^{x,k}(s_{1:n})\,\mathrm{e}^{-x\widehat{s_{1:n}}}$ with

$$
\begin{aligned}
G_1^{x,k}(s_{1:n}) &= \mathrm{e}^{-\alpha_k s_1^{\frac{k}{2}}} \\
G_n^{x,k}(s_{1:n}) &= \frac{\beta}{2}\widehat{s_{1:n}} \sum_{p=1}^{n-1} \int_0^x \mathrm{e}^{-\alpha_k (\widehat{s_{1:n}})^{\frac{k}{2}}(x-\xi)}\, G_p^{\xi,k}(s_{1:p}) G_{n-p}^{\xi,k}(s_{p+1:n})\, \mathrm{d}\xi
\end{aligned}
$$

- First kernels (k=1)

$$
\begin{aligned}
G_1^{x,1}(s_1) &= \mathrm{e}^{-\alpha_1 x\sqrt{s_1}}, \\
G_2^{x,1}(s_{1:2}) &= \frac{\beta\widehat{s_{1:2}}}{2\alpha_1} \frac{\mathrm{e}^{-\alpha_1 x\sqrt{s_1+s_2}} - \mathrm{e}^{-\alpha_1 x(\sqrt{s_1}+\sqrt{s_2})}}{-\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}}
\end{aligned}
$$

Outline

Deriving simple realizable structures

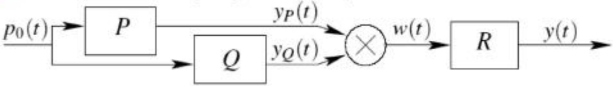
How to realize first kernels without multi-convolutions ?

n=1: linear filter (mono-conv.)

What about n=2 ?

Elementary 2nd order system

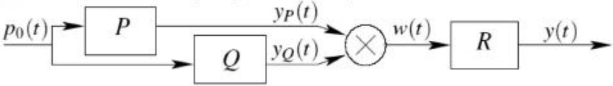
- Elementary system (P,Q,R: transfer fct):



$$K_2(s_{1:2}) = P(s_1)Q(s_2)R(\widehat{s_{1:2}}) \quad \text{if } n = 2,$$
$$K_n(s_{1:n}) = 0 \quad \text{otherwise.}$$

Elementary 2nd order system

- Elementary system (P,Q,R: transfer fct):



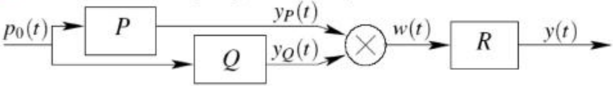
$$K_2(s_{1:2}) = P(s_1)Q(s_2)R(\widehat{s_{1:2}}) \quad \text{if } n = 2,$$
$$K_n(s_{1:n}) = 0 \quad \text{otherwise.}$$

- For k=0, $G_2^{x,0}(s_{1:2}) = \dfrac{\beta\,\widehat{s_{1:2}}}{2\alpha_0}\left(1-e^{-\alpha_0 x}\right)$

$$P(s) = Q(s) = 1, \text{ (identity systems)}$$
$$R(s) = \frac{\beta\left(1-e^{-\alpha_0 x}\right)}{2\alpha_0}\, s,$$

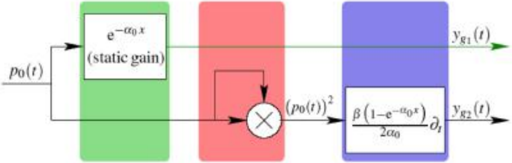
Elementary 2nd order system

- Elementary system (P,Q,R: transfer fct):



$$K_2(s_{1:2}) = P(s_1)Q(s_2)R(\widehat{s_{1:2}}) \quad \text{if } n=2,$$
$$K_n(s_{1:n}) = 0 \quad \text{otherwise.}$$

- For k=0, $G_2^{x,0}(s_{1:2}) = \frac{\beta\,\widehat{s_{1:2}}}{2\alpha_0}\left(1-e^{-\alpha_0 x}\right)$



Realistic case: k=1

- No straightforward identification:

$$G_2^{x,1}(s_{1:2}) = \frac{\beta \widehat{s_{1:2}}}{2\alpha_1} \frac{e^{-\alpha_1 x \sqrt{s_1+s_2}} - e^{-\alpha_1 x (\sqrt{s_1}+\sqrt{s_2})}}{-\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}}$$

Realistic case: k=1

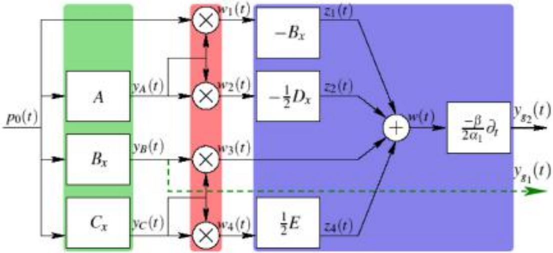
- No straightforward identification:

$$G_2^{x,1}(s_{1:2}) \;=\; \frac{\beta\widehat{s_{1:2}}}{2\alpha_1}\frac{\mathrm{e}^{-\alpha_1 x\sqrt{s_1+s_2}}-\mathrm{e}^{-\alpha_1 x(\sqrt{s_1}+\sqrt{s_2})}}{-\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}}$$

- Perfect squares & sum of elementary syst.:

$$\left[\frac{\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}}{\sqrt{s_1+s_2}+\sqrt{s_1}+\sqrt{s_2}}\cdot G_2^{x,k=1}(s_{1:2})\right]$$
$$= \Big[A(s_1)\mathbf{1}(s_2)B_x(\widehat{s_{1:2}})+\mathbf{1}(s_1)A(s_2)B_x(\widehat{s_{1:2}})$$
$$+A(s_1)A(s_2)D_x(\widehat{s_{1:2}})$$
$$-B_x(s_1)C_x(s_2)\mathbf{1}(\widehat{s_{1:2}})-C_x(s_1)B_x(s_2)\mathbf{1}(\widehat{s_{1:2}})$$
$$-C_x(s_1)C_x(s_2)E(\widehat{s_{1:2}})\Big]\frac{\beta}{4\alpha_1}\widehat{s_{1:2}},$$

Realistic case: 2nd order realization



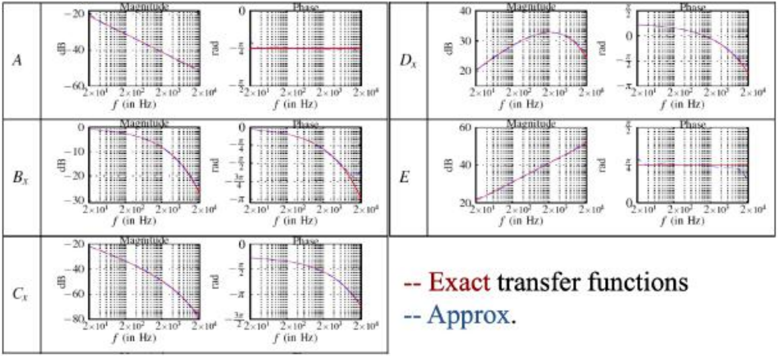
Structure composed of sums, products and linear filters with (irrational) transfer functions:

$A(s) = 1/\sqrt{s}$ $\quad B_x(s) = G_1^{x,1}(s) = \mathrm{e}^{-\alpha_1 x\sqrt{s}}$

$C_x(s) = \mathrm{e}^{-\alpha_1 x\sqrt{s}}/\sqrt{s}$ $\quad D(s) = \sqrt{s}\,\mathrm{e}^{-\alpha_1 x\sqrt{s}}$ $\quad E(s) = \sqrt{s}$

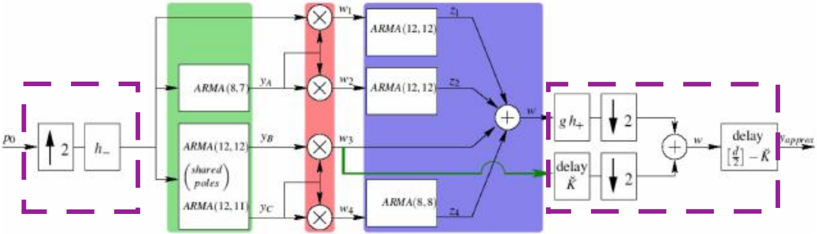
Outline

Bode diagrams of A,B,C,D,E for typical pipes



-- Exact transfer functions
-- Approx.

Digital 2nd order realization



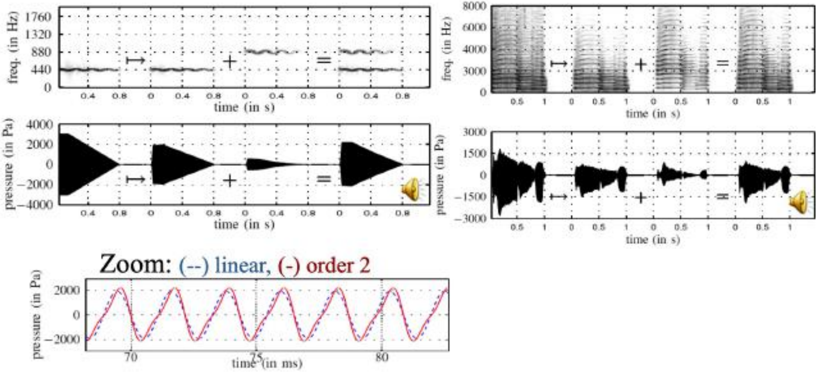
oversampling

irrational system

undersampling
& wave delay

Results for a typical trumpet pipe

- Ex.: 1.sinusoid with vibrato / 2.Chet Baker



Outline

Conclusion

Contributions

- Representation with poles and cuts of linear fractional/irrational systems
- Flexible method for the low-cost simulation based on approximation and optimization
- Suitable for real-time applications.
- Application to weakly nonlinear systems with Volterra series

Perspectives

- Open question: optimal choice of cut for approximation ?
- Open question: optimal placement of poles, once the cut has been chosen?

– The end –

Thank you for your attention

Acknowledgements: M. Hasler, D. Matignon, R. Mignot, V. Smet.