Stokes and Navier-Stokes equations with friction laws at the boundary of the domain and coupling of two fluids

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Problem 1. $\mathbf{u} = (u_1, \cdots)$: fluid velocity, p : pressure. Ω : flow domain, Γ : boundary of Ω.

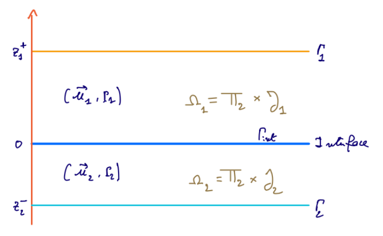
Navier-Stokes equations with a friction law

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in }]0,+\infty[\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in }]0,+\infty[\times\Omega, \\ \mathbf{u}\cdot\mathbf{n} = 0 & \text{on }]0,+\infty[\times\Gamma, \\ -\nu\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{u}H(|\mathbf{u}|) & \text{on }]0,+\infty[\times\Gamma, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, \end{cases}$$

- $\nu > 0$ is the kinematic viscosity,
- $\nabla\cdot = div$ is the divergence operator, Δ the Laplacian,
- $\Omega \subset \mathbb{R}^3$ is an open bounded Lipschitz domain, $\Gamma = \partial\Omega$,
- $\mathbf{f}$ is an external force, $\mathbf{u}_0$ the initial velocity.

Problem 2.

Coupling of two fluids thru a rigid interface (rigid lid hypothesis)



- $(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1(\mathbf{x}_h, z_1), \mathbf{u}_2(\mathbf{x}_h, z_2))$, $\mathbf{u}_i = (\mathbf{u}_{i,h}, w)$, $\mathbf{u}_{i,h} = (u_{i,x}, u_{i,y})$, velocities
- $\mathbf{x}_h \in \mathbb{T}_2$, two dimensional torus, $z_1 \in J_1 = [0, z_1^+]$, $z_2 \in J_2 = [z_2^-, 0]$, $z_1^+ > 0$ and $z_2^- < 0$,
- Interface : $\Gamma_{int} = \{(\mathbf{x}_h, 0), \mathbf{x}_h \in \mathbb{T}_2\}$,
- Top : $\Gamma_1 = \{(\mathbf{x}_h, z_1^+), \mathbf{x}_h \in \mathbb{T}_2\}$. Bottom : $\Gamma_2 = \{(\mathbf{x}_h, z_2^-), \mathbf{x}_h \in \mathbb{T}_2\}$,

$$\begin{cases} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \mathbb{T}_2 \times J_i, \\ \nabla \cdot \mathbf{u}_i = 0 & \text{in } \mathbb{T}_2 \times J_i, \\ w_i = \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_{int}, \\ \nu_i \dfrac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -\alpha(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}), & \text{on } \mathbb{T}_2 \times \Gamma_{int} \ (i \neq j), \\ \mathbf{u}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_i. \end{cases}$$

Questions :

- Behavior of the solutions when $\alpha \to \infty$?
- Stable numerical scheme and simulations

Problem 1. Friction law : assumptions

$-\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}|_\Gamma = \mathbf{u} H(|\mathbf{u}|)|_\Gamma = g(\mathbf{u})$. Assume

- $(\mathbf{H}_1)$: $\forall\, \mathbf{v} \in \mathbb{R}^3$, $0 < C_1 \leq H(|\mathbf{v}|) \leq C_2(1 + |\mathbf{v}|)$ or $C_1 |\mathbf{v}|^\alpha \leq H(|\mathbf{v}|) \leq C_2(1 + |\mathbf{v}|)$, $(0 < \alpha < 1)$,
- $(\mathbf{H}_2)$: $H \in C^1(\mathbb{R})$, $H' \in L^\infty(\mathbb{R})$, and $H' \geq 0$ *a.e.*

Typical examples

$$H(|\mathbf{v}|) = Cte = \alpha \text{ : Navier's Law}$$
$$H(|\mathbf{v}|) = C_D |\mathbf{v}| \text{ : Glauker-Manning's law}$$

C_D : drag coefficient.

Problem 1. Energy balance

Assume $\mathbf{u}_0 \in L^2(\Omega)$, $\mathbf{u}_0 \cdot \mathbf{n}|_\Gamma = 0$, $\mathbf{f} \in L^2_{loc}(\mathbb{R}_+, H^{-1}(\Omega)^d)$.
Let $(\mathbf{u}, p)$ such that $\nabla \cdot \mathbf{u} = 0$, $\mathbf{u} \cdot \mathbf{n}|_\Gamma = 0$ be a "strong" solution to the Navier-Stokes equations. Then, $\forall\, t > 0$,

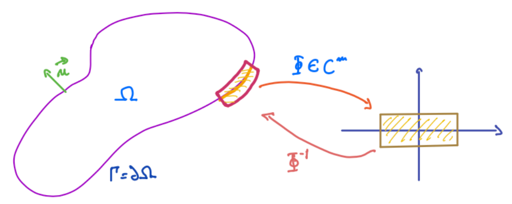
$$\frac{1}{2}\int_\Omega |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} + \nu \int_0^t \int_\Omega |\nabla \mathbf{u}(s,\mathbf{x})|^2 d\mathbf{x} ds +$$
$$\int_0^t \int_\Gamma |\mathbf{u}(s,\mathbf{x})|^2 H(|\mathbf{u}(s,\mathbf{x})|) d\mathbf{x} ds = \frac{1}{2}\int_\Omega |\mathbf{u}_0(x)|^2 d\mathbf{x} + \int_0^t \langle \mathbf{f}(s), \mathbf{u}(s,\cdot)\rangle ds$$

Hence for all $T > 0$, as $\mathbf{v} \to ||\nabla \mathbf{v}||^2_{\Omega;0,1} + ||\mathbf{v}||_{\Gamma;0,p}$ is a norm equivalent to $||\cdot||_{\Omega;1,2}$ on $H^1(\Omega)^3$, $1 < p \leq 4$,

$$||\mathbf{u}||_{L^2([0,T],H^1(\Omega)^d)} + ||\mathbf{u}||_{L^\infty([0,T],L^2(\Omega)^d)} \leq$$
$$C(||\mathbf{u}_0||_{\Omega;0,2}, H, ||\mathbf{f}||_{L^2([0,T],H^{-1}(\Omega)^d)}, \nu),$$

Problem 1. Function space

$\mathbf{u}\cdot\mathbf{n}|_\Gamma = 0$. Let $\Omega \subset \mathbb{R}^d$, bounded, of class C^m $(m \geq 1,\ d = 2, 3)$



$$\mathcal{W}_m = \{\varphi \in C^m(\overline{\Omega})^d,\ \varphi\cdot\mathbf{n}|_\Gamma = 0\},$$

$$W = \{\mathbf{u} \in H^1(\Omega)^d,\ \mathbf{u}\cdot\mathbf{n}|_\Gamma = 0\}$$

$$||\mathbf{u}||_W = \left(\int_\Omega |\nabla\mathbf{u}|^2 + \int_\Gamma |\mathbf{u}|^2\right)^{\frac{1}{2}} \approx ||\mathbf{u}||_{\Omega;1,2}$$

Theorem

(Chacón-Lewandowski. 2014) $\mathcal{W}(\Omega)$ is dense in $W(\Omega)$.

Problem 1. Properties of the Friction law

Let

$$G : \begin{cases} W \to W' \\ \mathbf{v} \to G(\mathbf{v}) \end{cases}$$

$$\forall\, \mathbf{w} \in W, \quad \langle G(\mathbf{v}), \mathbf{w} \rangle = \int_{\Gamma} \mathbf{v} \cdot \mathbf{w}\, H(|\mathbf{v}|) d\sigma.$$

Lemma

Assume that ($\mathbf{H}_1$) and ($\mathbf{H}_2$) hold. Then G satisfies, for all $\mathbf{v}, \mathbf{w} \in W$:

- $||G(\mathbf{v})||_{W'} \leq C(1 + ||\mathbf{v}||^2_W)$,
- $C||\mathbf{v}||^2_{\Gamma;0,2} \leq \langle G(\mathbf{v}), \mathbf{v} \rangle$ *(coercivity),*
- $0 \leq \langle G(\mathbf{v}) - G(\mathbf{w}), \mathbf{v} - \mathbf{w} \rangle$ *(monotony),*
- $||G(\mathbf{v}) - G(\mathbf{w})||_{W'} \leq C(||\mathbf{v}||_W + ||\mathbf{w}||_W)||\mathbf{v} - \mathbf{w}||_W$,
- *G is continous and compact.*

Problem 1. Transport-diffusion operators

Let

$$B : \begin{cases} W \times W \to W' \\ (\mathbf{v}, \mathbf{w}) \to B(\mathbf{v}, \mathbf{w}) \end{cases}, \quad \forall \mathbf{z} \in W,\ \langle B(\mathbf{v}, \mathbf{w}), \mathbf{z} \rangle = \int_\Omega (\mathbf{v} \cdot \nabla)\mathbf{w} \cdot \mathbf{z}.$$

Lemma

$\forall\, \mathbf{v}, \mathbf{w} \in W$,

- $||B(\mathbf{v}, \mathbf{w})||_{W'} \leq ||\mathbf{v}||_W ||\mathbf{w}||_W$,
- *When* $\nabla \cdot \mathbf{v} = 0$, *then* $\langle B(\mathbf{v}, \mathbf{w}), \mathbf{w} \rangle = 0$.

$$A : \begin{cases} W \to W' \\ \mathbf{v} \to A\mathbf{v} \end{cases}, \quad \forall \mathbf{w} \in W,\ \langle A\mathbf{v}, \mathbf{w} \rangle = \nu \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w}.$$

Momentum Equation : find $\mathbf{u}, p$ such that

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + B(\mathbf{u}, \mathbf{u}) + A\mathbf{u} + G(\mathbf{u}) + \nabla p = f, \quad \nabla \cdot \mathbf{u} = 0}$$

$Q = [0, T] \times \Omega$. The pressure is in $L^{4/3}(Q)$. More specifically : Let (u, p) be a strong solution to the NSE. Then

$$\left(\int_Q |p(t, \mathbf{x})|^{4/3} d\mathbf{x} dt \right)^{4/3} = ||p||_{Q;0,4/3} \leq C(T, H, \mathbf{u}_0, \mathbf{f}, \nu).$$

Essential ingredient of the proof. Solve

$$\begin{cases} -\Delta \Psi = p|p|^{-2/3} & \text{in } \Omega, \\ \dfrac{\partial \Psi}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega \Psi(\mathbf{x}) d\mathbf{x} = 0 & \end{cases}$$

then take $\mathbf{w} = \nabla \Psi$ as test function in the equation for the velocity.

Consequence :

$$\partial_t \mathbf{v} \in L^{4/3}([0, T], W_4'), \quad W_4 = W \cap H^4(\Omega)^4$$

Problem 1. Variational formulation and existence result

Weak solution : Find

$$(\mathbf{u}, p) \in L^2(W) \cap L^\infty(L^2) \times L^{4/3}(Q), \quad \text{s.t. } \partial_t \mathbf{u} \in L^{4/3}([0, T], W_4'),$$

such that $\forall\, (\mathbf{v}, q) \in L^4(W_4) \times L^2(L^2)$,

$$\int_0^T \langle \partial_t \mathbf{u}(t) + B(\mathbf{u}, \mathbf{u}) + A(\mathbf{u}) + G(\mathbf{u}), \mathbf{v} \rangle dt - \int_0^T p(t, \mathbf{x}) \nabla \cdot \mathbf{v}(t, \mathbf{x}) d\mathbf{x} dt +$$

$$\int_0^T q(t, \mathbf{x}) \nabla \cdot \mathbf{u}(t, \mathbf{x}) d\mathbf{x} dt = \int_0^T \langle \mathbf{f}, \mathbf{v} \rangle dt$$

Theorem

Chacón-Lewandowski, 2014. The NSE have a weak solution $(\mathbf{u}, p) \in L^2(W) \cap L^\infty(L^2) \times L^{4/3}(Q)$.

Inspired by the work of Bulíček, M. and Málek, J. and Rajagopal, K. R. 2007.

Problem 1. Proof by Regularization

Let

$$P_\varepsilon : \begin{cases} W \to H^1(\Omega) \\ \mathbf{v} \to p_\varepsilon = P_\varepsilon(\mathbf{v}) \end{cases}$$

where

$$\begin{cases} -\varepsilon \Delta p_\varepsilon + \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial p_\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma, \quad \displaystyle\int_\Omega p_\varepsilon(\mathbf{x}) d\mathbf{x} = 0 \end{cases}$$

and for a given mollifier $\rho_\alpha = \rho_\alpha(\mathbf{x})$, for $\mathbf{v} \in W$, $\overline{\mathbf{v}}$ its extension to $H^1(\mathbb{R}^3)^3$,

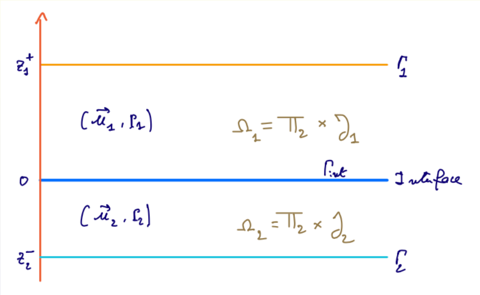
$$V_\alpha(\mathbf{v})(\mathbf{x}) = (\rho_\alpha \star \overline{\mathbf{v}})(\mathbf{x}) = \int_{\mathbb{R}^3} \rho_\alpha(\mathbf{y} - \mathbf{x}) \overline{\mathbf{v}}(\mathbf{y}) d\mathbf{y}$$

Then solve

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + B(V_\alpha(\mathbf{u}), \mathbf{u}) + A\mathbf{u} + G(\mathbf{u}) + \nabla P_\varepsilon(\mathbf{u}) = f.}$$

and take the limit as $(\alpha, \varepsilon) \to (0, 0)$.

Problem 2. Recall : coupled problem with friction



$$(Co./Fr.) : \begin{cases} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \mathbb{T}_2 \times J_i, \\ \nabla \cdot \mathbf{u}_i = 0 & \text{in } \mathbb{T}_2 \times J_i, \\ w_i = \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_{int}, \\ \nu_i \dfrac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = -\alpha(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}), & \text{on } \mathbb{T}_2 \times \Gamma_{int} \ \ (i \neq j), \\ \mathbf{u}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_i. \end{cases}$$

Problem 2. Functional framework

$$\mathcal{W}_i = \{\mathbf{u} \in C^\infty(\mathbb{T}_2 \times J_i), \quad \mathbf{u}_{|\Gamma_i} = 0, \quad \mathbf{u}\cdot\mathbf{n}_i|_{\Gamma_{int}} = 0, \quad \nabla\cdot\mathbf{u}_i = 0\},$$

equipped with $||\mathbf{u}||_{i,1} = ||\nabla\mathbf{u}||_{L^2(\mathbb{T}_2\times J_i)}$. Let

$$W_i = \overline{\mathcal{W}_i}^{||\cdot||_{i,1}}, \quad W = W_1 \times W_2,$$

$$W_0 = \{(\mathbf{u}_1, \mathbf{u}_2) \in W, \ \mathbf{u}_{1,h|\Gamma_{int}} = \mathbf{u}_{2,h|\Gamma_{int}} \ a.e. \text{ in } \Gamma_{int}\},$$

$\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W,$

$$\Lambda(\mathbf{U},\mathbf{V}) = \nu_1 \int_{\mathbb{T}_2\times J_1} \nabla\mathbf{u}_1\cdot\nabla\mathbf{v}_1 + \nu_2 \int_{\mathbb{T}_2\times J_2} \nabla\mathbf{u}_2\cdot\nabla\mathbf{v}_2.$$

$W_0 = \ker tr_{\Gamma_{int}}(u_1 - u_2)$, $\text{codim}(W_0) = 1$,

$$W = W_0 \oplus^\perp \text{vect}\,\mathbf{\Phi}, \quad \mathbf{\Phi} = (\phi_1, \phi_2)$$

Problem 2. Weak Solutions

Let

$$\int_{\mathbb{T}_2 \times J_1} \mathbf{f}_1 \cdot \mathbf{v}_1 + \int_{\mathbb{T}_2 \times J_2} \mathbf{f}_2 \cdot \mathbf{v}_2 = (\mathbf{F}, \mathbf{V})$$

Definition

(weak solution) A couple $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2) \in W$ is a weak solution to Problem (Co./Fr.) when $\forall\, \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W$,

$$\Lambda(\mathbf{U}, \mathbf{V}) + \alpha \int_{\Gamma_{int}} (\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) \cdot (\mathbf{v}_{1,h} - \mathbf{v}_{2,h}) = (\mathbf{F}, \mathbf{V}).$$

Theorem

Problem (Co./Fr.) has a unique weak solution, denoted $\mathbf{U}_\alpha$.

Straightforward by Lax-Milgram Theorem.

Question : behavior of $(\mathbf{U}_\alpha)_{\alpha > 0}$ when $\alpha \to \infty$?

The solution $\mathbf{U}_\alpha$ of Problem (Co./Fr.) verifies the energy balance :

$$\Lambda(\mathbf{U}_\alpha, \mathbf{U}_\alpha) + \alpha \int_{\Gamma_{int}} |\mathbf{u}_{1,h,\alpha} - \mathbf{u}_{2,h,\alpha}|^2 = (\mathbf{F}, \mathbf{U}_\alpha)$$

In particular :

- The familly $(\mathbf{U}_\alpha)_{\alpha>0}$ is bounded in W uniformly in α,
- $\displaystyle\int_{\Gamma_{int}} |\mathbf{u}_{1,h,\alpha} - \mathbf{u}_{2,h,\alpha}|^2 \to 0$ as $\alpha \to \infty$.

Consequence : There exists $(\alpha_n)_{n\in\mathbb{N}}$, $\alpha_n \xrightarrow[n\to\infty]{} +\infty$, $\mathbf{U} \in W_0$, such that $(\mathbf{U}_{\alpha_n})_{n\in\mathbb{N}}$ weakly converges to $\mathbf{U}$ in W.

Problem 2. Limit problem when $\alpha \to \infty$: continuous BC's

$$(Co./Con.) : \begin{cases} -\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i & \text{in } \mathbb{T}_2 \times J_i, \\ \nabla \cdot \mathbf{u}_i = 0 & \text{in } \mathbb{T}_2 \times J_i, \\ w_i = \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_{int}, \\ \mathbf{u}_1 = \mathbf{u}_2, & \text{on } \mathbb{T}_2 \times \Gamma_{int} \\ \mathbf{u}_i = 0 & \text{on } \mathbb{T}_2 \times \Gamma_i. \end{cases}$$

Theorem

(Legeais-Lewandowski, Applied Maths Letters, 2022.) The familly $(\mathbf{U}_\alpha)_{\alpha>0}$ strongly converges in W to the unique weak solution $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2) \in W_0$ of Problem (Co./Con.), in the sense :

$$\forall \mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in W_0, \quad \Lambda(\mathbf{U}, \mathbf{V}) = (\mathbf{F}, \mathbf{V}).$$

Initialization : We solve the following initial problem on the upper part to get a first field $(\mathbf{u}_1^{\alpha,0}, P_1^{(0)})$:

$$-\nu_1 \Delta \mathbf{u}_1^{\alpha,0} + \nabla P_1^{(0)} = \mathbf{f}_1, \qquad (1)$$

$$\nabla \cdot \mathbf{u}_1^{\alpha,0} = 0, \qquad (2)$$

$$\nu_1 \frac{\partial \mathbf{u}_{1,h}^{\alpha,0}}{\partial \mathbf{n}_1}|_{\Gamma_{int}} = -\alpha \mathbf{u}_{1,h}^{\alpha,0}, \quad \mathbf{u}_{1,h}^{\alpha,0} \cdot \mathbf{n}_1|_{\Gamma_{int}} = 0. \qquad (3)$$

$$\mathbf{u}_{1,h}^{\alpha,0}|_{\Gamma_1} = 0, \qquad (4)$$

Problem 2. Schwarz like Algorithm-iterations

Once the upper field $(\mathbf{u}_1^{\alpha,n}, P_1^{(n)})$ is calculated for a given $n \geq 0$, determine the bottom field $(\mathbf{u}_2^{\alpha,n}, P_2^{(n)})$ by solving :

$$-\nu_2 \Delta \mathbf{u}_2^{\alpha,n} + \nabla P_2^{(n)} = \mathbf{f}_2, \quad \nabla \cdot \mathbf{u}_2^{\alpha,n} = 0, \tag{5}$$

$$\nu_2 \frac{\partial \mathbf{u}_{2,h}^{\alpha,n}}{\partial \mathbf{n}_2}|_{\Gamma_I} = -\alpha(\mathbf{u}_{2,h}^{\alpha,n} - \mathbf{u}_{1,h}^{\alpha,n}), \quad \mathbf{u}_{2,h}^{\alpha,n} \cdot \mathbf{n}_2 = 0. \tag{6}$$

$$\mathbf{u}_{2,h}^{\alpha,n}|_{\Gamma_2} = 0, \tag{7}$$

and go from n to $n+1$ in calculating $(\mathbf{u}_1^{\alpha,n+1}, P_1^{(n+1)})$ by solving

$$-\nu_1 \Delta \mathbf{u}_1^{\alpha,n+1} + \nabla P_1^{(n+1)} = \mathbf{f}_1, \quad \nabla \cdot \mathbf{u}_1^{\alpha,n+1} = 0, \tag{8}$$

$$\nu_1 \frac{\partial \mathbf{u}_{1,h}^{\alpha,n+1}}{\partial \mathbf{n}_1}|_{\Gamma_{int}} = -\alpha(\mathbf{u}_{1,h}^{\alpha,n+1} - \mathbf{u}_{2,h}^{\alpha,n}), \quad \mathbf{u}_{1,h}^{\alpha,n+1} \cdot \mathbf{n}_1 = 0 \tag{9}$$

$$\mathbf{u}_{1,h}^{\alpha,n+1}|_{\Gamma_1} = 0. \tag{10}$$

Problem 2. Numerical convergence

The simulations were carried out with the software FreeFem $++$

α	n
10	4026
10^2	984
10^3	408
10^4	221
10^5	137
10^6	54
10^9	9

To check the numerical convergence of the method, we study the error term $\|\mathbf{U}^{\alpha,n+1} - \mathbf{U}^{\alpha,n}\|_{L^2}$. On the left, for a given α, we have the first value of n for which

$$\|\mathbf{U}^{\alpha,n+1} - \mathbf{U}^{\alpha,n}\|_{L^2} < 10^{-3}.$$

The method is always converging, and the convergence is almost instantaneous for large α ($> 10^6$).

The complete theoretical proof of the convergence is in progress.

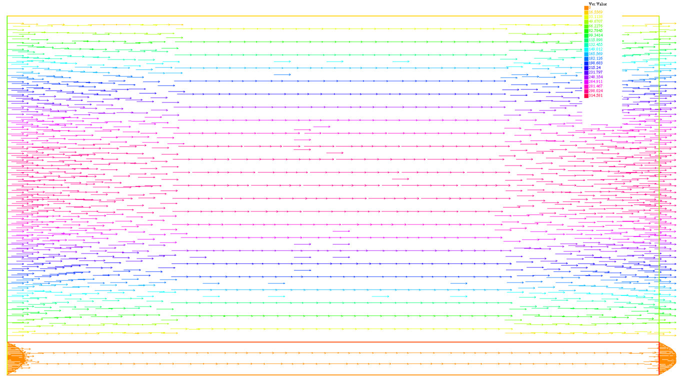


FIGURE – Velocities, $f_1 = (1, -1) = f_2$, $n = 9$, $\alpha = 10^9$