Projection scheme for the resolution of the Navier-Stokes equation in Discrete Exterior Calculus

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La Rochelle Université

KAUST
King Abdullah University of Science and Technology

Geometric integrators

- Schemes preserving the geometrical structure of equations
- Better reproduction of physical properties

▷ (Multi)-symplectic integrators

Hamiltonian formulation $\iota_X \omega = \mathrm{d}\, H \quad \omega \in \Lambda^2$

☞ Preservation of ω, $\Phi^* \omega = \omega$

▷ Invariant schemes

Flows and dissipative systems

☞ Schemes that preserve symmetries at the discrete level

D. Razafindralandy et al, A review of some geometric integrators, *AMSES Journal*, vol 5, 2018.

▷ Discrete Exterior Calculs

Follows exactly the cohomological structure

☞Preservation of the property $\mathrm{d}^2 = 0$

Illustration, relation between Exterior Calculus and tensor calculus

Exterior Calculus		Calculus in $\mathbb{R}^3$
Differential forms $\omega \in \Lambda^k$		**vector fields/tensors**
$\mathbf{u}^\flat = (u_i = g_{ij}u^j) \in \Lambda^1$	$\xleftrightarrow{\sharp,\flat}$	$\mathbf{u} = (u^i) \in \mathbb{R}^3$
$\Lambda^0 \xrightarrow{\mathrm{d}} \Lambda^1 \xrightarrow{\mathrm{d}} \Lambda^2 \xrightarrow{\mathrm{d}} \Lambda^3$		$\mathbb{R} \xrightarrow{\mathrm{grad}} \mathbb{R}^3 \xrightarrow{\mathrm{curl}} \mathbb{R}^3 \xrightarrow{\mathrm{div}} \mathbb{R}$
$\mathrm{d}^2 = 0$		$\mathrm{curl}\,\mathrm{grad} = 0, \mathrm{div}\,\mathrm{curl} = 0$
$\int_{\partial V} \omega = \int_V d\omega$	Stokes	$\int_{\partial V} \iota\mathbf{u}.dS = \int_V \mathrm{div}\, u dV, \ldots$

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Hodge Star operator

- Hodge Operator $\star$ is an isomorphisme of $\Lambda^k(M) \longrightarrow \Lambda^{n-k}(M)$

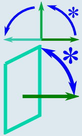
  By definition $\tau \wedge \star\omega = g(\tau, \omega) Vol$ where g metric and Vol volume form

  In $\mathbb{R}^2$ using Euclidean metric and $Vol = dx \wedge dy$

  $\star 1 = dx \wedge dy, \quad \star dx = dy, \quad \star dy = -dx$

  In $\mathbb{R}^3$ using Euclidean metric and $Vol = dx \wedge dy \wedge dz$

  $\star 1 = dx \wedge dy \wedge dz, \quad \star dx = dy \wedge dz, \quad \star(dx \wedge dy) = dz, \ldots$

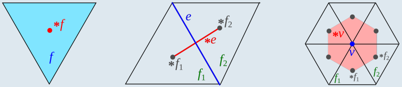


Domain Discretization: Primal mesh and Dual mesh

- Simplicial complex K in $\mathbb{R}^N \equiv$ Collection of simplices

  - Each k-simplex is associated to a $(n-k)$-cell.

  ▷ Primal simplices and their corresponding dual cells in 2D



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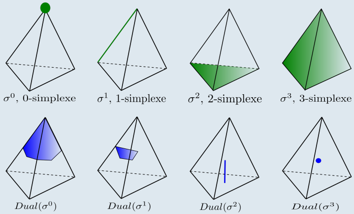

- The simplices and the dual cells are oriented.

  K_k: set of k-oriented simplices $\qquad *K_k$: set of $(n-k)$-oriented cells

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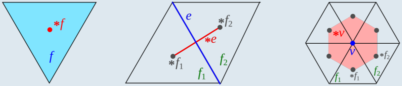

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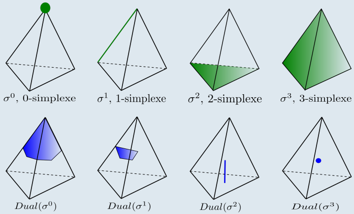

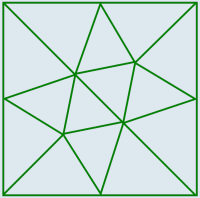

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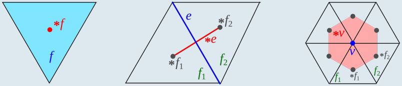

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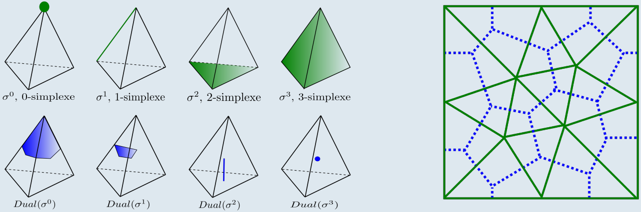

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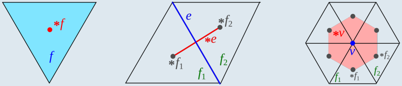

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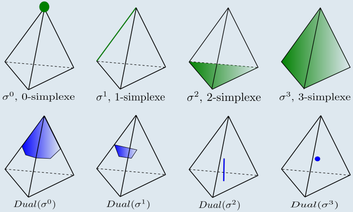

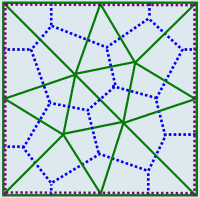

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Discretization of differential forms

- From K differential forms to cochains: De Rham

$$R : \begin{array}{ccc} \Lambda^k(M) & \longrightarrow & \Lambda^k(K) \\ \omega & \longmapsto & R\omega \end{array} \quad \text{Where } R\omega : \sigma \in K_k \longmapsto \langle R\omega, \sigma \rangle = \begin{cases} \int_\sigma \omega & \text{si } k \geq 1 \\ \omega(\sigma) & \text{si } k = 0 \end{cases}$$

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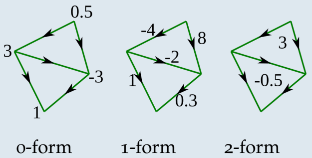
Primal k-cochain

$$\Lambda_k(K) = \{ \sum_{\sigma \in K_k} z_\sigma \sigma, \quad z_\sigma \in \mathbb{Z} \}$$

$$\Lambda^k(K) = Hom(\Lambda_k(K), \mathbb{R})$$

$$\omega \in \Lambda^k(K) : \quad \sigma \in K_k \longmapsto \langle \omega, \sigma \rangle \in \mathbb{R}$$



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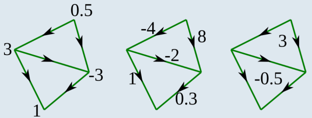


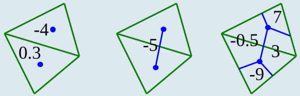
0-form 1-form 2-form

Dual k-cochain

$$\Lambda_k(*K) = \{ \sum_{*\sigma \in *K_k} z_{*\sigma} * \sigma, \quad z_{*\sigma} \in \mathbb{Z} \}$$

$$\Lambda^k(*K) = Hom(\Lambda_k(*K), \mathbb{R})$$

$$\omega \in \Lambda^k(*K): \quad \sigma \in *K_k \longmapsto \langle \omega, *\sigma \rangle \in \mathbb{R}$$



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Discrete Exterior Derivative and Boundary operator

Stokes

$$\int_\sigma \mathrm{d}\,\omega = \int_{\partial\sigma} \omega$$

$$\langle \mathrm{d}\omega, \sigma \rangle = \langle \omega, \partial\sigma \rangle$$

Boundary operator ∂

$\Lambda_k(K)$=SpanK_k = $\{k$-chaines$\}$.

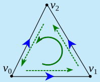
$\partial_k : \Lambda_K \longrightarrow \Lambda_{k-1} \qquad \partial_k[v_0...v_k] = \sum_{i=0}^k (-1)^i [v_0...\hat{v}_i...v_k]$

$\partial[v_0v_1] = v_1 - v_0$

$\partial[v_1v_2] = v_2 - v_1$

$\partial[v_0v_2] = v_2 - v_0$

$\partial[v_0v_1v_2] = [v_1v_2] - [v_0v_2] + [v_0v_1]$

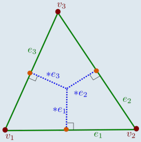


Discrete Hodge Operator

- Diagonal Hodge $\star : k$-forms$\longrightarrow n-k$-forms

$$\frac{< \star\omega, *\sigma >}{|*\sigma|} = \frac{< \omega, \sigma >}{|\sigma|}$$

$$[H]_{ij} = \frac{|*\sigma_i^k|}{|\sigma_j^k|}, \qquad H_1 = \begin{pmatrix} \frac{|*e_1|}{|e_1|} & 0 & 0 \\ 0 & \frac{|*e_2|}{|e_2|} & 0 \\ 0 & 0 & \frac{|*e_3|}{|e_3|} \end{pmatrix}$$



- Analytical discrete Hodge

R. AYOUB et al, 'A new Hodge operator in Discrete Exterior Calculus, Application to fluid mechanics', 2020- *Pure and Applied Analysis*

Tensor Formulation ⇒ Exterior Calculus Formulation (Stream Function)

Tensor formulation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) + \dfrac{1}{\rho}\operatorname{grad} p - \nu \Delta u + \beta g \theta \, e_y = 0, \\ \operatorname{div} u = 0, \\ \dfrac{\partial \theta}{\partial t} + \operatorname{div}(u\theta) - k\Delta\theta = 0. \end{cases}$$

$\flat$, $(v = u^\flat)$

$\operatorname{grad} p^\flat = \mathrm{d}\, p, \quad (\operatorname{curl} u)^\flat = \star \mathrm{d}\, v, \quad (\Delta u)^\flat = \mathrm{d}\,\delta v + \delta\,\mathrm{d}\, v$

Exterior calculus formulation

$$\begin{cases} \dfrac{\partial v}{\partial t} + i_{v^\sharp}\,\mathrm{d}\, v + \dfrac{1}{\rho}\,\mathrm{d}\,\overline{p} - \nu \delta dv + \beta g \theta \,\mathrm{d}\, y = 0, \\ \delta v = 0, \\ \dfrac{\partial \theta}{\partial t} - \delta(\theta v) + k\delta\,\mathrm{d}\,\theta = 0. \end{cases}$$

$\overline{p} = p + \frac{1}{2}\rho u^2$

Stream function formulation

$\delta v = 0 \implies v = -\star d\psi$

d to eliminate the pressure $(\mathrm{d}^2\, \overline{p} = 0)$

$$\begin{cases} \dfrac{\partial}{\partial t}\mathrm{d}\star\mathrm{d}\,\psi + \mathrm{d}(i_{v^\sharp}\,\mathrm{d}\star\mathrm{d}\,\psi) - \nu\,\mathrm{d}\,\delta\,\mathrm{d}\star\mathrm{d}\,\psi - \beta g\,\mathrm{d}(\theta\,\mathrm{d}\, y) = 0, \\ v = -\star\mathrm{d}\,\psi, \\ \dfrac{\partial \theta}{\partial t} - \delta(\theta v) + k\delta\,\mathrm{d}\,\theta = 0. \end{cases}$$

Tensor Calculus

Step 1:

$$\begin{cases} \dfrac{\tilde{u}^{n+1}-u^n}{\Delta t} + \dfrac{3}{2}\operatorname{div}(u^n \otimes u^n) - \dfrac{1}{2}\operatorname{div}(u^{n-1} \otimes u^{n-1}) \\ \qquad + \dfrac{1}{\rho}\operatorname{grad} p^n - \nu \Delta \tilde{u}^{n+1} = g_M^{n+1}, \quad \text{in } M \\ \\ \tilde{u}^{n+1} = u_\Gamma^{n+1}, \qquad \text{on } \Gamma \end{cases}$$

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A.J. Chorin, 'Numerical solution of the Navier–Stokes equations', *mathematics and computing journal*
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Where N is the exterior normal to Γ.

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Exterior Calculus

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$$\begin{cases} \dfrac{\tilde{v}^{n+1} - v^n}{\Delta t} - \nu(\delta\,\mathrm{d} + \mathrm{d}\,\delta)\tilde{v}^{n+1} = g_M^{n+1} - \\ \left(\frac{3}{2} i_{u^n}\,\mathrm{d}\,v^n - \frac{1}{2} i_{u^{n-1}}\,\mathrm{d}\,v^{n-1}\right) - \dfrac{1}{\rho}\,\mathrm{d}\,\bar{p}^n, & \text{in } M \\ \tilde{v}^{n+1} = u_\Gamma^{\flat\, n+1}, & \text{sur } \Gamma \end{cases}$$

Step 2:

$$\begin{cases} \star\,\mathrm{d}\star\mathrm{d}\,\tilde{\bar{p}}^{n+1} = \dfrac{\rho}{\Delta t}\delta \tilde{v}^{n+1}, & \text{in } M \\ \mathrm{tr}(\star\,\mathrm{d}\,\tilde{\bar{p}}^{n+1}) = 0, & \text{sur } \Gamma \end{cases}$$

Where tr trace operator on Γ $\quad (\mathrm{tr}\,\omega = \omega_{|\Gamma})$

Step 3:

$$\begin{aligned} \bar{p}^{n+1} &= \tilde{\bar{p}}^{n+1} + \bar{p}^n \\ v^{n+1} &= \tilde{v}^{n+1} - \frac{\Delta t}{\rho}\,\mathrm{d}\,\tilde{\bar{p}}^{n+1} \end{aligned}$$

A.J. Chorin, 'Numerical solution of the Navier–Stokes equations', *mathematics and computing journal*
R. Temam, 'Navier–Stokes Equations: Theory and Numerical Analysis', *North-Holland, Amsterdam*

Formulation of the Neumann boundary condition in DEC

$$\begin{cases} \star\, \mathrm{d} \star \mathrm{d}\, p = g_M, & \text{in } M \\ \mathrm{tr} \star \mathrm{d}\, p = g_\Gamma\, vol_\Gamma, & \text{on } \Gamma \end{cases}$$

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p primal cochain

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***p* primal cochain**

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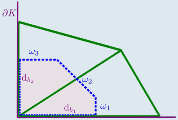

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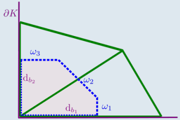

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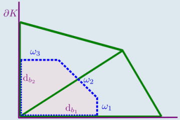

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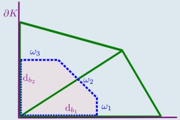

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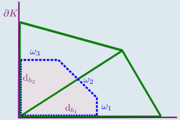

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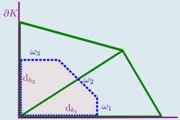

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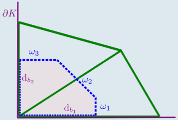

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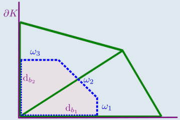

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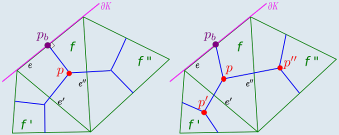

Taylor-Green Vortex solution in primary variables in DEC

$$\begin{cases} u = -\cos(x)\sin(y)e_x + \sin(x)\cos(y)e_y \\ p = -\frac{\rho}{4}(\cos(2x) + \cos(2y)) \end{cases}$$

Domaine $[-\pi, \pi] \times [-\pi, \pi]$

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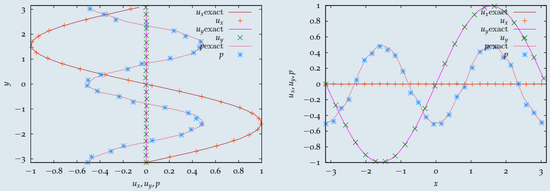
- Velocity and pressure profliles along $x = 0$ and $y = 0$ (Barycentric dual)



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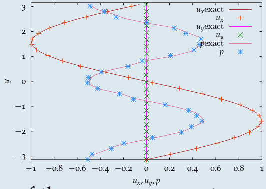
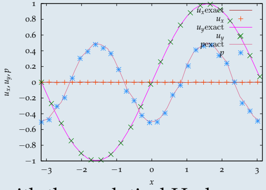
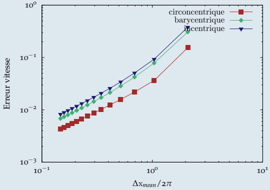
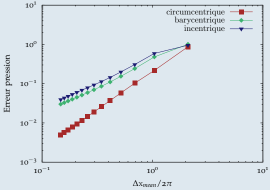




- Evolution of the error on an acute mesh with the analytical Hodge





Lid-driven cavity, solution for $Re = 100$

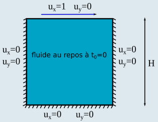
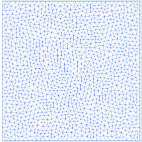
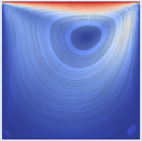


Illustration of the Lid-driven cavity problem.

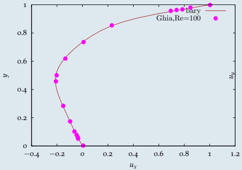
Urmila Ghia et al, High- Resolutions for incompressible flow using the Navier-Stokes equations and a multigrid method, Application to fluid mechanics, *Journal of Computational Physics*



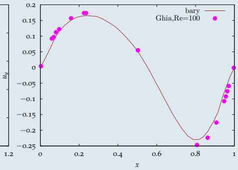
Delaunay mesh



streamlines

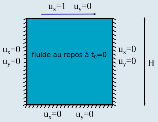


horizontal velocity profile



vertical velocity profile

Lid-driven cavity, solution for $Re = 100$



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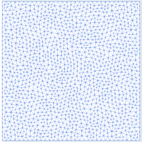

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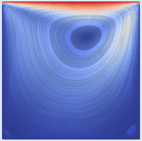

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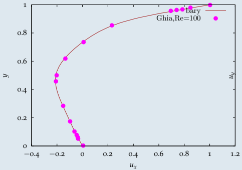
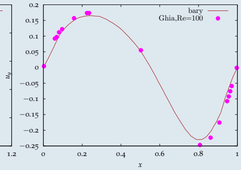
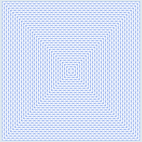


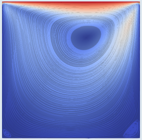
horizontal velocity profile



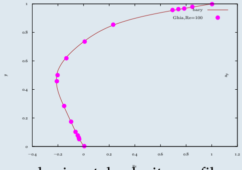
vertical velocity profile



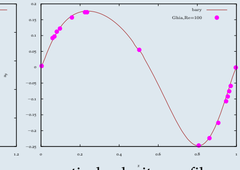
Right mesh



streamlines



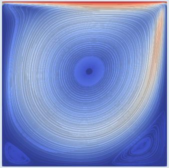
horizontal velocity profile



vertical velocity profile

Lid-driven cavity, solution for $Re = 1000$

- Solution on a right mesh with dual barycentric



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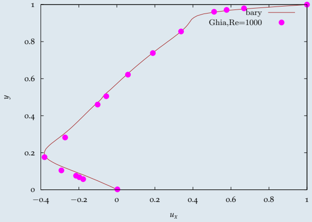
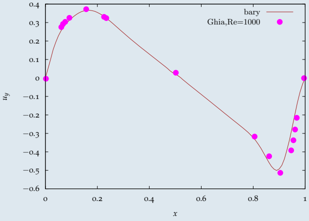


Vertical velocity profile



Horizontal velocity profile

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Mamdouh S. Mohamed 'Discrete exterior calculus discretization of incompressible Navier–Stokes equations over surface simplicial meshes', 2016 - *Journal of Computational Physics*

R. AYOUB et al, 'A new Hodge operator in Discrete Exterior Calculus, Application to fluid mechanics', 2020- *Pure and Applied Analysis*

Thank You