Kirchhoff plate theory revisited for large deformations

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An old theory

Sophie GERMAIN, 1776-1831

Gustav KIRCHHOFF, 1824-1887

$$\frac{Eh^3}{12(1-\nu^2)}\Delta^2 w = q$$

Basic assumptions of KPT

- Linearity (in the thickness). Not essential
- Conservation of normality: deduced from…
- Plane stress

$$\sigma_{\alpha 3} \approx 0 \quad \Rightarrow \quad \mu\left(\frac{\partial u_\alpha}{\partial Z} + \frac{\partial u_3}{\partial X_\alpha}\right) \approx 0$$

Plane stress **Conservation of normality**

Solution of this equation

$$u_\alpha(X_1, X_2, Z) = \bar{u}_\alpha(X_1, X_2) + Z\, d_\alpha(X_1, X_2) \qquad d_\alpha = -\frac{\partial u_3}{\partial X_\alpha}$$

$$\boxed{u_3(X_1, X_2, Z) = \bar{u}_3(X_1, X_2)} \qquad ??$$

Contradiction of KPT

- **If** $\mathrm{u}_3 = \bar{\mathrm{u}}_3(\mathrm{X}_1, \mathrm{X}_2)$, **then** $\epsilon_{33} = \frac{\partial \mathrm{u}_3}{\partial \mathrm{Z}} = 0$ **Plane strain !**

- **If** $u_3 = \bar{u}_3(X_1, X_2) + \mathrm{Z}\, d_3(X_1, X_2)$, **then** $\epsilon_{\alpha 3} \neq 0$

Normality not satisfied !

- Impossible to deduce exactly the strain from the assumed displacement

Asymptotic analyses

CIARLET (1980), DESTUYNDER (1982), etc

- Displacement $u_\alpha = O(\boldsymbol{\varepsilon}^2), \quad u_3 = O(\boldsymbol{\varepsilon})$???
- Plane Stress $\sigma_{\alpha 3} = O(\varepsilon), \quad \sigma_{33} = O(\varepsilon^2)$
- Normality in any case (Kirchhoff)
- Main deflection $u_3 = \varepsilon \bar{u}_3(X_1, X_2) + O(\varepsilon^2)$
- Linearity of $\mathbf{u}$ and ε, not of $\boldsymbol{\sigma}$

Principle of a new plate model

- Principle : Enhanced Assumed Strain (EAS) and plane stress
- EAS (Simo-Rifai 1990, Ramm et al 1994-1998) : disconnect partially displacement and strain
- Plane stress

Enhanced Assumed Strain

- Assumed displacement

$$\boldsymbol{x} = \bar{\mathbf{x}}(X_1, X_2) + Z\,\mathbf{d}(X_1, X_2)$$

- Deduced deformation

$$\mathbf{F} = \nabla\boldsymbol{x} = \nabla\bar{\mathbf{x}} + \mathbf{d} \otimes E_3 + Z\,\nabla\mathbf{d}$$

- Enhanced deformation: $\mathbf{B}(X_1, X_2)$ new unknown vector field

$$\mathbf{F} = \nabla\bar{\mathbf{x}} + \mathbf{d} \otimes E_3 + Z\,\nabla\mathbf{d} + Z\,\mathbf{B} \otimes E_3$$

A hyperelastic plate model

- Kinematics (3D $\rightarrow$ 2D): 3 unknown vectors

$$\boldsymbol{x} = \bar{\mathbf{x}} + Z\,\mathbf{d}$$

$$\mathbf{F} = \nabla\bar{\mathbf{x}} + \mathbf{d} \otimes \mathbf{E}_3 + Z\,(\nabla\mathbf{d} + \mathbf{B}) \otimes \mathrm{E}_3$$

- Constitutive law (3D)

$$\begin{cases} \mathbf{C} = {}^{t}\mathbf{F}\,.\,\mathbf{F} \\ \mathbf{S} = \widehat{\mathrm{G}}(\mathbf{C}) \\ \mathbf{P} = \mathbf{F}.\,\mathbf{S} \end{cases}$$

- Plane stress assumption

$$\mathbf{S}.\,\mathbf{E}_3 = 0 \quad \text{or } \mathbf{P}.\,\mathbf{E}_3 = 0, \quad \frac{d\mathbf{S}}{dZ}.\,\mathbf{E}_3 = 0 \quad \text{for } Z=0$$

Reduction of the number of unknowns

- Reduction at the incremental level $(\,.\,)^*$
- Incremental constitutive law at Z=0:

$$P^*_{ij} = L_{ijk\alpha}\,\bar{x}^*_{k,\alpha} + L_{ijk3}\,d^*_k$$

- First plane stress condition

$$P^*_{i3} = L_{i3k\alpha}\,\bar{x}^*_{k,\alpha} + L_{i3k3}\,d^*_k = \mathbf{0}$$

Acoustic tensor A_{ik}

- Elimination of the director (D. Steigmann):

$$d^*_i = -A^{-1}_{ij} L_{j3k\alpha}\,\bar{x}^*_{k,\alpha} = -\,K_{ik\alpha}\bar{x}^*_{k,\alpha}$$

$$\mathbf{d}^* = -\mathbf{K}:\nabla\bar{\mathbf{x}}^*$$

Tensor K giving the director (normality and …)

Example of reduction

- **First plane stress condition:** $P_{i3} = L_{i3k\alpha}\bar{x}_{k,\alpha} + L_{i3k3}d_k = 0$

- **Isotropic material, <u>close to the stress state</u>** $(a = {}^{1}/_{2} - \nu,\ b = 1 - \nu)$

$$\begin{bmatrix} 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a \\ \nu & 0 & 0 & 0 & \nu & 0 \end{bmatrix} \{\nabla\bar{x}\} + \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- **First one recovers the normality conservation** $d_\alpha = -\bar{x}_{3,\alpha}$

- **Next one gets the transverse strain** $d_3 = \epsilon_{zz} = -\dfrac{\nu}{1-\nu}\left(\bar{x}_{1,1} + \bar{x}_{2,2}\right)$

Elimination of the EAS vector

- The second plane stress condition

$$\frac{d\mathbf{S}}{dZ}.\mathbf{E}_3 = 0 \quad \text{for } Z=0$$

$$\Rightarrow \quad F_{ni}\frac{dS_{i3}}{dZ} = L_{i3k\alpha}d_{k,\alpha} + L_{i3k3}B_k = \mathbf{0}$$

- $\Rightarrow$ The same relation, the same tensor **K**

$$\mathbf{B} = -\mathbf{K} : \nabla\mathbf{d}$$

Assembling kinematics and material law

Kinematics

$$\begin{cases} \mathbf{F}^* = \mathfrak{I}_M : \nabla\bar{\mathbf{x}}^* + \mathrm{Z}\mathfrak{I}_B : \nabla\mathbf{d}^* = \mathfrak{I} : \begin{pmatrix} \nabla\bar{\mathbf{x}}^* \\ \mathrm{Z}\,\nabla\mathbf{d}^* \end{pmatrix} \\ \text{with} \qquad \mathbf{d}^* = -\mathbf{K} : ?\bar{\mathbf{x}}^* \end{cases}$$

Constitutive law:

$$\mathbf{P}^* = \mathbf{L}(\mathrm{Z}) : \mathfrak{I} : \begin{pmatrix} \nabla\bar{\mathbf{x}}^* \\ \mathrm{Z}\,\nabla\mathbf{d}^* \end{pmatrix}$$

Stiffness (bilinear form)

$$\iint_{\omega} (\nabla\boldsymbol{\delta}\bar{\mathbf{x}}, \nabla\boldsymbol{\delta}\mathbf{d}) : {}^{t}\mathfrak{I} : \int_{-h/2}^{h/2} \begin{bmatrix} \mathbf{L} & \mathrm{Z}\mathbf{L} \\ \mathrm{Z}\ {}^{t}\mathbf{L} & \mathrm{Z}^2\mathbf{L} \end{bmatrix} \mathrm{dZ} : \mathfrak{I} : \begin{pmatrix} \nabla\bar{\mathbf{x}}^* \\ \nabla\mathbf{d}^* \end{pmatrix} \mathrm{d}\omega$$

3 PDE's from the balance equations

Internal power: $-P_{int} = \int_{\omega} (\langle \mathbf{P} \rangle : \delta\bar{\mathbf{F}} + \langle Z\mathbf{P} \rangle : \delta\bar{\mathbf{F}}')\, d\omega$

$\delta\bar{\mathbf{F}} = \mathfrak{I}_M : \nabla\delta\bar{\mathbf{x}}$ $\quad$ $\delta\bar{\mathbf{F}}' = \mathfrak{I}_B : \nabla\delta\mathbf{d}$

Plate stress tensors: $\mathbf{N} = {}^t\mathfrak{I}_M : \langle \mathbf{P} \rangle, \qquad \mathbf{M} = {}^t\mathfrak{I}_B : \langle Z\mathbf{P} \rangle$

$$\Rightarrow \quad -P_{int} = \int_{\omega} \left(N_{i\alpha}\, \delta\bar{x}_{i,\alpha} + M_{i\alpha}\, \delta d_{i,\alpha} \right) d\omega$$

The PDE's:

$$\frac{\partial N_{i\alpha}}{\partial X_\alpha} + \frac{\partial}{\partial X_\beta}\left(K_{ji\beta} \frac{\partial M_{j\alpha}}{\partial X_\alpha} \right) + f_i = 0$$

Summary

- A true Kirchhoff plate for large strains with the help of the EAS concept
- Finite elements?
- Extension to shells?
- Comparison with the approach by Taylor series (Steigmann, H.H. Dai, …)?
- Mathematical foundations?