Spectral Methods for fractional Allen-Cahn equation (et autres)

M. Azaiez, S. Lin, X. Zhou, C. Xu

Xiamen University, Bordeaux university
azaiez@u-bordeaux.fr

August 30, 2022

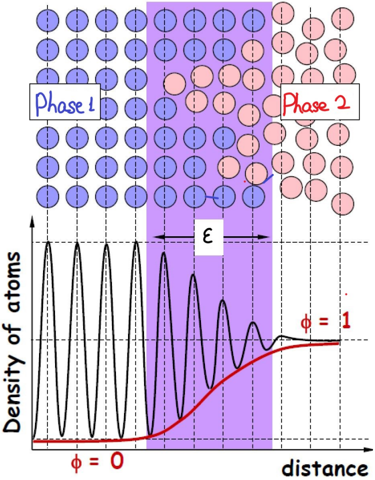
Overview

Introduction of the phase field

- ϕ is a phase-field function that characterizes the phase state. ϕ takes values in $[0, 1]$ (density).
- the parameter ϵ depends on the interfacial width between the two phases.
- rescaling: replace ϕ by

$$\tilde{\phi} = 2\phi - 1,$$

$\tilde{\phi}$ takes values in $[-1, 1]$.



Gradient flow models

- Models of the gradient flow take the form:

$$\frac{\partial \phi}{\partial t} = -\mathrm{grad}_H E(\phi), \tag{1.1}$$

where $E[\phi(\boldsymbol{x}, t)]$ is the free energy functional associated to the physical problem, $\mathrm{grad}_H E(\phi)$ is the functional derivative of E in the Sobolev space H.

- Multiplying both sides of (1.1) by $\delta E/\delta\phi$ and integrating the resulting equation gives following energy dissipation law:

$$\frac{d}{dt}E(\phi) = \left(\mathrm{grad}_H E(\phi), \frac{\partial \phi}{\partial t}\right) = -\|\mathrm{grad}_H E(\phi)\|_0^2, \tag{1.2}$$

The gradient flow in L^2 : Allen Cahn Equation

In this talk we will consider the Ginzburg-Landau energy $E : H^1(\Omega) \to \mathbb{R}$

$$E(\phi) = \int_\Omega \frac{\epsilon^2}{2} |\nabla \phi|^2 + F(\phi) \boldsymbol{dx}. \tag{1.3}$$

According to the definition of the gradient flow in L^2,we have

$$(grad_{L^2} E(\phi), v)_{L^2} = \frac{\delta E}{\delta \phi}(u)(v) = \int_\Omega (-\epsilon^2 \Delta \phi + F^{'}(\phi)) v \boldsymbol{dx}. \tag{1.4}$$

Then,

$$grad_{L^2} E(\phi) = -\epsilon^2 \Delta \phi + F^{'}(\phi). \tag{1.5}$$

Allen-Cahn equation is the L^2-gradient flow of the Ginzburg-Landau energy

$$\partial_t \phi = -grad_{L^2} E(\phi) = -(-\epsilon^2 \Delta \phi + F^{'}(\phi)) \tag{1.6}$$

Classic Allen-Cahn Equation

The Allen-Cahn equation with homogeneous Neumann B.C. is defined by

$$\begin{cases} \frac{\partial \phi}{\partial t} - \varepsilon \Delta \phi + f(\phi) = 0, & (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ \nabla \phi \cdot \boldsymbol{n}\big|_{\partial \Omega} = 0, & \\ \phi(t=0) = \phi_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{1.7}$$

- $\Omega = (-1, 1)^2$
- $f(\phi) = F'(\phi)$ with $F(\phi)$ being a given energy potential.

Fractional Allen-Cahn equation (FACE)

Our model is a Fractional Allen-Cahn equation ($0 < s \leq 1$)

$$\begin{cases} \frac{\partial \phi}{\partial t} + \varepsilon(-\Delta)^s \phi + f(\phi) \ = \ 0, \quad (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ \nabla \phi \cdot \boldsymbol{n}\big|_{\partial \Omega} \ = \ 0, \\ \phi(t=0) \ = \ \phi_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega. \end{cases} \tag{1.8}$$

The Allen-Cahn equation has been widely used in many complicated moving interface problems in materials science and fluid dynamics through a phase-field approach.

Spectral fractional Laplacian $(-\Delta)^s$

We adopt the *spectral decomposition* approach to define fractional Laplacian.

Let $(\lambda_i, \varphi_i)_{i=1}^{\infty}$, be eigenpairs of the Laplacian $-\Delta$:

- $-\Delta\varphi_i = \lambda_i\varphi_i$, $\lambda_i \geq 0$ with corresponding B.C.
- $\{\varphi_i\}$ is a complete orthonormal basis, in the sense

$$(\varphi_i, \varphi_j) = \delta_{ij}$$

If $u(\boldsymbol{x}) = \sum_{i=1}^{\infty} c_i\varphi_i(\boldsymbol{x})$, the define

$$(-\Delta)^s u(\boldsymbol{x}) := \sum_{i=1}^{\infty} \lambda_i^s c_i \varphi_i(\boldsymbol{x}), \quad (0 < s \leq 1). \tag{1.9}$$

Remark

The operator $T = (-\Delta)^s$ *is linear and self-adjoint, i.e., if* $f = \sum_{i=1}^{\infty} a_i\varphi_i$ *and* $g = \sum_{i=1}^{\infty} b_i\varphi_i$*, then* $\langle Tf, g\rangle = \sum_{i=1}^{\infty} a_i b_i \varphi_i = \langle f, Tg\rangle$.

Scalar auxiliary variable approach for (SAV) FACE

We assume $\exists C_1$ such that $\int_\Omega F(\phi)\mathrm{d}\boldsymbol{x} + C_1 > 0$. And, introducing a *scalar* auxiliary variable

$$r(t) := \sqrt{\int_\Omega F(\phi)\,\mathrm{d}\boldsymbol{x} + C_1}$$

Then, we rewrite the phase-field equation (1.7) as

$$\left\{\begin{aligned} \frac{\partial\phi}{\partial t} &= \mu, \qquad \nabla\phi\cdot\boldsymbol{n}\big|_{\partial\Omega} \;=\; 0 \\ \mu &= -\varepsilon(-\Delta)^s\phi - \frac{r(t)}{\sqrt{\int_\Omega F(\phi)d\boldsymbol{x}+C_1}}f(\phi), \\ r_t &= \frac{1}{2\sqrt{\int_\Omega F(\phi)d\boldsymbol{x}+C_1}}\int_\Omega f(\phi)\frac{\partial\phi}{\partial t}\,d\boldsymbol{x}. \end{aligned}\right. \tag{1.10}$$

Energy dissipation law for (1.10)

Theorem

If $\phi \in L^2\big((0,T], H^s(\Omega)\big)$, $0 < s < 1$, *is the solution of equations* (1.10), *then we have the following energy dissipation law*

$$\frac{d}{dt}\left(r^2 + \frac{\varepsilon}{2}|\phi|^2_{s/2}\right) = -\int_\Omega \|\mu\|_0^2 \, d\boldsymbol{x}$$

.

Remark

If $u, v \in H^s(\Omega)$, $0 < s \leq 1$, *it holds* $((-\Delta)^s u, v) = \sum_{i=1}^{\infty} \tilde{u}_i \tilde{v}_i \lambda_i^s = (u, (-\Delta)^s v) = ((-\Delta)^{s/2} u, (-\Delta)^{s/2} v)$. *We define*

$$|v|_{s/2} = ((-\Delta)^{s/2} v, (-\Delta)^{s/2} v) = \Big(\sum_{i=1}^{\infty} |\tilde{v}_i|^2 \lambda_i^s\Big)^{1/2}$$

SAV/BDF1 scheme

We construct a First-order backward difference (BDF1) semi-implicit scheme for (1.10):

Given $\phi^0 = \phi_0$, find $\phi^{n+1} \in H^s(\Omega)$ such that,

$$\begin{cases} \frac{\phi^{n+1}-\phi^n}{\Delta t} = \mu^{n+1}, \qquad \nabla\phi^{n+1}\cdot \boldsymbol{n}\big|_{\partial\Omega} = 0, \\ \mu^{n+1} = -\varepsilon(-\Delta)^s\phi^{n+1} - \frac{r^{n+1}}{\sqrt{\int_\Omega F(\phi^n)d\boldsymbol{x}+C_1}} f(\phi^n), \\ \frac{r^{n+1}-r^n}{\Delta t} = \frac{1}{2\sqrt{\int_\Omega F(\phi^n)d\boldsymbol{x}+C_1}} \int_\Omega f(\phi^n)\frac{\phi^{n+1}-\phi^n}{\Delta t}\, d\boldsymbol{x}. \end{cases} \tag{1.11}$$

Discrete energy law

Theorem

The scheme is unconditionally energy stable in the sense that

$$\frac{1}{\Delta t}\left(H(\phi^{n+1}, r^{n+1}) - H(\phi^n, r^n)\right)$$
$$= -\|\mu^{n+1}\|_0^2 - \frac{\varepsilon}{2\Delta t}|\phi^{n+1} - \phi^n|_{s/2}^2 - \frac{1}{\Delta t}(r^{n+1} - r^n)^2.$$

with the modified energy

$$H(\phi, r) = r^2 + \frac{\varepsilon}{2}|\phi|_{s/2}^2.$$

Implement

To summarize, we implement (1.11)(Scalar Auxiliary Variable approach with a first-order semi-implicit scheme) as follows: Find $\alpha^{n+1}, \beta^{n+1} \in H^s$ such that

1. Compute $c^n = \int_\Omega F(\phi^n)d\boldsymbol{x} + C_1, \quad \tilde{c}_0 = (f(\phi^n), \phi^n), \quad \tilde{c}_1 = \frac{r^n}{\sqrt{c^n}} - \frac{\tilde{c}_0}{2c^n};$
2. Compute $g^n := \frac{1}{\Delta t}\phi^n - \tilde{c}_1 f(\phi^n);$
3. Solve $\dfrac{1}{\Delta t}\beta^{n+1} + \varepsilon(-\Delta)^s\beta^{n+1} = f(\phi^n);$
4. Solve $\dfrac{1}{\Delta t}\alpha^{n+1} + \varepsilon(-\Delta)^s\alpha^{n+1} = g^n;$
5. Compute $\tilde{c}_2 = (f(\phi^n), \beta^{n+1}), \quad \tilde{c}_3 = (f(\phi^n), \alpha^{n+1}), \quad \tilde{c}_4 = \tilde{c}_3/(2c^n + \tilde{c}_2);$
6. Compute $\phi^{n+1} = \alpha^{n+1} - \tilde{c}_4\beta^{n+1}; \qquad r^{n+1} = r^n + \dfrac{(f(\phi^n), \phi^{n+1}) - \tilde{c}_0}{2\sqrt{c^n}}.$

Implement for SAV/BDF2

Ps: Implement Scalar Auxiliary Variable approche with a semi-implicit second-order scheme based on BDF (SAV/BDF2) as follows:

1. Compute $\bar{\phi}^{n+1} = 2\phi^n - \phi^{n-1}$, $\quad \bar{c}^{n+1} = \int_\Omega F(\bar{\phi}^{n+1})\mathrm{d}\boldsymbol{x} + C_1$, $\quad \tilde{c}_0 = (f(\bar{\phi}^{n+1}), 4\phi^n - \phi^{n-1})/3$, $\quad \tilde{c}_1 = \frac{4r^n - r^{n-1}}{3\sqrt{\bar{c}^{n+1}}} - \frac{\tilde{c}_0}{2\bar{c}^{n+1}}$;
2. Compute $\quad g^n := \frac{4\phi^n - \phi^{n-1}}{2\Delta t} - \tilde{c}_1 f(\bar{\phi}^{n+1})$.
3. Solve $\quad \dfrac{3}{2\Delta t}\beta^{n+1} + \varepsilon(-\Delta)^s\beta^{n+1} = f(\bar{\phi}^{n+1})$;
4. Solve $\quad \dfrac{3}{2\Delta t}\alpha^{n+1} + \varepsilon(-\Delta)^s\alpha^{n+1} = g^n$;
5. Compute $\quad \tilde{c}_2 = (f(\bar{\phi}^{n+1}), \beta^{n+1})$, $\quad \tilde{c}_3 = (f(\bar{\phi}^{n+1}), \alpha^{n+1})$, $\quad \tilde{c}_4 = \tilde{c}_3/(2\bar{c}^{n+1} + \tilde{c}_2)$;
6. Compute $\quad \phi^{n+1} = \alpha^{n+1} - \tilde{c}_4\beta^{n+1}$, $\quad r^{n+1} = \dfrac{4r^n - r^{n-1}}{3} + \dfrac{(f(\bar{\phi}^{n+1}), \phi^{n+1}) - \tilde{c}_0}{2\sqrt{\bar{c}^{n+1}}}$.

Spatial discretizations

- $\Sigma = \{(\xi_i,\, \rho_i);\, 0 \leq i \leq N\}$ denote the sets of GLL formula.

$$\forall \phi \in I\!P_{2N-1}(\Lambda :=]-1,+1[), \qquad \int_{-1}^{+1} \phi(\xi)\, d\xi = \sum_{j=0}^{N} \phi(\xi_j)\, \rho_j, \tag{1.12}$$

where $I\!P_N(\Lambda)$ denotes the space of polynomials of degree $\leq N$.

- Lagrange basis $h_i(x) \in I\!P_N(\Lambda)$ built on Σ by

$$h_i(\xi_j) \;=\; \delta_{ij}, \qquad\qquad 0 \leq i, j \leq N, \tag{1.13}$$

- ϕ will be approximated polynomial functions ϕ_N as follows

$$\phi_N(\boldsymbol{x}, t) = \sum_{i=0}^{N} \sum_{j=0}^{N} \alpha_{i,j}(t) h_i(x) h_j(y). \tag{1.14}$$

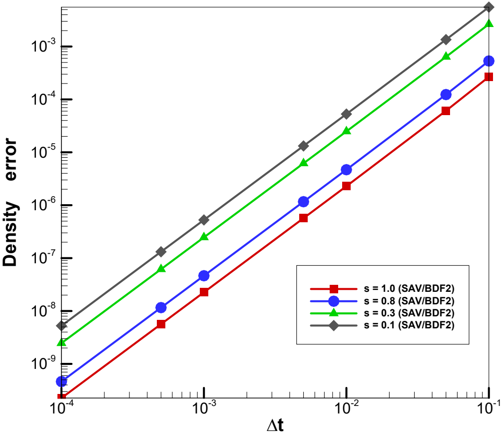
- The L^2-inner products will be achieved using GLL :

$$(\varphi, \psi) \approx (\varphi, \psi)_N := \sum_{i=0}^{N} \sum_{j=0}^{N} \varphi(\xi_i)\psi(\xi_j)\rho_i\rho_j. \tag{1.15}$$

SAV/BDF2 scheme with different fractional order s

FiG.1.

- SAV/BDF2 for Ex.1
- log(err)-log(Δt)
- error in L^2 norm at $T = 2.0$
- Spectral method in space with $N_x = N_y = 32$ (LGL).
- $\phi(t,x,y) = sin(t)\,cos(\pi x)\,cos(\pi y)$,
- $f(\phi) = \phi\,(\phi^2 - 1)$.



The initial state is a circular phase interface of the radius $R_0 = 100$ in the rectangular domain $]-128, 128[^2$.

$$\phi(\boldsymbol{x}, 0) = \begin{cases} 1, & |\boldsymbol{x}|^2 < 100^2, \\ -1, & |\boldsymbol{x}|^2 \geq 100^2. \end{cases}$$

Such a circular interface is unstable and the driving force will make it shrink and eventually disappear. It has been shown the velocity and the radius of the moving interface are given by

$$V = \frac{dR}{dt} = -\frac{1}{R}, \quad R(t) = \sqrt{R_0^2 - 2t}.$$

- We map the computational domain $]-128, 128[^2$ to $]-1, 1[^2$. Therefore actually we are led to solve the fractional Allen-Cahn equation (1.7) with the coefficients $\gamma = 1/128^2$ and $\varepsilon = 0.0078$.
- the space resolution is set to $N = 512$, and the time step size is $\Delta t = 0.1$.
- We use the Spectral Galerkin method to express ϕ as

$$\phi = \sum_{n_1, n_2 \leq N} \hat{\phi}_{n_1,n_2} e_{n_1}(x) e_{n_2}(y),$$

with $N = 512$, and $\left\{e_{n_1}(x)e_{n_2}(y)\right\}$ are the numerical orthonormal eigenfunctions of the Laplacian $-\Delta$ in $(-1, 1)^2$ with homogeneous Neumann boundary condition.

Benchmark test: calculus

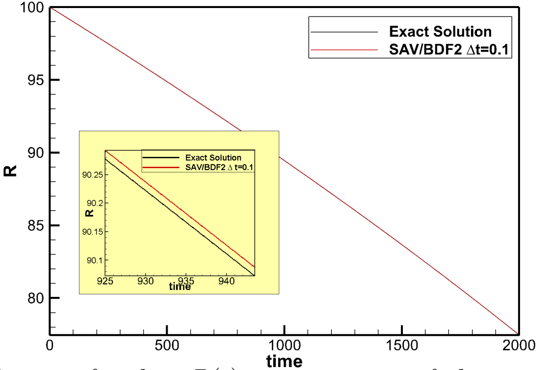


Figure: *The evolution of radius $R(t)$: comparison of the exact solution and numerical result in the case $s = 1$.*

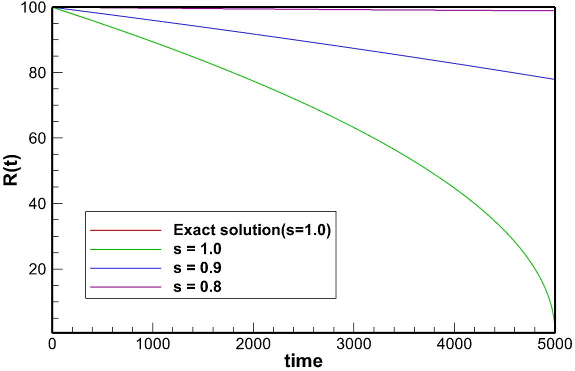
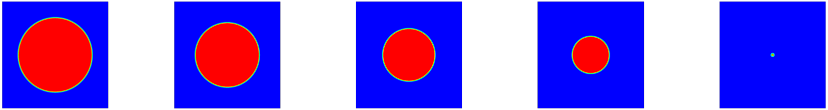
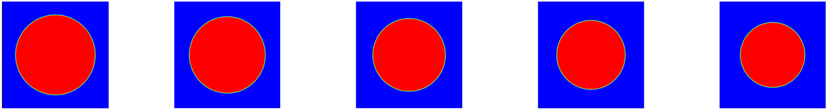


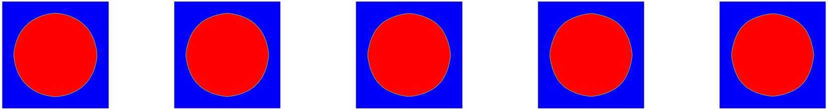
Figure: *Evolution of the radius for different fractional order s: impact of the order on the radius decay rate.*



(a) s=1



(b) s=0.9



(c) s=0.8

Figure: *Temporal evolution of a circular domain from left to right at times* $t = 1000, 2000, 3000, 4000, 5000$, *for fractional order* $s = 1, 0.9, 0.8$, *for the top, middle and bottom rows, respectively.*

Fractional Stokes problem

Definition of fractional calculus

Fractional integral

Let $\Omega = [a, b]$ (a, b be finite or infinite). The left-sided and right-sided Riemann-Liouville fractional integral of order $s > 0$ are defined by

$$
{}_aI_x^s f(x) := \frac{1}{\Gamma(s)} \int_a^x (x-t)^{s-1} f(t)\, dt
$$

$$
{}_xI_b^s f(x) := \frac{1}{\Gamma(s)} \int_x^b (t-x)^{s-1} f(t)\, dt
$$

Properties

If $s > 0, \sigma > 0$ then

$$
{}_aI_x^{s+\sigma} f(x) := {}_aI_x^s\, {}_aI_x^\sigma f(x),
$$

$$
{}_xI_b^{s+\sigma} f(t) := {}_xI_b^s\, {}_xI_b^\sigma f(x)
$$

Riemann-Liouville fractional derivatives

RL fractional derivatives of order $s > 0$ are defined by

$$
\begin{aligned}
{}_aD_x^s f(x) :&= \frac{d^n}{dx^n} {}_aI_x^{n-s} f(x) \quad (n = [s] + 1) \\
&= \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-s-1} f(t)\, dt
\end{aligned}
$$

and

$$
\begin{aligned}
{}_xD_b^s f(x) :&= (-1)^n \frac{d^n}{dx^n} {}_xI_b^s f(x) \quad (n = [s] + 1) \\
& \frac{(-1)^n}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-s-1} f^{(n)}(t)\, dt
\end{aligned}
$$

Fractional Stokes equation

For $1 < \alpha \leq 2$, the fractional incompressible Stokes problem reads : Find $(\mathbf{u}, p)$

$$\begin{aligned} -\nu \Delta^{\frac{\alpha}{2}} \mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

where

- $\Omega := \Lambda^d$ denote the 2- and 3-D domains, i.e $d = 2, 3$,
- $\boldsymbol{x}$ is a generic point of Ω.
- $\mathbf{f} = \mathbf{f}(\boldsymbol{x})$ describes the body force
- ν is a positive parameter which represents the kinematic viscosity

- We consider a symmetric definition of fractional laplacian

$$\Delta^{\frac{\alpha}{2}} := \frac{1}{4}\sum_{j=1}^{d}\left(\partial_{x_j}^{\frac{\alpha}{2}} - {}_{x_j}\partial^{\frac{\alpha}{2}}\right)\left({}^{C}\partial_{x_j}^{\frac{\alpha}{2}} - {}_{x_j}^{C}\partial^{\frac{\alpha}{2}}\right)$$

- For $0 < s < 1$ we define fractional gradient and divergence operators by

$$\nabla^s g := \frac{1}{2}\left(\left({}^{C}\partial_{x_1}^{s} - {}_{x_1}^{C}\partial^{s}\right)g, \cdots, \left({}^{C}\partial_{x_d}^{s} - {}_{x_d}^{C}\partial^{s}\right)g\right)^T,$$

$$\nabla^s \cdot \mathbf{v} := \frac{1}{2}\sum_{i=1}^{d}\left(\partial_{x_i}^{s} - {}_{x_i}\partial^{s}\right)v^i.$$

Then the fractional laplacian can be written as

$$\Delta^{\frac{\alpha}{2}} = \nabla^{\frac{\alpha}{2}} \cdot \nabla^{\frac{\alpha}{2}}.$$

Fractional Stokes equation

To express the variational formulation of the Stokes equation we introduce the Sobolev spaces :

- For the velocity :

$$\mathrm{X} := \{\mathbf{v} \in H_0^{\frac{\alpha}{2}}(\Omega)^d;\ \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

  endowed with the norm

$$\|\mathbf{v}\|_{\mathrm{X}} := \left(\|\mathbf{v}\|_{\frac{\alpha}{2}}^2 + \|\nabla \cdot \mathbf{v}\|_0^2\right)^{\frac{1}{2}}.$$

  Obviously, if $\mathbf{v} \in H^1(\Omega)^d$, we have $\|\mathbf{v}\|_{\mathrm{X}} \le \|\mathbf{v}\|_1$.

- For pressure p, we define the space

$$\mathrm{Q} := \left\{q \in L^2(\Omega) : \int_\Omega q(\mathbf{x})d\mathbf{x} = 0\right\}.$$

Fractional Stokes equation

The weak formulation reads : for a given $f \in \mathrm{X}'$, find a pair $(\mathbf{u}, p)$ in $\mathrm{X} \times \mathrm{Q}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle f, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathrm{X} \\ b(\mathbf{u}, q) &= 0, \quad \forall q \in \mathrm{Q} \end{aligned}$$

where bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) :&= \frac{\nu}{4} \sum_{i=1}^{d} \sum_{j=1}^{d} \Big[\Big(\partial_{x_j}^{\frac{\alpha}{2}} u^i, \partial_{x_j}^{\frac{\alpha}{2}} v^i \Big) + \Big({}_{x_j}\partial^{\frac{\alpha}{2}} u^i, {}_{x_j}\partial^{\frac{\alpha}{2}} v^i \Big) \\ &\quad - \Big(\partial_{x_j}^{\frac{\alpha}{2}} u^i, {}_{x_j}\partial^{\frac{\alpha}{2}} v^i \Big) - \Big({}_{x_j}\partial^{\frac{\alpha}{2}} u^i, \partial_{x_j}^{\frac{\alpha}{2}} v^i \Big) \Big] \\ b(\mathbf{v}, q) :&= -(\nabla \cdot \mathbf{v}, q) \end{aligned}$$

- For a fixed integer $N \geq 2$, $\mathcal{P}_N(\Omega)$ denotes the space of polynomials of degree $\leq N$ with respect to any space variable.
- For $s, \sigma > -1$, the Jacobi polynomials, denoted by $J_n^{s,\sigma}(x)$, are orthogonal with respect to the Jacobi weight function $\omega^{s,\sigma}(x) := (1-x)^s(1+x)^\sigma$ over Λ, namely,

$$\int_{-1}^{1} J_n^{s,\sigma}(x) J_m^{s,\sigma}(x) \omega^{s,\sigma}(x)\, dx = \gamma_n^{s,\sigma} \delta_{mn},$$

where δ_{mn} is the Kronecker-delta symbol and

$$\gamma_n^{s,\sigma} = \frac{2^{s+\sigma+1}\Gamma(n+s+1)\Gamma(n+\sigma+1)}{(2n+s+\sigma+1)n!\Gamma(n+s+\sigma+1)}.$$

- We introduce the Jacobi-Gauss-Lobatto nodes $(\xi_{N,i}^{s,\sigma})_{0\le i\le N}$ roots of $(1-\xi^2)J_{N-1}^{s+1,\sigma+1}$ and corresponding weights $(\rho_{N,i}^{s,\sigma})_{0\le i\le N}$
- the Jacobi-Gauss nodes $(\hat{\xi}_{N,i}^{s,\sigma})_{0\le i\le N}$ roots of $J_{N+1}^{s,\sigma}$ and corresponding weights $(\hat{\rho}_{N,i}^{s,\sigma})_{0\le i\le N}$.
- We define three discrete scalar product:

$$(u,v)_{NS} := \sum_{j=0}^{N} \int_{\Lambda} u(x,\xi_{N,j}^{0,0})v(x,\xi_{N,j}^{0,0})\rho_{N,j}^{0,0}\,dx,$$

$$(u,v)_{NF} := \sum_{i=0}^{N} \int_{\Lambda} u(\xi_{N,i}^{0,0},y)v(\xi_{N,i}^{0,0},y)\rho_{N,i}^{0,0}\,dy,$$

$$(u,v)_{NA} := \sum_{i,j=0}^{N} u(\xi_{N,i}^{0,0},\xi_{N,j}^{0,0})v(\xi_{N,i}^{0,0},\xi_{N,j}^{0,0})\rho_{N,i}^{0,0}\rho_{N,j}^{0,0}.$$

We set $\mathrm{X}_N = \mathcal{P}_N(\Omega)^d \cap \mathrm{X}$, and $\mathrm{Q}_M = \mathcal{P}_M(\Omega) \cap \mathrm{Q}$. The discrete problem reads : find $(\mathbf{u}_N, p_M) \in \mathrm{X}_N \times \mathrm{Q}_M$ such that

$$\begin{array}{rcll} a_N(\mathbf{u}_N, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_M) & = & (\mathbf{f}, \mathbf{v}_N)_{NA}, & \forall \mathbf{v}_N \in \mathrm{X}_N, \\ b_N(\mathbf{u}_N, q_M) & = & 0, & \forall q_M \in \mathrm{Q}_M. \end{array}$$

where

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{v}_N) =& \frac{\nu}{4} \sum_{i=1}^{d} \Big[\Big(\partial_x^{\frac{\alpha}{2}} u_N^i, \partial_x^{\frac{\alpha}{2}} v_N^i \Big)_{NS} + \Big({}_x\partial^{\frac{\alpha}{2}} u_N^i, {}_x\partial^{\frac{\alpha}{2}} v_N^i \Big)_{NS} \\ & - \Big(\partial_x^{\frac{\alpha}{2}} u_N^i, {}_x\partial^{\frac{\alpha}{2}} v_N^i \Big)_{NS} - \Big({}_x\partial^{\frac{\alpha}{2}} u_N^i, \partial_x^{\frac{\alpha}{2}} v_N^i \Big)_{NS} \\ & + \Big(\partial_y^{\frac{\alpha}{2}} u_N^i, \partial_y^{\frac{\alpha}{2}} v_N^i \Big)_{NF} + \Big({}_y\partial^{\frac{\alpha}{2}} u_N^i, {}_y\partial^{\frac{\alpha}{2}} v_N^i \Big)_{NF} \\ & - \Big(\partial_y^{\frac{\alpha}{2}} u_N^i, {}_y\partial^{\frac{\alpha}{2}} v_N^i \Big)_{NF} - \Big({}_y\partial^{\frac{\alpha}{2}} u_N^i, \partial_y^{\frac{\alpha}{2}} v_N^i \Big)_{NF} \Big] \\ b_N(\mathbf{v}_N, q_M) =& -(\nabla \cdot \mathbf{v}_N, q_M)_{NA} \end{aligned}$$

Spurious modes and SEM

A naive choice leads to an ill-posed approximation due to the pollution induced by the spurious modes set

$$\mathrm{Z}_{N,M} = \{q_M \in \mathrm{Q}_M : \forall \mathbf{v}_N \in \mathrm{X}_N,\ b_N(\mathbf{v}_N, q_M) = 0\}.$$

- When $M = N$ and for $\alpha = 2$, the dimension of $\mathrm{Z}_{N,N}$ is 7 for $d = 2$ and $(12N + 3)$ for $d = 3$.
- Maday and Patera suggested for the case of $\alpha = 2$, to reduce the pressure space to Q_{N-2}
- This choice solves partially the problem since such a mixed element has the so-called weak spurious modes. A weak spurious mode is a pressure mode $q_N \in \mathrm{Q}_{N-2}$ such that,

$$\lim_{N\to\infty} \left(\sup_{\mathbf{v}_N \in \mathrm{X}_N} \frac{b(q_N, \mathbf{v}_N)}{\|\mathbf{v}_N\|_{\mathrm{X}}} \right) = 0.$$

Stable SEM : Error estimation

Discret Inf-Sup condition

For any $q_N \in \mathrm{Q}_{N-2}$, $\exists \mathbf{w}_N \in \mathbb{P}_N(\Omega)^d \cap H_0^1(\Omega)^d$ s.t.

$$(\nabla \cdot \mathbf{w}_N, p_N) = -(q_N, p_N), \quad \forall p_N \in \mathbb{P}_{N-2}(\Omega),$$

and

$$\|\mathbf{w}_N\|_1 \lesssim \beta_N^{-1} \|q_N\|_0,$$

where $\beta_N = N^{-\frac{d-1}{2}}$ is call the *Inf-Sup* constant.

Convergence

Let s, σ, γ be 3 positive real numbers. Assume the solution $(\mathbf{u}, p)$ of the Stokes problem belongs to $H^s(\Omega)^d \times H^\sigma(\Omega)$ and $\mathbf{f}$ in $H^\gamma(\Omega)^d$, then

$$\|\mathbf{u} - \mathbf{u}_N\|_{\frac{\alpha}{2}} + \beta_N \|p - p_N\|_0 \lesssim N^{\frac{\alpha}{2}-s} \|\mathbf{u}\|_s + N^{-\sigma} \|p\|_\sigma + N^{-\gamma} \|\mathbf{f}\|_\gamma$$

Uzawa Algorithm

The equivalent matrix formulation of the Fractional Stokes problem is

$$\begin{aligned} \mathbf{A}_N \underline{\mathbf{u}}_N + \mathbf{D}_M \underline{p}_M &= \mathbf{B}_N \underline{\mathbf{f}}_N, \\ \mathbf{D}_M^T \underline{\mathbf{u}}_N &= 0, \end{aligned}$$

where

- $\underline{\mathbf{u}}_N$ of unknowns for the velocity consists of the values of all the components at the nodes $(\xi_{i,N}^{0,0}, \xi_{j,N}^{0,0})_{1 \le i,j \le N-1}$.
- The vector $\underline{p}_M$ corresponding to the unknowns for the pressure is made of the values of p_M at the nodes $(\hat{\xi}_{i,N}^{0,0}, \hat{\xi}_{j,N}^{0,0})_{0 \le i,j \le M}$.
- $\mathbf{A}_N$ is the discrete Fractional Laplace operator
- $\mathbf{D}_M$ is the discrete Gradient operator.

Uzawa Algorithm

$\Rightarrow$ A Block Gaussian elimination is performed to uncouple the pressure and the velocity.

- Then the pressure is solved :

$$\mathbf{D}_M^T \mathbf{A}_N^{-1} \mathbf{D}_M \underline{p}_M \ = \ \mathbf{D}_M^T \mathbf{A}_N^{-1} \mathbf{B}_N \ \underline{\mathbf{f}}_N.$$

- The velocity $\underline{\mathbf{u}}_N$ is computed by solving

$$\mathbf{A}_N \underline{\mathbf{u}}_N \ = \ \mathbf{B}_N \underline{\mathbf{f}}_N - \mathbf{D}_M \underline{p}_M.$$

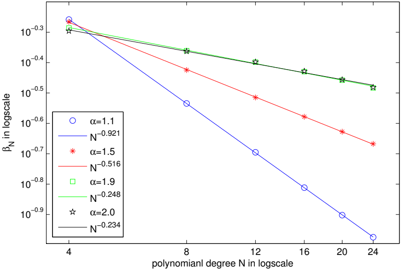
The Uzawa matrix is

- of dimension $(M+1)^d$,
- full, symmetric and positive definite

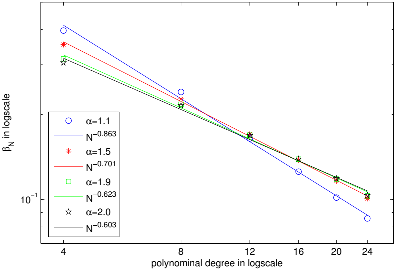
Numerical results: Inf-Sup bihavior

β_N is inferred from κ_N of the preconditioned Uzawa operator $\mathbf{B}_M^{-1}\mathbf{S}_\alpha$ through the estimate

$$C\beta_N^{-2} \leq \kappa_N \leq C'\beta_N^{-2}$$



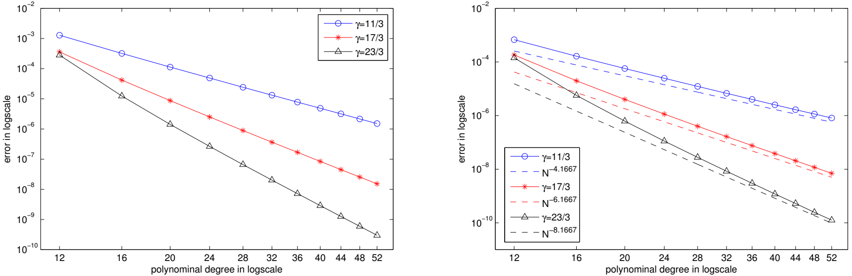
(a) $d = 2$



(b) $d = 3$

Figure: β_N versus N for $\alpha \in \{1.1,\ 1.5,\ 1.9,\ 2\}$

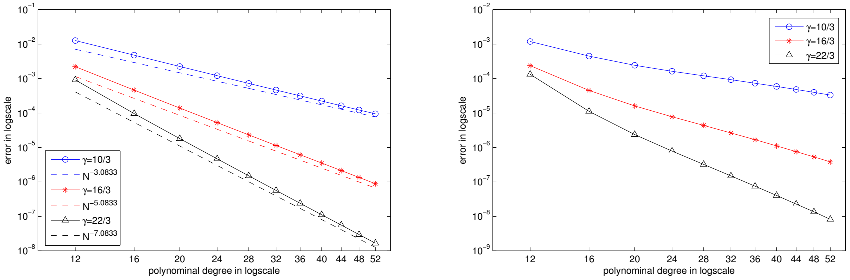
$$\mathbf{u} = \begin{pmatrix} \cos(\pi x)\cos(\pi y) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix}, \qquad p = x^\gamma + y^\gamma$$



(a) $\mathbf{u}$ (b) p

Figure: Error as a function of polynomial degree $d = 2$

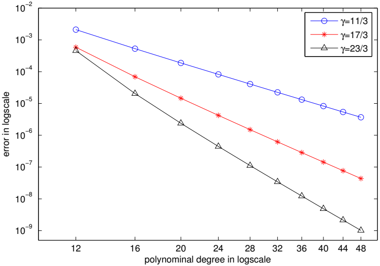
$$\mathbf{u} = \begin{pmatrix} \pi x^{\gamma+1} \sin(\pi y) \\ (\gamma+1) x^{\gamma} \cos(\pi y) \end{pmatrix}, \quad p = \sin(\pi(x+y))$$



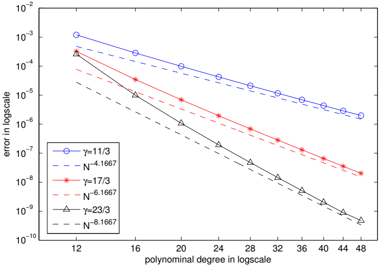
(a) $\mathbf{u}$ (b) p

Figure: Error as a function of polynomial for $d = 2$

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x)\sin(\pi y)\sin(\pi z) \\ \sin(\pi x)\sin(\pi y)\sin(\pi z) \\ \sin(\pi(x+y))\cos(\pi z) \end{pmatrix}, \quad p = x^\gamma + y^\gamma + z^\gamma$$



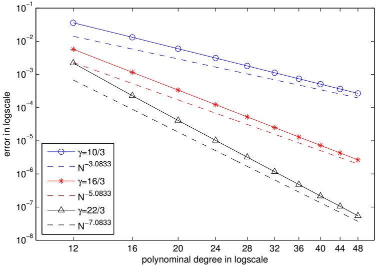
(a) $\mathbf{u}$



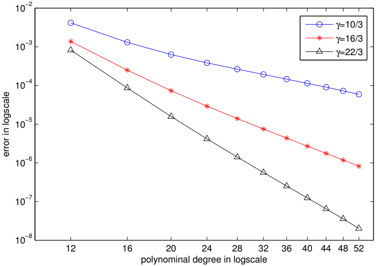
(b) p

Figure: Error as a function of polynomial degree $d = 3$

$$\mathbf{u} = \begin{pmatrix} \pi x^{\gamma+1} y^{\gamma} \sin(\pi z) \\ \pi x^{\gamma} y^{\gamma+1} \sin(\pi z) \\ 2(\gamma+1) x^{\gamma} y^{\gamma} \cos(\pi z) \end{pmatrix}, \quad p = \sin(\pi(x+y+z))$$



(a) $\mathbf{u}$



(b) p

Figure: Error as a function of polynomial degree $d = 3$

Fractional Navier - Stokes problem

NS Equation

Consider the following problem formulate on $\Omega \times [0,T]$: find $\mathbf{u}$ and pressure p such that

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta^{\frac{\alpha}{2}} \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega \times [0,T], \\
\nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega \times [0,T], \\
\mathbf{u} &= 0, \quad \text{on } \partial\Omega \times [0,T], \\
\mathbf{u}(\cdot, t=0) &= 0, \quad \text{on } \Omega,
\end{aligned}$$

$\mathbb{P}_N - \mathbb{P}_{N-2}$ method based on Uzawa solution algorithm can be applied but we turn to pressure correction method....

Goda first order method

For $k \geq 0$ compute $(\tilde{\mathbf{u}}^{k+1}, \mathbf{u}^{k+1}, p^{k+1})$ s.t.

$$\begin{cases} \dfrac{\tilde{\mathbf{u}}^{k+1} - \mathbf{u}^k}{\Delta t} - \nu \Delta^{\frac{\alpha}{2}} \tilde{\mathbf{u}}^{k+1} + (\mathbf{u}^k \cdot \Delta)\mathbf{u}^k + \nabla p^k = \mathbf{f}^{k+1}, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{in } \partial\Omega, \end{cases}$$

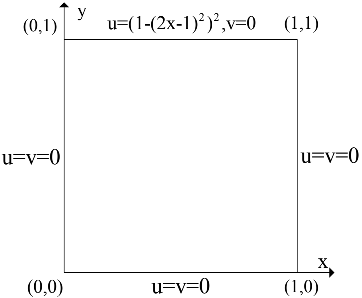
$$\begin{cases} \dfrac{\mathbf{u}^{k+1} - \tilde{\mathbf{u}}^{k+1}}{\Delta t} + \nabla \phi = 0, & \text{in } \Omega \\ \nabla \cdot \mathbf{u}^{k+1} = 0, & \text{in } \Omega \\ \mathbf{u}^{k+1} = 0, & \text{in } \partial\Omega, \end{cases}$$

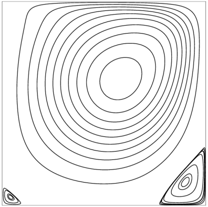
$$p^{k+1} = \phi + p^k,$$

where ϕ is solution of laplace equation

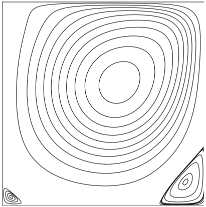
$$\begin{cases} -\Delta \phi = -\dfrac{\nabla \cdot \tilde{\mathbf{u}}^{k+1}}{\Delta t}, & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} = 0, & \text{in } \partial\Omega. \end{cases}$$

Driven cavity problem is considered:

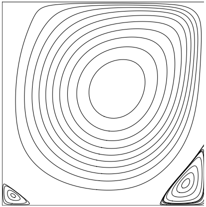




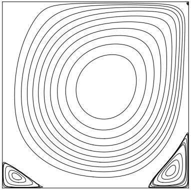
(a) $\alpha = 2.0$



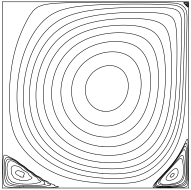
(b) $\alpha = 1.9$



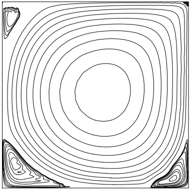
(c) $\alpha = 1.7$



(d) $\alpha = 1.5$



(e) $\alpha = 1.3$



(f) $\alpha = 1.1$

α	PV	LV	RV
1.6	0.05827	-4.50227E-5	-5.37374E-4
1.7	0.06565	-2.16562E-5	-4.80161E-4
1.8	0.07264	-1.11905E-5	-4.06845E-4
1.9	0.07912	-6.64049E-6	-3.24419E-4
2.0	0.08499	-4.19722E-6	-2.43440E-4

Table: Driven cavity

Here

- PV: value in the center of principal vortex
- LV(resp. RV):value in the center of left(resp. right) secondary vortex

Merci