Cinématique d'un materiau microstructuré

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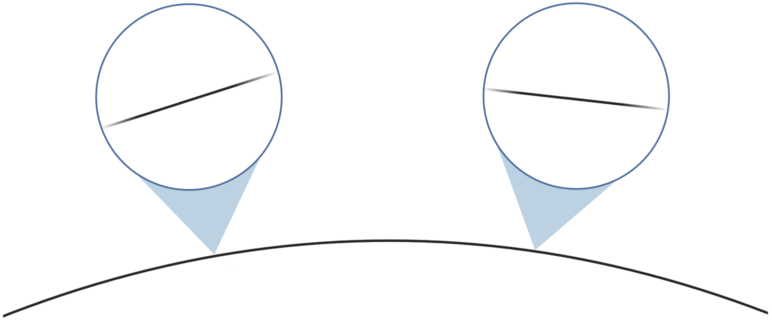
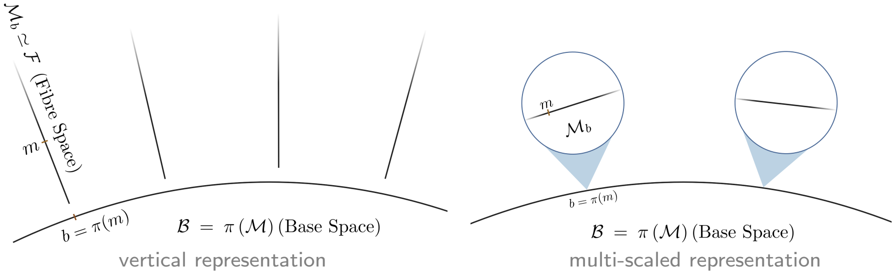
Introduction



Introduction – Fibre bundles



Definition 1: Fibre bundle

Smooth manifolds: total space $\mathcal{M}$, base space $\mathcal{B}$ and fibre $\mathcal{F}$
Smooth projection

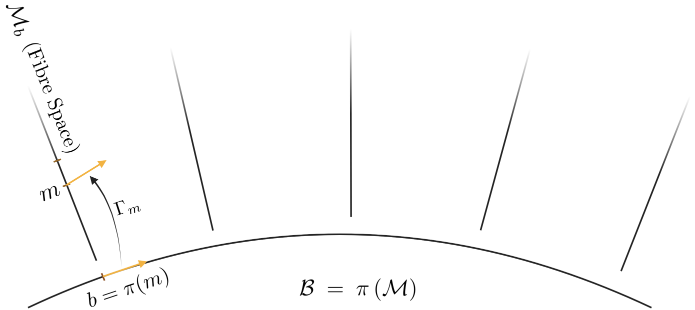
$$\pi : \mathcal{M} \to \mathcal{B}$$

such that $\forall b \in \mathcal{B}$ and $\mathcal{U} \subset \mathcal{B}$ with $b \in \mathcal{U}$ such that

$$\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathcal{F}$$

Fibre $\mathcal{F}$ are vector space

Ehresmann Connections



Representation of a Ehresmann connection via the lifting operator Γ

Definition 2: Connection – An horizontal/macroscopic lift

Smooth assignment at each $m \in \mathcal{M}$ of a smooth linear right inverse Γ_m of $\mathrm{d}\pi$:

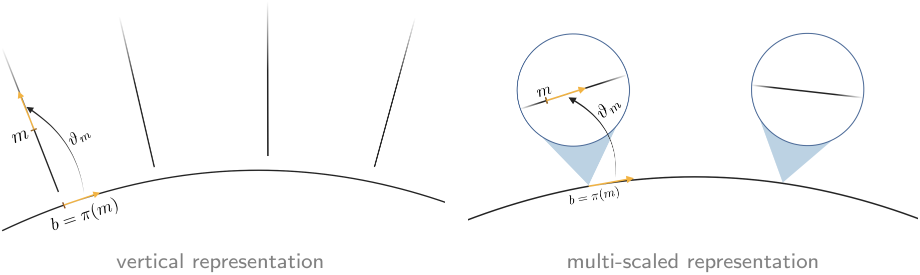
$$\forall m \in \mathcal{M}, \qquad \mathrm{d}\pi \circ \Gamma_m = \mathrm{Id}_{\mathrm{T_b}\mathcal{B}}$$

This assignement defines a sub vector space $\mathrm{H}_m(\mathcal{M})$ ($\simeq T_b\mathcal{B}$) of the tangent space $T_m\mathcal{M}$ at m:

$$T_m\mathcal{M} = \mathrm{V}_m(\mathcal{M}) + \mathrm{H}_m(\mathcal{M})$$

Horizontal vectors represent macroscopical vectors.

Solder forms



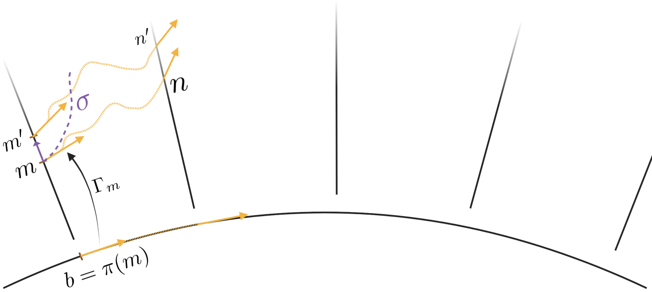
Definition 3: Solder form — A vertical/microscopic lift

Smooth assignment at each $m \in \mathcal{M}$ of a smooth linear isomorphism:

$$\vartheta_m : \mathrm{T}_b\mathcal{B} \simeq \mathrm{V}_m(\mathcal{M}) \qquad \text{where} \qquad \mathrm{V}_m(\mathcal{M}) = \ker \mathrm{d}\pi_{|\mathrm{T}_m\mathcal{M}}$$

- $\mathrm{V}_m(\mathcal{M})$ is used for purely microscopical vectors.
- Connexion and solder forms *encode the physical dimension* of macroscopic and microscopic vectors, respectively.

Lifts and parallel transports



Let choose $\mathbf{u} \in \mathrm{T}_b\mathcal{B}$, $\sigma \in \mathrm{Sect}(\mathcal{M})$ and $\mathbf{A} : \mathcal{M} \to \mathcal{M}'$ then

$$\nabla_{\mathbf{u}}\sigma \in \mathrm{V}_{\sigma(b)}(\mathcal{M})$$

$$\nabla \cdot \sigma : \mathcal{B} \longrightarrow \mathbb{R}, \quad \text{with} \quad (\nabla \cdot \sigma)_b = \mathrm{tr}\left((\nabla\sigma)_b\right)$$

$$\nabla_{\mathbf{u}}\mathbf{A} : \mathcal{M}_b \longrightarrow \mathcal{M}'_b \qquad \nabla \cdot \mathbf{A} \in \mathrm{Sect}(\mathcal{M}')$$

with: $$(\nabla_{\mathbf{u}}\mathbf{A})_b\sigma(b) = \nabla_{\mathbf{u}}(\mathbf{A}\sigma) - \mathbf{A}(\nabla_{\mathbf{u}}\sigma) \ \in \mathrm{V}_{\sigma(b)}(\mathcal{M}')$$

The Euclidean case

Transformation

$$\varphi : \mathbb{B} \longrightarrow \mathbb{E} \qquad \text{and its derivative} \quad \mathbf{F} = \mathrm{d}\varphi : \mathrm{T}\mathbb{B} \longrightarrow \mathrm{T}\mathbb{E}$$

Hyper-elastic material

$$\begin{aligned} \overline{\mathbf{W}} : \ \mathcal{L}_{\varphi}\left(\mathrm{T}\mathbb{B}, \mathrm{T}\mathbb{E}\right) &\longrightarrow \mathbb{R} \\ \mathbf{F}_X &\longmapsto \overline{\mathbf{W}}\left(\mathbf{F}_X\right) \end{aligned}$$

this energy is computed over a given point X.

Gradient of the energy

$$\left[\mathrm{D}\overline{\mathbf{W}}\left(\mathbf{F}\right)\right]_X = \mathrm{d}\overline{\mathbf{W}}_X\left(\mathbf{F}_X\right) \in \mathcal{L}\left(\mathrm{T}^*_X\mathbb{B}, \mathrm{T}^*_x\mathbb{E}\right)$$

$$\text{where} \qquad \mathrm{Mat}\left(\left[\mathrm{D}\overline{\mathbf{W}}\left(\mathbf{F}\right)\right]_X\right)_{ij} = \left.\frac{\partial \overline{\mathbf{W}}\left(\mathbf{F}\right)}{\partial \mathrm{Mat}\left(\mathbf{F}_X\right)_{ij}}\right|_{\mathrm{Mat}\left(\mathbf{F}_X\right)}$$

this gradient is computed over X and associated to a variation of $\mathbf{F}_X$

A geometric formalism

Metrics (linear, symmetrical and non-degenerate)

Spatial metric:

$$\mathbf{g} : \mathrm{T}\mathbb{E} \longrightarrow \mathrm{T}^*\mathbb{E}$$

Material metric (right Cauchy-Green):

$$\mathbf{C} = \mathbf{F}^T \mathbf{g} \mathbf{F} \in T_2^0(\mathbb{B})$$

where $T_2^0(\mathbb{B}) = \mathcal{L}\left((T^*\mathbb{B})^p \times (T\mathbb{B})^q, \mathbb{R}\right)$ is the space of p-contravariant and q-covariant tensors.

Isotropic behaviour

$$\mathbf{W} : \begin{array}{rcl} \mathcal{L}_{\mathrm{Id}}(\mathrm{T}\mathbb{B}, \mathrm{T}\mathbb{B}) & \longrightarrow & \mathbb{R} \\ \mathbf{C}_X & \longmapsto & \mathbf{W}(\mathbf{C}_X) \end{array} \qquad \mathrm{D}\mathbf{W} : \begin{array}{rcl} T_2^0(\mathbb{B}) & \longrightarrow & T_0^2(\mathbb{B}) \\ \mathbf{C} & \longmapsto & \mathrm{D}\mathbf{W}(\mathbf{C}) \end{array}$$

where $[\mathrm{D}\mathbf{W}(\mathbf{C})]_X = \mathrm{d}\mathbf{W}_X(\mathbf{C}_X) = \dfrac{1}{2}\mathbf{S}_X$

Summary *Euclidean case*

Geometry

$$(\mathbb{B}, \mathbf{G}) \text{ and } (\mathbb{E}, \mathbf{g})$$

Transformation

$$\mathbf{F} : \mathrm{T}\mathbb{B} \to \mathrm{T}\mathbb{E}$$

Behaviour

$$\mathbf{W} : \mathcal{L}_{\mathrm{Id}}\left(\mathrm{T}\mathbb{B}, \mathrm{T}\mathbb{B}\right) \longrightarrow \mathbb{R}$$

Equilibrium (without external forces) $\nabla \cdot \sigma = \mathbf{0}$

$$\nabla \cdot \left(\mathbf{F} \cdot \mathbf{S} \cdot \frac{\mathbf{F}^{-T}}{\det\left(\mathbf{F}\right)}\right) = 0$$

$$\nabla \cdot \left(\mathbf{F} \cdot 2\mathrm{D}\mathbf{W}\left(\mathbf{C}\right) \cdot \frac{\mathbf{F}^{-T}}{\det\left(\mathbf{F}\right)}\right) = 0$$

$$\nabla \cdot \left(\mathbf{F} \cdot 2\mathrm{D}\mathbf{W}\left(\mathbf{F}^{T}\mathbf{g}\mathbf{F}\right) \cdot \frac{\mathbf{F}^{-T}}{\det\left(\mathbf{F}\right)}\right) = 0$$

Summary *Fibre bundle case*

Geometry

$(\mathcal{M}, \mathbf{G})$ and $(\mathcal{E}, \mathbf{g})$

con. & sold. on $\mathrm{T}\mathcal{M}$
connexion on $\mathrm{T}\mathcal{E}$

Transformation

$$\mathbf{F} : \mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{E}$$

Behaviour

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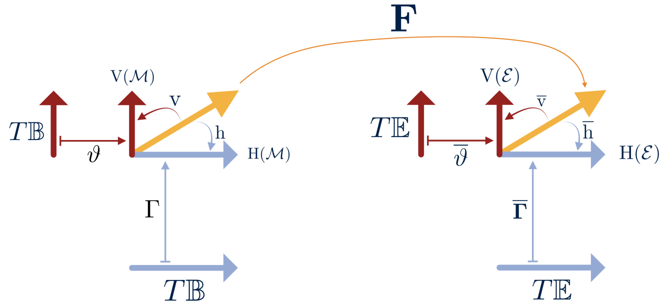
Scaled material modelling

$\mathcal{M}$ and $\mathcal{E}$ are vector bundles, with $\dim(\mathcal{F}) = 3$.

$$\pi : \mathcal{M} \longrightarrow \mathbb{B}$$
$$\overline{\pi} : \mathcal{E} \longrightarrow \mathbb{E}$$



$$\mathbf{F}_{|m} = \left(\overline{\mathbf{\Gamma}}_{\overline{m}} \quad \overline{\vartheta}_{\overline{m}}\right) \cdot \begin{pmatrix} \mathrm{d}\varphi_{|m} & 0 \\ \mathbf{F}^{\mathrm{v}}_{\mathrm{h}}{}_{|m} & \mathbf{F}^{\mathrm{v}}_{\mathrm{v}}{}_{|m} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{\Gamma}_m^{-1}\mathrm{h} \\ \vartheta_m^{-1}\mathrm{v} \end{pmatrix} \qquad \varphi : \mathbb{B} \to \mathbb{E}$$

- $\overline{\mathbf{\Gamma}}$, $\mathbf{\Gamma}$
- $\overline{\vartheta}$, ϑ

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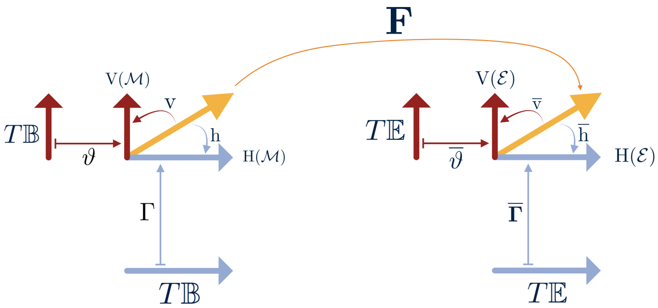

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- $\overline{\mathbf{\Gamma}}$, $\mathbf{\Gamma}$ $\Rightarrow$ Levi-Civita connexion multiplied by a macroscopic scaling factor L
- $\overline{\vartheta}$, ϑ $\Rightarrow$ canonical solder form multiplied by a microscopic scaling factor ℓ

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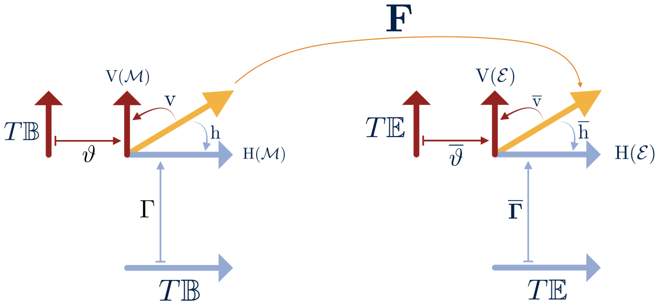

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$\Rightarrow$ **Scaling ratio:** $\zeta = \dfrac{l}{L} \in]0, 1]$

Scaled material modelling

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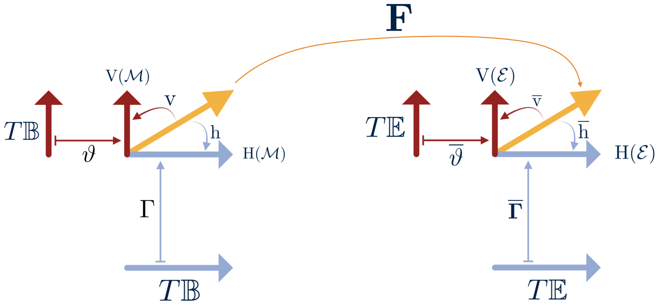

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- φ, $\mathbf{F}^{\mathrm{v}}_{\mathrm{h}}$, $\mathbf{F}^{\mathrm{v}}_{\mathrm{v}}$

Scaled material modelling

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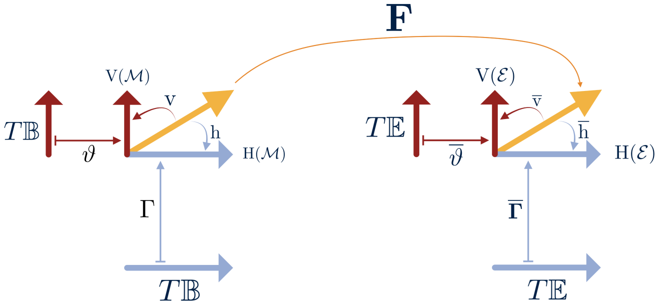

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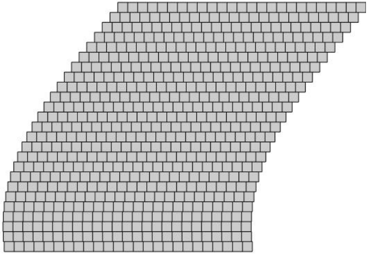
- φ, $\mathbf{F}^{\mathrm{v}}_{\mathrm{h}}$, $\mathbf{F}^{\mathrm{v}}_{\mathrm{v}}$

$\Rightarrow$ **Coupling term:** $\mathbf{F}^{\mathrm{v}}_{\mathrm{h}} = \mathrm{d}\varphi - \mathbf{F}^{\mathrm{v}}_{\mathrm{v}}$

Example

$$\varphi : \begin{pmatrix} X \\ Y \end{pmatrix} \longmapsto \begin{pmatrix} X - \mathbf{u}(Y) \\ Y \end{pmatrix}$$

$$\mathbf{F}^{\mathrm{v}}_{\mathrm{v}} = \mathrm{Id} \qquad \|\mathbf{u}(Y)\| \ll 1$$



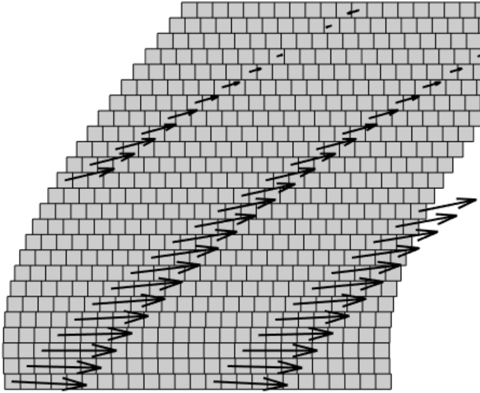
St. Vernant-Kirchhoff type of energy

$$\mathbf{W}(\mathbf{C}) = \frac{\lambda}{2}\mathrm{tr}\left[\frac{\mathbf{C}-\mathbf{G}}{2}\right]^2 + \mu\mathrm{tr}\left[\left(\frac{\mathbf{C}-\mathbf{G}}{2}\right)^2\right]$$

$$D\mathbf{W}(\mathbf{C})(\Delta\mathbf{C}) = \frac{\lambda}{2}\mathrm{tr}\left[\frac{\mathbf{C}-\mathbf{G}}{2}\right]\mathrm{tr}\left[\Delta\mathbf{C}\right] + \mu\mathrm{tr}\left[\frac{\mathbf{C}-\mathbf{G}}{2}\mathbf{G}^{-1}\Delta\mathbf{C}\right]$$

Example

$$[\nabla \cdot \sigma]_{(X,Y)} = [\nabla \cdot \sigma_{\text{elastic}}]_{(X,Y)} + \zeta\,\mathbf{u}''(Y) \begin{pmatrix} \frac{3}{2}\zeta\,(2\mu+\lambda)\,(\mathbf{u}'(Y))^2 \\ \zeta\,(2\mu+\lambda)\,\mathbf{u}'(Y) \\ \mu + \frac{3}{2}\,(1+\zeta^2)\,(2\mu+\lambda)\,(\mathbf{u}'(Y))^2 \\ 0 \end{pmatrix}$$



$\zeta = 0.0$

Example

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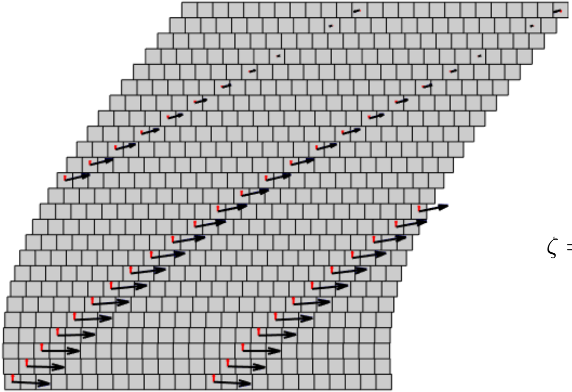
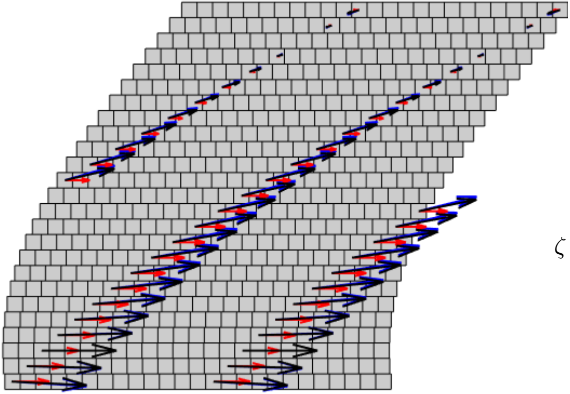


$\zeta = 0.2$

Example

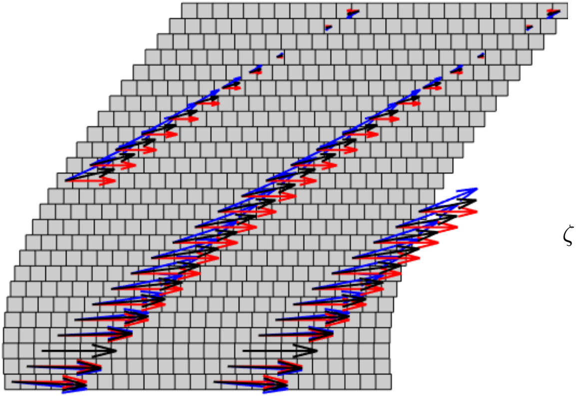
$$[\nabla \cdot \sigma]_{(X,Y)} = [\nabla \cdot \sigma_{\text{elastic}}]_{(X,Y)} + \zeta \, \mathbf{u}''(Y) \begin{pmatrix} \frac{3}{2}\zeta\,(2\mu+\lambda)\,(\mathbf{u}'(Y))^2 \\ \zeta\,(2\mu+\lambda)\,\mathbf{u}'(Y) \\ \mu + \frac{3}{2}\,(1+\zeta^2)\,(2\mu+\lambda)\,(\mathbf{u}'(Y))^2 \\ 0 \end{pmatrix}$$



$\zeta = 0.5$

Example

$$[\nabla \cdot \sigma]_{(X,Y)} = [\nabla \cdot \sigma_{\text{elastic}}]_{(X,Y)} + \zeta\, \mathbf{u}''(Y) \begin{pmatrix} \frac{3}{2}\zeta\,(2\mu+\lambda)\,(\mathbf{u}'(Y))^2 \\ \zeta\,(2\mu+\lambda)\,\mathbf{u}'(Y) \\ \mu + \frac{3}{2}\,(1+\zeta^2)\,(2\mu+\lambda)\,(\mathbf{u}'(Y))^2 \\ 0 \end{pmatrix}$$



$\zeta = 1.0$

Conclusion

- A temptation to extend standard elasticity applied on metrizable manifold to fiber bundle.
- The objective is to model the coupling at two distinct scales
- Considering vector bundle is a first step
  - Covariant derivation and divergence have been defined
  - Proper kinematics $\mathbf{F} : \mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{E}$ is construct
  - Proper metrics and strain-tensor are used
- A first example allows to gives simulation (but is not yet an equilibrium equation)

Van Hoi Nguyen, Guy Casale, Lo$\hat{\text{"}}$c Le Marrec. *On tangent geometry and generalized continuum with defects,*. Mathematics and Mechanics of Solids (2021)