Introduction to fractional calculus

Jacky CRESSON

Université de Pau et Pays des pays de l'Adour

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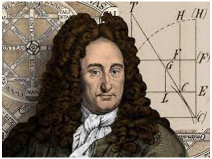
Fractional calculus ? Beginning with a curious question



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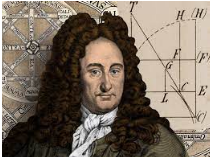

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L'Hopital receives the letter of Leibniz and writes

What if n be $1/2$?

Why not ?

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Classical reference: S. Samko, A. Kilbas and O. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Beach, Yverdon, 1993.

The algebraic problem

Finding an operator D^{α}, $\alpha \in \mathbb{R}^{+}$, which is an extension of the classical derivative, i.e. satisfying for $n \in \mathbb{N}$

$$D^{n}[f] = \frac{d^{n}f}{dt^{n}}. \tag{1}$$

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We want also that all these operators are **linear**
satisfies a semi-group property:

$$D^\alpha \circ D^\beta = D^{\alpha+\beta}. \tag{2}$$

Iterated left antiderivative

Let $I = [a, b]$ be an interval and $f : I \to \mathbb{R}^d$ be a sufficiently smooth function. We denote by $I^1_{a+}[f]$ the antiderivative of f vanishing at $t = a$, i.e.

$$\forall\ t \in I, \quad I^1_{a+}[f](t) = \int_a^t f(s)ds. \tag{3}$$

We denote by I^k_{a+} the operator defined for all $k \in \mathbb{N}^*$ by

$$I^k_{a+} = I^1_{a+} \circ \cdots \circ I^1_{a+} \quad (k \text{ times}). \tag{4}$$

Using the Fubini's theorem, we have

$$\begin{aligned} I^2_{a+}[f](t) &= \int_a^t \left(\int_a^s f(u)du\right) ds = \int_a^t \left(\int_u^t ds\right) f(u)du, \\ &= \int_a^t (t-u)f(u)du. \end{aligned} \tag{5}$$

We can prove by induction and the Fubini theorem that for all $t \in I$, we have

$$I^k_{a+}[f](t) = \frac{1}{(k-1)!} \int_a^t (t-u)^{k-1} f(u)du. \tag{6}$$

For every $k \in \mathbb{N}^*$, the quantity quantity $I^k_{a+}[f]$ is called the **left integral** with inferior limit a of order k of f.

Algebraic definition of iterated antiderivative

The left integral of order k with inferior limit a of f is the unique solution g of the following problem

$$\forall\ 0 \leq n \leq k-1, \quad g^{(n)}(a) = 0, \quad g^{(k)} = f. \tag{7}$$

The terminology for left comes from the fact that the integral is evaluated using value of $f(s)$ on the left hand side, i.e. with $s < t$.

Iterated derivatives

The k-th derivative of f, satisfies for all $t \in I$,

$$\left(\frac{d}{dt}\right)^{k}[f] = \left(\frac{d}{dt}\right)^{k+1}\left[I^{1}_{a+}[f]\right], \tag{8}$$

and

$$\left(\frac{d}{dt}\right)^{k}[f](t) - \left(\frac{d}{dt}\right)^{k}[f](a) = I^{1}_{a+}\left[\left(\frac{d}{dt}\right)^{k+1}[f]\right](t). \tag{9}$$

Right iterated integrals

Let $b \in \mathbb{R}$ and $f : I \to f$ be a sufficiently smooth function. We denote by $I^1_{b-}[f]$ the minus antiderivative of f vanishing at $t = b$, i.e. for all $t \in I$,

$$I^1_{b-}[f](t) = \int_t^b f(s)ds. \tag{10}$$

For every $k \in \mathbb{N}^*$, we denote by $I^k_{b-} = I^1_{b-} \circ \dots I^1_{b-}$ (k-times). We easily prove that for all $t \in I$, we have

$$I^k_{b-}[f](t) = \frac{1}{(k-1)!} \int_t^b (u-t)^{k-1} f(u)du. \tag{11}$$

The quantity $I^k_{b-}[f]$ is usually called the **right integral** of order k with superior limit b of f.

Right iterated derivatives

The k-th antiderivative satisfies for all $t \in I$ the relations

$$\left(-\frac{d}{dt}\right)^{k} [f] = \left(-\frac{d}{dt}\right)^{k+1} \left[I^{1}_{b-}[f] \right], \tag{12}$$

and

$$\left(-\frac{d}{dt}\right)^{k} [f](t) - \left(-\frac{d}{dt}\right)^{k} [f](b) = I^{1}_{b+} \left[\left(-\frac{d}{dt}\right)^{k+1} [f] \right] (t). \tag{13}$$

Main strategy

The basic idea behind fractional integrals and derivatives is that having a generalization of the left (resp. right) integral of order k for positive real values of k, then one can obtain a generalization of the notion of k-th derivative of f using:

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relations (8) leading to the left fractional Riemann-Liouville derivative
relations (9) leading to the left fractional Caputo derivative respectively.
The right analogue is also possible.

Left Riemann-Liouville fractional integrals

The **Gamma function of Euler** denoted by Γ and defined for all $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1} e^{-s} ds. \tag{14}$$

We have for all $k \in \mathbb{N}^*$ that $\Gamma(k) = (k-1)!$.

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The **Riemann-Liouville α-fractional integrals**, $\mathbb{R} \ni \alpha > 0$, for $f : [a, b]\mathbb{R}^d$, $d \in \mathbb{N}$, a $AC^2([a, b])$ function and $[a, b] \subset \mathbb{R}$, $0 \leq a < b$:

$$\begin{aligned} I_-^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t \in (a, b], \\ I_+^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau, \quad t \in [a, b), \end{aligned} \tag{15}$$

We set $I_-^0 f = I_+^0 f = f$.

Example

The left fractional integral of Riemann-Liouville for the function $f(t) = (t-a)^\beta$ with $\beta > -1$. We have

$$I_+^\alpha[f](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \tag{16}$$

for every $t \in [a, +\infty[$ if $\beta + \alpha \geq 0$ and $t \in]a, +\infty[$ if $\beta + \alpha < 0$.

Riemann-Liouville fractional derivatives

Restricting to $\alpha \in [0.1]$, we obtained what is called the Riemann-Liouville α-fractional derivatives:

$$\begin{aligned} D^{\alpha}_{-}[f] &= \frac{d}{dt}\left[I^{1-\alpha}_{-}[f]\right],, \\ D^{\alpha}_{+}[f] &= -\frac{d}{dt}\left[I^{1-\alpha}_{+}[f]\right]. \end{aligned} \tag{17}$$

It is easy to see that $D^0_- f = D^0_+ f = f$, whereas it can be proven that

$$D^1_- f = -D^1_+ f = df/dt. \tag{18}$$

Some examples

The left fractional Riemann-Liouville derivative of a constant is not zero. Indeed, we have

$$D_+^\alpha[c] = .\frac{c}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad (19)$$

for $\alpha \in]0,1[$ and $t > a$. This properties in particular implies that a geometric interpretation of the fractional derivative is not possible. Indeed, from a geometrical view point, for an arbitrary constant $c \in \mathbb{R}$, the fonction f and $f + c$ have the exactly the same geometric properties but not the same fractional derivatives.

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The left fractional Riemann-Liouville derivative of $f(t) = (t-a)^{\beta}$ is given by

$$D_{+}^{\alpha}[f](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \quad (20)$$

for every $t \in [a, +\infty[$ if $\beta - \alpha \geq 0$ and for every $t \in]a, +\infty[$ if $-1 < \beta < 0$.

Left Caputo fractional derivative

$$
\begin{aligned}
{}_cD_+^{\alpha}[f] &= I_+^{1-\alpha}\left[\frac{df}{dt}\right], \\
{}_cD_-^{\alpha}[f] &= I_-^{1-\alpha}\left[\frac{df}{dt}\right]
\end{aligned}
\tag{21}
$$

$$
{}_cD_\sigma^{\alpha}[c] = 0, \alpha \in]0,1[, \ \sigma = \pm. \tag{22}
$$

Connection between the RL and Caputo fractional derivative

Lemma

For every $\alpha \in]0,1[$ and every $f \in AC([a,b],\mathbb{R}^d)$, $D^\alpha_+[f]$ and ${}_cD^\alpha_+[f]$ are defined almost everywhere on $[a,b]$ and

$$D^\alpha_+[f](t) = {}_cD^\alpha_+[f](t) + \frac{1}{\Gamma(1-\alpha)}\frac{f(a)}{(t-a)^\alpha}, \tag{23}$$

or more simply that

$${}_cD^\alpha_+[f] = D^\alpha_+[f - f(a)]. \tag{24}$$

Properties of fractional deriavtives

- **Integration by part**:

$$\int_a^b f(t) D_\sigma^\alpha g(t) dt = \int_a^b D_{-\sigma}^\alpha f(t) g(t) dt, \ \sigma = \pm. \tag{25}$$

- **Semi-group property**:

$$D_\sigma^\alpha D_\sigma^\beta = D_\sigma^{\alpha+\beta}, \quad 0 \le \alpha, \beta \le 1/2, \tag{26}$$

where we assume both functions $f, g \in AC^2([a, b])$.

In particular, when $\alpha = \beta = 1/2$, we have the specialization

$$D_-^{1/2} D_-^{1/2} = d/dt, \quad D_+^{1/2} D_+^{1/2} = -d/dt. \tag{27}$$

Interpreting fractional derivatives

Let T_t be a stochastic process with a probability density ρ_t. We assume that the Laplace transform $\mathcal{L}$ of its probability density satisfies:

$$\mathcal{L}[\rho_t](s) = E_\alpha(-st^\alpha), \tag{28}$$

where $0 < \alpha < 1$ and E_α is the Mittag-Leffler function

$$E_\alpha(z) = \sum_{k \geq 0} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{29}$$

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Let f be a sufficiently smooth function $f : t \to f(t)$, we denote by $f_\alpha(t)$ the quantity:

$$f_\alpha(t) = \mathbb{E}(f(T_t)) = \int_0^\infty \rho_t(\tau) f(\tau)\, d\tau, \tag{30}$$

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The main property of this new dynamical variable is that it satisfies :

Theorem

Let $f : \mathbb{R}^+ \to \mathbb{R}$ a sufficiently smooth function, then we have

$$\forall t \geq 0,\ \ \mathbb{E}\left(\frac{df}{dt}(T_t)\right) = {}_cD_-^\alpha\left(\mathbb{E}(f(T_t))\right), \tag{31}$$

where ${}_cD_-^\alpha$ is the fractional derivative of Caputo with inferior limit 0.

Fractionalization of PDEs

Theorem

Let $u : (t,x) \in \mathbb{R} \times \mathbb{R}^d \to u(t,x)$ be a solution of the following partial differential equation

$$\frac{\partial f}{\partial t}(t) + A(f(t)) = 0, \tag{32}$$

where A is an operator satisfying the transfer property

$$\mathbb{E}\left(A(f(T_t))\right) = A\left(\mathbb{E}(f(T_t))\right). \tag{33}$$

Then, the function

$$u_\alpha(t,x) = \mathbb{E}(u(T_t,x)), \tag{34}$$

satisfies the fractional partial differential equation

$$ {}_c\partial^{\alpha}_{t,-} u_\alpha(t,x) + A.(u_\alpha(t,x)) = 0. \tag{35}$$

where ${}_c\partial^{\alpha}_{t,-} u_\alpha$ is the Caputo derivative with respect to t of u_α.

Discrete variational derivative and integrals

For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we denote by $x^{(n)}$ and $(x)_n$ the *Pochhammer symbols* defined by $x^{(0)} = (x)_0 = 1$ and for all $n \in \mathbb{N}^*$ by

$$x^{(n)} = x(x+1)\cdots(x+n-1), \qquad (x)_n = x(x-1)\cdots(x-n+1). \tag{36}$$

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Definition (Discrete fractional derivatives)

Let $0 < \alpha < 1$. The **right discrete fractional derivative of Grünwald-Letnikov** of superior limit b of order $\alpha > 0$ is the mapping from $C(,\mathbb{R}^d)$ to $C(^+,\mathbb{R}^d)$ defined for all $f \in C(,\mathbb{R}^d)$ by

$$\Delta_+^\alpha[f](t_k) = \frac{1}{h^\alpha}\sum_{n=0}^{N-k}\frac{(-1)^n}{n!}(\alpha)_n f_{k+n}, \quad \forall\ k = 0, ..., N-1. \tag{37}$$

The left discrete fractional derivative of Grünwald-Letnikov of inferior limit a of order $\alpha > 0$ is the mapping from $C(,\mathbb{R}^d)$ to $C(_-,\mathbb{R}^d)$ defined for all $f \in C(,\mathbb{R}^d)$ by

$$\Delta_-^\alpha[f](t_k) = \frac{1}{h^\alpha}\sum_{n=0}^{k}\frac{(-1)^n}{n!}(\alpha)_n f_{k-n}, \quad \forall\ k = 1, ..., N, \tag{38}$$

Discrete fractional integrals and properties

The previous formula keeps sense even for $\alpha < 0$. Following K. Diethelm ([**?**],§.2.4), we denote by $J_+^\alpha = \Delta_+^{-\alpha}$ and $J_-^\alpha = \Delta_-^{-\alpha}$ the right and left discrete fractional integrals.

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$$\Delta_-^\alpha[f] = \Delta_-^1\left[J_-^{1-\alpha}[f]\right], \quad \Delta_+^\alpha[f] = \Delta_+^1\left[J_+^{1-\alpha}[f]\right] \tag{39}$$

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Lemma (Semi-group property)

For all $\alpha, \beta > 0$, we have

$$\Delta_\sigma^\alpha\left[\Delta_\sigma^\beta\right] = \Delta_\sigma^{\alpha+\beta}, \quad J_\sigma^\alpha\left[J_\sigma^\beta\right] = J_\sigma^{\alpha+\beta}. \tag{40}$$

Moreover, we have

$$\Delta_\sigma^\alpha\left[J_\sigma^\alpha\right][f] = f \tag{41}$$

Variational structure for dissipative systems

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- Variational formulations can be obtained if one accepts to add to a given dissipative system a complementary set of equations as proved by H. Bateman in 1931. The main observation was that due to the irreversibility of a dissipative system, a given equation must be considered as physically incomplete as the dynamics is not invariant under time-reversing.

Another strategy ?

The Helmholt'z theorem uses crucially the framework of the classial differential calculus and is no longer valid when one use a different setting. The idea to use the **fractional calculus** as a possible framework to overcome this difficulty is originally due to F. Riewe in 1996.

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Let L be a C^2 Lagrangian function $L : \mathbb{R}^d \times \mathbb{R}^d \mathbb{R}$, $(x, v) \mapsto L(x, v)$, given $x, v \in \mathbb{R}^d$. The fractional action integral $L : C^2([a, b], \mathbb{R}^d)\mathbb{R}$ is defined by

$$L_\alpha(q) = \int_a^b L(q(t), D_-^\alpha q(t))\, dt. \tag{42}$$

When $\alpha = 1$, we have $D_-^1 = D_+^1 = d/dt$ so that (42) reduces to the classical action functional.

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Extremals of L_α are solutions of a fractional differential systems (see [?]) called the **fractional Euler-Lagrange equation** denoted by $(EL)_\alpha$ and given by

$$D_+^\alpha \frac{\partial L}{\partial v}(q, D_-^\alpha q) + \frac{\partial L}{\partial x}(q, D_-^\alpha q) = 0. \tag{43}$$

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Problem: $D_-^{1/2} \circ D_+^{1/2} \neq d/dt$!

Restoring causality

H. Bateman idea: the classical set of variables used to described a dissipative system has to be doubled in order to take into account the irreversibility of the dynamics. The new variable is understood as encoding the time reversed dynamics.

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doubling state space coupled with the use of fractional derivatives proposed by J. Cresson and P. Inizan in 2010.

$$\mathbb{L}(x, y, v_x, v_y, w_x, w_y), \tag{44}$$

and fractional Lagrangian functionals of the form

$$L_{\alpha,\beta}(x, y) = \int_a^b \mathbb{L}(x, y, \dot{x}, \dot{y}, D_-^{\alpha} x, D_+^{\beta} y)\, dt. \tag{45}$$

Time reversal

$$x_\star(x)(t) = x(a+b-t), \quad \forall\ t \in [a,b]. \tag{46}$$

Lemma (Time reversal duality)

Let $x \in AC([a,b])$, the following diagram commutes

$$\begin{array}{ccc} x & \xrightarrow{D_-^\alpha} & D_-^\alpha[x] \\ \Big\downarrow{\scriptstyle \star_{a,b}} & & \Big\downarrow{\scriptstyle \star_{a,b}} \\ x_\star & \xrightarrow{D_+^\alpha} & D_+^\alpha[x_\star] \end{array} \quad , \tag{47}$$

which implies the equality

$$D_+^\alpha[x_\star] = \left(D_-^\alpha[x]\right)_\star \tag{48}$$

Constraining the extended Lagrangian

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$$\mathbb{L}(x, x_\star, v, v_\star, w, w_\star) = L(x, v) + L(x_\star, v_\star) + P(w, w_\star), \qquad (49)$$

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$$\mathbb{L}(x, x_\star, v, v_\star, w, w_\star) = L(x, v) + L(x_\star, v_\star) + P(w, w_\star), \tag{49}$$

- The dynamics of x (resp. $x_\star$) depends only on x (resp. $x_\star$) and derivatives of the form $D_-^\gamma x$ (resp. $D_+^\gamma x_\star$).
- The dynamics satisfied by $x_\star$ is the time reversed dynamics of x.

Restriction on variations ?

Considering reversible variation, i.e. such that

$$h = h_\star \tag{50}$$

and the previous constraints, we have

$$\mathbb{L}(x, x_\star, v, v_\star, w, w_\star) = L(x, v) + L(x_\star, v_\star) + \gamma w w_\star, \tag{51}$$

If

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(x, \dot{x})\right) - \frac{\partial L}{\partial x}(x, \dot{x}) + \gamma D_{-}^{\alpha+\beta} x = 0. \tag{52}$$

then $(x, x_\star)$ is a critical point of $\mathbb{L}$!