Un principe variationnel de minimum pour les équations de Navier-Stokes

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Introduction

State of the Art

- **related topics :** Hamiltonian fluid dynamics [Salmon 1988]
- **Variational formulation :** anti-selfdual Lagrangians
  applied to steady incompressible Navier-Stokes equations [Ghoussoub 2006]

Key-ideas

- In [Buliga & de Saxcé MMS 2016], we proposed a symplectic version of Brezis-Ekeland-Nayroles principle based on the concepts of **symplectic subdifferential and polar functions**

- The object of this work is to generalize the previous formalism to dissipative media in **large deformations** in three steps :

  - a **Lagrangian formalism** for the reversible media based on the calculus of variation by jet theory

  - a corresponding **Hamiltonian formalism** for such media

  - a **symplectic minimum principle** for dissipative media

The original picture

The original picture :
a fragance of symplectic mechanics

- We are working in the "phase-space" of $z = \begin{pmatrix} x \\ \pi \end{pmatrix}$

  where x are the dof and π the momenta
- Symplectic form (or Lagrange's brackets)

  $\omega(z, z') = z^T J\, z' = (x, \pi) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} x' \\ \pi' \end{pmatrix}$
- Symplectic gradient (Hamiltonian vector field) $\dot{z} = X_H(t, z) = J\, \nabla_z H(t, z)$
- restitues the canonical equations

$$\dot{x} = grad_\pi H$$

$$\dot{\pi} = -grad_x H$$

- more generally $\quad \dot{z} = X_H \quad \text{s.t.} \quad \forall\, \delta z, \qquad \omega(\dot{z}, \delta z) = \delta H$

  If the dof are fields, the symplectic gradient is a functional derivative

The original picture : convex turning symplectic

- Decompose the evolution into reversible and irreversible parts :

$$\dot{z} = \dot{z}_R + \dot{z}_I, \quad \dot{z}_R = X_H, \quad \dot{z}_I = \dot{z} - X_H$$

- Dissipation potential $\phi = \phi(\dot{z})$, convex (but not differentiable !)
- Symplectic subdifferential

$$\partial^\omega \phi(\dot{z}) = \{\dot{z}_I \quad \text{s.t.} \quad \forall \dot{z}', \quad \phi(\dot{z} + \dot{z}') - \phi(\dot{z}) \geq \omega(\dot{z}_I, \dot{z}')\}$$

- Symplectic polar function

$$\phi^{*\omega}(\dot{z}_I) = \sup \{\omega(\dot{z}_I, \dot{z}) - \phi(\dot{z}) : \dot{z}\}$$

- Satisfying a symplectic Fenchel inequality

$$\phi(\dot{z}) + \phi^{*\omega}(\dot{z}_I) - \omega(\dot{z}_I, \dot{z}) \geq 0$$

  and the equality is reached for the constitutive law

$$\dot{z}_I \in \partial^\omega \phi(\dot{z})$$

The original picture :
the symplectic Brezis-Ekeland-Nayroles principle



Following [Brezis & Ekeland CRAS 1976, Nayroles CRAS 1976],
we proposed **a symplectic version of their principle :**

- the evolution curve $t \mapsto z(t)$ minimizes the functional

$$\Pi(z) = \int_0^T \left[\phi(\dot{z}) + \phi^{*\omega}(\dot{z} - X_H) - \omega(\dot{z} - X_H, \dot{z})\right] \, \mathrm{dt}$$

  among all curves such that $z(0) = z_0$ and the minimum is zero.

Applications

- **Plasticity** $x = (u, \varepsilon^p)$ with numerical applications
  [Cao, Oueslati, An Danh & de Saxcé *Comput. Mech.* 2020],
  [Cao et al. *Appl. Math. Model.* 2021], [Cao et al. CMAME 2021]
- **Fracture Mechanics** $x = (u, \psi)$ [de Saxcé IJSS 2022]

Lagrangian formalism

Lagrangian formalism :
modelling the matter and its motion

- Event occuring at position x and at time t

$$X = \begin{pmatrix} t \\ x \end{pmatrix} \in \textbf{time-space } \mathcal{M}$$

- Material point x_0 located at position x and at time t

$$x_0 = \kappa(t, x) = \kappa(X)$$

preserved along its trajectory

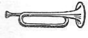
Lagrangian formalism :
calculus of variation by jet theory

1D heuristic :

- Vary both the value y and the variable x of the function $y(x)$

$$\delta\left(\frac{dy}{dx}\right)=\frac{dx\,d(\delta y)-d(\delta x)\,dy}{(dx)^2}$$

$$\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}(\delta y)-\frac{dy}{dx}\,\frac{d}{dx}(\delta x)$$

 Unlike the usual rule,
the variation of the derivative **is not equal to** the derivative of the variation

Lagrangian formalism : calculus of variation by jet theory

- **Rigorous version**
  - Action

$$\alpha\,[x_0] = \int_\Omega \mathcal{L}\left(X, x_0, \frac{\partial x_0}{\partial X}\right) d^4X$$

  - New parameterization $X = \psi(Y)$
  - Calculate the variation of

$$\alpha\,[X, x_0] = \int_{\Omega'} \mathcal{L}\left(\psi(Y), x_0, \frac{\partial x_0}{\partial Y}\,\frac{\partial Y}{\partial X}\right)\ \det\left(\frac{\partial X}{\partial Y}\right) d^4Y$$

  - Consider $Y = X$
  - $\boxed{\delta X \Rightarrow div_X\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial x_0}{\partial X}\right)}\,\frac{\partial x_0}{\partial X} - \mathcal{L}\ 1_{\mathbb{R}^4}\right) + \frac{\partial \mathcal{L}}{\partial X} = 0}$

For details, see the book [de Saxcé & Vallée, Galilean Mechanics and Thermodynamics of Continua, Wiley 2016]

Lagrangian formalism :
explicit form of variation equations

- Lagrangian $\mathcal{L} = \rho \left(\frac{1}{2} \parallel v \parallel^2 - e_{int} \right)$ with
  - the right Cauchy strains $C = F^T F$
  - the density $\rho = \frac{\rho_0(x_0)}{\sqrt{\det(C)}}$
  - the internal energy $e_{int}(x_0, C)$

Lagrangian formalism :
explicit form of variation equations

Introducing

- the linear momentum $\pi = grad_v \mathcal{L} = \rho\, v$
- the Hamiltonian density $\mathcal{H} = \pi \cdot v - \mathcal{L} = \rho \left(\frac{1}{2} \parallel v \parallel^2 + e_{int} \right)$
- the reversible stresses $\sigma_R = 2\rho\, F \, \frac{\partial e_{int}}{\partial C}\, F^T$

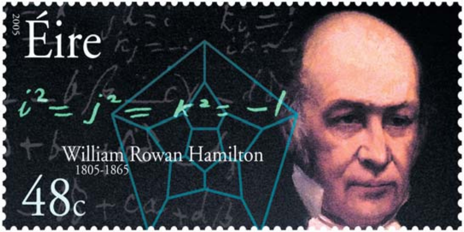
In absence of gravity, we recover the **balance equations** of :

- energy : $\dfrac{\partial \mathcal{H}}{\partial t} + div\ \left(\mathcal{H} v - \sigma\, v \right) = 0$
- linear momentum : $\boxed{-\dfrac{\partial \pi}{\partial t} + div\, (\sigma_R - v\, \pi^T) = 0}$

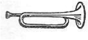
After simplification $\rho\, \frac{dv}{dt} = div\ \sigma_R$

For a barotropic fluid $\rho\, \frac{dv}{dt} = -grad\ q$

Hamiltonian formalism



Hamiltonian formalism : variation by jet theory

- We want to calculate the symplectic gradient by $\omega(X_H, \delta z) = \delta H$
- Total energy at time t : $H = \int_{\Omega_t} \mathcal{H}\left(x, x_0, \frac{\partial x_0}{\partial x}, \pi\right) d^3 x$
-  As v are the components of a 1-contravariant tensor and $\mathcal{H} = \pi \cdot v - \mathcal{L}$, $\pi \cdot v$ is a density

  Then π are the components of a 1-covariant and antisymmetric 3-contravariant tensor
- New parameterization $x = \psi(y)$
- Calculate the variation of
  $H\left[x, x_0, \pi\right] = \int_{\Omega'_t} \mathcal{H}\left(\psi(y), x_0, \frac{\partial x_0}{\partial y}\ \frac{\partial y}{\partial x}, \det\left(\frac{\partial y}{\partial x}\right)\ \left(\frac{\partial y}{\partial x}\right)^T \pi'\right)\ \det\left(\frac{\partial x}{\partial y}\right) d^3 y$
- Consider $y = x$

Hamiltonian formalism : symplectic gradient

-  In the original picture, $z(t)$ is an element of a manifold and $\dot{z}(t)$ is the tangent vector to the evolution curve $t \mapsto \boldsymbol{z}(t)$.

  In large deformations, there is not a canonical way to define $\dot{z}(t)$.

  Hence we replace $\dot{z}$ by

$$\zeta = \left(\frac{dx}{dt}, \frac{\partial \pi}{\partial t} \right)$$

- **Symplectic form**

$$\omega(\zeta, \zeta') = \int_{\Omega_t} \left(\frac{dx}{dt} \cdot \frac{\partial \pi}{\partial t}' - \frac{\partial \pi}{\partial t} \cdot \frac{dx}{dt}' \right) d^3 x$$

- The **canonical equations** read $\zeta = X_H$
  where the symplectic gradient is calculated by $\omega(X_H, \delta z) = \delta H$
  with arbitrary variations of the Hamiltonian performed by the jet theory

Hamiltonian formalism : symplectic gradient

- The corresponding **canonical equations** are

$$\frac{dx}{dt} = grad_\pi \mathcal{H}$$

$$\frac{\partial \pi}{\partial t} = -grad_x \mathcal{H}$$

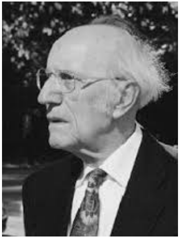
$$-div\left(\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial x_0}{\partial x}\right)}\,\frac{\partial x_0}{\partial x} - \left(\mathcal{H} - grad_\pi \mathcal{H}\cdot\pi\right) 1_{\mathbb{R}^3} + grad_\pi \mathcal{H}\,\pi^T\right)$$

with the extra terms of the jet theory in red

- For the previous Lagrangian, we recover by the last equation

$$\frac{dx}{dt} = \frac{\pi}{\rho}, \qquad -\frac{\partial \pi}{\partial t} + div\left(\sigma_R - v\,\pi^T\right) = 0$$

A symplectic minimum principle for dissipative media



A symplectic minimum principle for dissipative media

- An evolution curve $t \mapsto (\zeta, \kappa)$ is said **admissible** if it satisfies the initial and boundary conditions

- the **natural** curve minimize the functional

$$\Pi[\zeta, \kappa] = \int_T^0 \{\phi(\zeta) + \phi^{*\omega}(\zeta - X_H) - \omega(\zeta, \zeta - X_H)\} \, dt$$

among all admissible curves, and the minimum is zero

A symplectic minimum principle for dissipative media : additional hypothesis on the dissipation potential

- $\frac{\partial \pi}{\partial t}$ is **ignorable** in ϕ : $\phi\left(v, \frac{\partial \pi}{\partial t}\right) = \varphi(v)$
- Reminder : $\zeta_I = \zeta - X_H$ given by : $v_I = v - \frac{\pi}{\rho}$, $\quad \pi_I = \frac{\partial \pi}{\partial t} - div\left(\sigma_R - v\,\pi^T\right)$
- The symplectic polar function $\phi^{*\omega}(v_I, \pi_I)$ has a finite value equal to the Fenchel polar function $\varphi^*(-\pi_I)$ if $\pi = \rho\, v$
- The functional reads

$$\Pi[\zeta, \kappa] = \int_T^0 \left\{\varphi(v) + \varphi^*\left(-\frac{\partial \pi}{\partial t} + div\left(\sigma_R - v\,\pi^T\right)\right) - \omega(\zeta, \zeta - X_H)\right\} dt$$

- φ depends on v through the strain velocity $\quad D = \mathcal{D}(v) = grad_s\, v$
- $\varphi(v) = \int_{\Omega_t} W(\mathcal{D}(v))\, d^3x \qquad$ is quadratic in v
- Viscous dissipative stresses $\sigma_I = \frac{\partial W}{\partial D}(\mathcal{D}(v)) = 2\mu\, dev\,(D)$
- For the minimizer, $\zeta_I \in \partial^\omega \phi(\zeta)$ then $-\pi_I \in \partial\phi(v)$
  $-\frac{\partial \pi}{\partial t} + div\left(\sigma_R - v\,\pi^T\right) = -div\,\sigma_I$
  and we recover **Navier-Stokes equations**

$$\rho\,\frac{dv}{dt} = \frac{\partial \pi}{\partial t} + div\left(v\,\pi^T\right) = div(\sigma_R + \sigma_I) = -grad\, q + \mu \triangle v + \frac{\mu}{3}\, grad\,(div\, v)$$

Conclusion

- **The incompresible case** is obtained
  by introducing a constraint in the minimum principle.

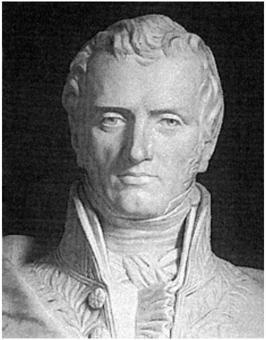
  The pressure appears as a Lagrange multiplier

- **Advantages of the present formulation**
  - It paves the way to providing variational approximations of the solutions
  - The functional is not convex but there is convexity
    that is good for the minimization procedure

- **Perspective**
  - Our approach is developped in the nonsmooth mechanics framework
    and can be used for Bingham fluids
  - Analytical and numerical applications
  - Functional analysis aspects

Cooperations with researchers
in fluid mechanics and mathematics are welcome

Thank you !



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