On some identification problems in fractional differential equations

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CFM2022 – Nantes

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From Continuous Time Random Walk (CTRW) to Fractional Derivatives

$(T_n)_{n\geq 1}$: positive *iid* random waiting times having a *pdf* $\psi(t)$, $t > 0$.
$(X_n)_{n\geq 1}$: *iid* random jumps having a *pdf* $w(x)$, $x \in \mathbb{R}$.

Setting $t_0 := 0$, $t_n := \sum_{k=1}^{n} T_k$.

The wandering particle :

- Starts at point $x = 0$ in instant $t = 0$.
- Makes a jump X_n in instant t_n.
- $x = 0$ for $0 \leq t < T_1 = t_1$.
- $x = \sum_{k=1}^{n} X_k$ for $t_n \leq t < t_{n+1}$.

Hypothesis : $(T_n)_{n\geq 1}$ and $(X_n)_{n\geq 1}$ are independent.

The master equation of Montroll & Weiss (1965)

Probabilistic arguments $\Longrightarrow$ The master Equation (Montroll & Weiss, 1965)

$$p(x,t) = \delta(x) \int_t^{+\infty} \psi(\tau)\, d\tau \;+ \int_0^t \psi(t-\tau) \left(\int_{-\infty}^{+\infty} w(x-\xi) p(\xi,\tau)\, d\xi \right) d\tau, \quad (1)$$

where

- $\delta(x)$ is the Dirac measure.
- $\displaystyle\int_t^{+\infty} \psi(\tau)\, d\tau \;= \mathbb{P}(T > t)$: (Survival probability : probability that the waiting time is $\geq t$ at a given position).
- $p(x, 0^+) = \delta(x)$.

From Master Equation to Fractional Differential Equations

Theorem 1 (R. Gorenflo & F. Mainardi)

Assume that :

- $w(x) \sim c_1\, |x|^{-(\alpha+1)}$ *as* $|x| \longrightarrow +\infty$, *with* $\alpha \in]0,2[$.
- $\psi(t) \sim c_2\, t^{-(\beta+1)}$ *as* $t \longrightarrow +\infty$, *with* $\beta \in]0,1[$.

Then, up to scaling the variables :

- $x \longleftarrow (\Delta x) \times x$,
- $t \longleftarrow (\Delta t) \times t$,
- *with* $(\Delta x)^\alpha = c_3\, (\Delta t)^\beta$,

the master equation (1) goes over to the space-time fractional diffusion equation :

$$\begin{cases} \mathscr{D}_t^\beta\, p(x,t) \; - \; {}_0D_1^\alpha\, p(x,t) = 0, \quad 0 < \alpha < 2, \quad 0 < \beta < 1, \\ u(x,0^+) = \delta(x), \quad x \in \mathbb{R}, \quad t > 0, \end{cases}$$

where the fractional differential operators $\mathscr{D}_t^\beta$ *and* ${}_0D_1^\alpha$ *will be specified.*

Definitions and notations

Definition 2 (Riemann-Liouville's fractional integral and derivative of order $\alpha > 0$)

1. The left and right sided Riemann-Liouville fractional integrals of order α :

$$\begin{cases} {}_aI^{\alpha}\, u(x) & = & \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x \frac{u(t)}{(x-t)^{1-\alpha}} dt, \\ I_b^{\alpha}\, u(x) & = & \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_x^b \frac{u(t)}{(t-x)^{1-\alpha}} dt. \end{cases}$$

2. The left and right sided Riemann-Liouville fractional derivatives of order α :

$$\begin{cases} {}_aD^{\alpha}\, u(x) & = & \dfrac{d^n}{dx^n} \left[{}_aI^{n-\alpha} u(x) \right], \\ D_b^{\alpha}\, u(x) & = & (-1)^n \dfrac{d^n}{dx^n} \left[I_b^{n-\alpha} u(x) \right], \end{cases}$$

where $\Gamma(x) = \displaystyle\int_0^{+\infty} t^{x-1}\, e^{-t}\, dt$ is the Euler's Gamma function and $n = [\alpha] + 1$.

For $0 < \alpha < 1$, we get :

- $_aD^\alpha\, u(x) = \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \left(\displaystyle\int_a^x \dfrac{u(t)}{(x-t)^\alpha} dt \right).$
- $D_b^\alpha\, u(x) = \dfrac{-1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \left(\displaystyle\int_x^b \dfrac{u(t)}{(t-x)^\alpha} dt \right).$

By permuting the operators $\dfrac{d^n}{dx^n}$ and $I^{n-\alpha}$, with $n = [\alpha] + 1$, we obtain :

Definition 3 (Caputo's fractional derivative)

- Left-sided Caputo fractional derivative : $_a\mathscr{D}^\alpha\, u(x) =_a I^{n-\alpha} u^{(n)}(x)$,
- Right-sided Caputo fractional derivative : $\mathscr{D}_b^\alpha\, u(x) = (-1)^n\, I_b^{n-\alpha} u^{(n)}(x)$.

Riemann-Liouville vs Caputo derivatives (functional analysis approach)

H. Emamirad, A. Rougirel. *Abstract differential triplet and boundary restriction operators with application to fractional differential operators*. 2021. https:hal.archives-ouvertes.frhal-02958830v2

Consider for example the pde problem

$$L_{\vec{\alpha}}\, u = f(u) \quad (+ \text{ Boundary and initial conditions}), \tag{2}$$

where $L_{\vec{\alpha}}$ is a linear combination of fractional differential operators D^{α_i}.

Let $u_{\vec{\alpha}}$ be the solution of (2) and assume that $u_{\vec{\alpha}}$ is sufficiently smooth with respect to its variables and with respect to $\vec{\alpha}$.

Then

$$\begin{aligned}
\frac{\partial}{\partial \alpha_i}\left[D^{\alpha_i}_{x_i} u_{\vec{\alpha}}\right](x_i) &= \frac{\partial}{\partial \alpha_i}\left[\frac{\partial}{\partial x_i}\left(\int_0^{x_i} \frac{(x_i - t)^{-\alpha_i}}{\Gamma(1-\alpha_i)} u_{\vec{\alpha}}\,(t)\; dt\right)\right] \\
&= \frac{\partial}{\partial x_i}\left(\int_0^{x_i} \frac{(x_i - t)^{-\alpha_i}}{\Gamma(1-\alpha_i)} \ln\left(\frac{1}{x_i - t}\right) u_{\vec{\alpha}}(t)\; dt\right) + \\
&+ \frac{\Gamma'(1-\alpha_i)}{\Gamma(1-\alpha_i)}\, D^{\alpha_i}_{x_i} u_{\vec{\alpha}}(x_i) + \; D^{\alpha_i}_{x_i} \frac{\partial u_{\vec{\alpha}}}{d\alpha}(x) \\
&= D^{\alpha_i}_{x_i}\left[\frac{\partial u_{\vec{\alpha}}}{\partial \alpha_i}\right](x_i) + \frac{\Gamma'(1-\alpha_i)}{\Gamma(1-\alpha_i)}\, D^{\alpha_i}_{x_i} u_{\vec{\alpha}}(x_i) + \mathbb{D}^{\alpha_i}_{x_i} u_{\vec{\alpha}}(x_i)
\end{aligned}$$

It follow that the problem satisfied by the function $\dfrac{\partial}{\partial \alpha_i}\left[D_{x_i}^{\alpha_i} u_{\overrightarrow{\alpha}}\right]$ is

$$L_{\overrightarrow{\alpha}}\, w = f'(u_{\overrightarrow{\alpha}})\, w \;+\; \mathbb{L}(u_{\overrightarrow{\alpha}}) \quad (+\ \text{Homogeneous boundary and initial conditions}), \tag{3}$$

Therefore, we can perform descent gradient methods to identify the vector derivation order $\overrightarrow{\alpha}$ from a cost function

$$J(\overrightarrow{\alpha}) = \|u_{\overrightarrow{\alpha}} - u_{\exp}\|^2$$

Application to the Taylor-Couette flow

We consider an application to coaxial annular flow in which a fluid is confined between two cylinders of radii R_{in} and R_{out}. A fractional order model governing the fluid velocity $u_\theta(r,t)$:

$$\begin{cases} \frac{\rho}{\mathbb{V}}\frac{\partial u_\theta}{\partial t} + \frac{\rho}{\mathbb{G}}\mathscr{D}_t^{2-\beta}u_\theta = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}, \quad R_{in} < r < R_{out},\ 0 < t \le T \\ u_\theta(R_{in},t) = \phi_i(t),\ u_\theta(R_{out},t) = \phi_o(t), \quad 0 \le t \le T, \\ u_\theta(r,0) = \frac{\partial u_\theta(r,0)}{\partial t} = 0, \quad R_{in} < r < R_{out}, \end{cases}$$

with $\beta \in]0,1[$. The constants ρ, $\mathbb{V}$ and $\mathbb{G}$ are given positive physical parameters.

Taylor-Couette numerical simulations

We choose $\widehat{\beta} = 0.7$ which will be supposed to be unknown. We are interested to find $\widehat{\beta}$ by considering a measure z of the solution $u_\theta(\widehat{\beta})$ on $]R_{in}, R_{out}[\times]0, T[$.

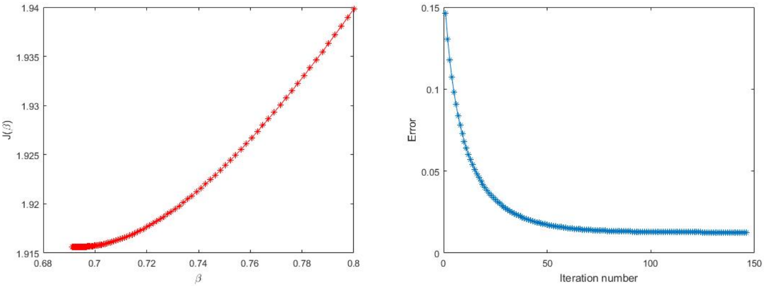

Figure 1: Left: The cost function $J(\beta)$, with initialization $\beta_0 = 0.8$. Right: Relative error for β in the steepest descent scheme.

Preliminaries

One-parametric Mittag-Leffler's function

For any $\gamma \in]0,1[$,

- Cauchy problem & one-parametric Mittag-Leffler function

$$\begin{cases} {}_0\mathscr{D}^\gamma\, u(t) &= u(t), \quad t > 0 \\ u(0) &= u_0 \end{cases} \implies u(t) = u_0\, E_\gamma(t^\gamma).$$

- E_θ is the one-parametric Mittag-Leffler function $(\theta > 0)$:

$$E_\theta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\theta k + 1)}, \text{ for any } z \in \mathbb{C},$$

Two-parametric Mittag-Leffler's function

For any $\alpha \in]1, 2[$,

- Boundary eigenvalue problem & two-parametric Mittag-Leffler function

$$\begin{cases} {}_0D^{\gamma}\, u(x) &= \lambda\, u(x), \quad 0 < x < 1 \\ u(0) = u(1) &= 0 \end{cases} \quad \Longrightarrow \quad u(x) = x^{\alpha-1}\, E_{\alpha,\alpha}(\lambda\, x^{\alpha}).$$

- $E_{\theta,\nu}$ is the two-parametric Mittag-Leffler function $(\theta > 0,\ \nu > 0)$:

$$E_{\theta,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\theta k + \nu)}, \text{ for any } z \in \mathbb{C}.$$

Theorem 4

Let $\theta\ \in]0\,,2\,[$, $\nu > 0$ *and* μ *be such that* $\frac{\theta\,\pi}{2} < \mu < \min(\pi, \theta\pi)$. *Then, there is a constant* $c = c(\theta, \nu, \mu) \geq 0$ *such that*

$$\left|E_{\theta,\nu}(z)\right| \leq \frac{c}{1+|z|}, \quad \forall\, z \in \mathbb{C}, \quad \mu \leq |\arg(z)| \leq \pi. \tag{4}$$

Theorem 5

Let $(f_n)_{n\in\mathbb{N}}$, *be a sequence of functions defined on the interval* $]a, b]$. *Suppose the following conditions are fulfilled:*

1. *the fractional derivative* $D_x^\alpha f_n(x)$ *exists for all* $n \in \mathbb{N}$ *and* $x \in]a, b]$;
2. *both series* $\sum_{n=1}^{\infty} f_n(x)$ *and* $\sum_{n=1}^{\infty} D_x^\alpha f_n(x)$ *are uniformly convergent on the interval* $[a+\varepsilon, b]$ *for any* $\varepsilon > 0$.

Then the function defined by the series $\sum_{n=1}^{\infty} f_n(x)$ *is* α*–differentiable on* $]a, b]$ *and satisfies*

$$D_x^\alpha\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} D_n^\alpha f_n(x). \tag{5}$$

Spectral problem

Consider the spectral problem:

$$D_x^{\alpha} X(x) = \lambda\, X(x),\ 0 \leq x \leq 1, \quad \text{with} \;\; X(0) = X(1) = 0. \tag{6}$$

The eigenfunctions of (6) are given by [1]:

$$X_n(x) = x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha}), \quad n \geq 1, \tag{7}$$

where λ_n are the associated eigenvalues respectively. Notice that these eigenvalues form a countable set that is denoted by $(\lambda_n)_{n\geq 1}$. Moreover, they satisfy the following

- (P1) There are only finitely many real eigenvalues and the rest appears as complex conjugate pairs.
- (P2) $|\lambda_n| \; \sim \; (2\,\pi\, n)^{\alpha}$ as $n \to +\infty$.
- (P3) $\dfrac{\alpha\,\pi}{2} < \arg(\lambda_n) \leq \pi$ for n sufficiently large and $\arg(\lambda_n) \; \sim \; \dfrac{\alpha\,\pi}{2}$ as $n \to +\infty$.

[1] Gorenflo, R. And Kilbas, A. And Mainardi, F. And Rogosin, S.,
Mittag-Leffler Functions, Related Topics and Applications. Springer (2014)

Spectral problem

- The family $(X_n)_{n\geq 1}$ is a basis of $L^2((0,1),\mathbb{C})$.
- This basis is not orthogonal (because of : D_x^α not self-edjoint)
- The family $\left(\widehat{X}_n\right)_{n\geq 1}$ defined by

$$\widehat{X}_n(x) = X_n(1-x) = (1-x)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (1-x)^\alpha\right) \quad (8)$$

correspond to the eigenfunctions of the adjoint operator of D_x^α with respect to the inner product in $L^2((0,1),\mathbb{C})$: $\langle f\,,\,g\rangle_{L^2((0,1),\mathbb{C})} = \int_0^1 f(x)\,\overline{g}(x)\,dx,$

- The adjoint operator of D_x^α is the right sided RL α-fractional derivative :

$${}_xD^{\alpha_i} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_x^1 (t-x)^{n-1-\alpha} f(t)\,dt, \;\; n = [\alpha]+1. \quad (9)$$

- The family $(\widehat{X}_n)_{n\geq 1}$ is biorthogonal to $(X_n)_{n\geq 1}$ in $L^2((0,1),\mathbb{C})$ and satisfy

$$\langle X_n\,,\,\widehat{X}_m\rangle_{L^2(0,1)} = 0 \;\; \text{for} \;\; n\neq m \;\; \text{and} \;\; \langle X_n\,,\,\widehat{X}_n\rangle_{L^2(0,1)} = C_n > 0. \quad (10)$$

[2] T. S. Aleroev, M. V. Khasmaev, *Boundary Value Problem for one-dimensional differential advection-dispersion equation*. Vestnik MGSU [Proceedings of Moscow State University of Civil Engineering] (2014) 71–76.

Source identification problem

Let us consider the source identification problem

$$\begin{cases} \mathcal{D}_t^{\gamma} u(x,t) - \sum_{i=1}^{d} D_{x_i}^{\alpha_i} u(x,t) = p(x)\, f(t), \ (x,t) \in \Omega \times]0,T[, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = \varphi(x), \ x \in \overline{\Omega}, \\ \int_{\Omega} u(x,t)\, dx = \psi(t), \ t \in [0,T]. \end{cases} \tag{11}$$

where the functions p, φ and ψ are given and the functions u and f are unknown.

Source identification problem

We integrate the equation on Ω and expand of $p(x)$ and $\varphi(x)$ in the biorthogonal bases:

$$p(x) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} p_{n_1 \cdots n_d} \bigotimes_{i=1}^{d} X^i_{n_i}, \tag{12}$$

$$\varphi(x) = \sum_{n_1 \geq 1} \cdots \sum_{n_d \geq 1} \varphi_{n_1 \cdots n_d} \bigotimes_{i=1}^{d} X^i_{n_i}, \tag{13}$$

with
$$p_{n_1 \cdots n_d} = \frac{\left\langle p \, , \bigotimes_{i=1}^{d} \widehat{X}^i_{n_i} \right\rangle_{L^2(\Omega)}}{\left\langle \bigotimes_{i=1}^{d} X^i_{n_i} \, , \bigotimes_{i=1}^{d} \widehat{X}^i_{n_i} \right\rangle_{L^2(\Omega)}}, \tag{14}$$

and
$$\varphi_{n_1 \cdots n_d} = \frac{\left\langle \varphi \, , \bigotimes_{i=1}^{d} \widehat{X}^i_{n_i} \right\rangle_{L^2(\Omega)}}{\left\langle \bigotimes_{i=1}^{d} X^i_{n_i} \, , \bigotimes_{i=1}^{d} \widehat{X}^i_{n_i} \right\rangle_{L^2(\Omega)}}. \tag{15}$$

Source identification problem

It follows that f satisfies the integral equation :

$$f(t) = \frac{1}{a}\left(\mathcal{D}_t^{\gamma}\psi(t) - \phi(t) - \int_0^t (t-s)^{\gamma-1}\, f(s)\, \rho(t-s)ds\right), \tag{16}$$

where

$$\Gamma_{n_1\cdots n_d} = \prod_{i=1}^{d} E_{\alpha_i,\alpha_i+1}\left(\sum_{k=1}^{d}\lambda_{n_k}^{k}\right).$$

$$a = \sum_{n_1\geq 1}\cdots\sum_{n_d\geq 1}\Gamma_{n_1\cdots n_d}\, p_{n_1\cdots n_d} \tag{17}$$

$$\phi(t) = \sum_{n_1\geq 1}\cdots\sum_{n_d\geq 1}\left(\sum_{i=1}^{d}\lambda_{n_i}^{i}\right)\varphi_{n_1\cdots n_d}\,\Gamma_{n_1\cdots n_d}\, E_{\gamma,1}\left(\left(\sum_{i=1}^{d}\lambda_{n_i}^{i}\right)t^{\gamma}\right) \tag{18}$$

and

$$\rho(t) = \sum_{n_1\geq 1}\cdots\sum_{n_d\geq 1}\left(\sum_{i=1}^{d}\lambda_{n_i}^{i}\right)p_{n_1\cdots n_d}\,\Gamma_{n_1\cdots n_d}\, E_{\gamma,\gamma}\left(\left(\sum_{i=1}^{d}\lambda_{n_i}^{i}\right)t^{\gamma}\right). \tag{19}$$

Source identification problem

Theorem 6

Let

- $p \in \mathcal{C}\left(\overline{\Omega}, \mathbb{R}\right)$ *such that* $D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d} p \in \mathcal{C}\left(\overline{\Omega}, \mathbb{R}\right)$ *and* $\displaystyle\int_\Omega p(x)\, dx \neq 0$.
- *Let* $\psi \in \mathcal{C}([0,T], \mathbb{R})$ *such that* $\mathcal{D}_t^\gamma \psi \in \mathcal{C}([0,T], \mathbb{R})$.
- $\varphi \in \mathcal{C}\left(\overline{\Omega}, \mathbb{R}\right)$ *such that* $D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d} \varphi \in \mathcal{C}\left(\overline{\Omega}, \mathbb{R}\right)$ *and* $\varphi|_{\partial\Omega} = 0$.

Then there exists at least one solution to Problem (11).

Idea of the proof. Schauder fixed point theorem in Banach spaces with the Arzela-Ascoli compactness result :

$$\begin{array}{rcl} B \ : \quad C([0,T],\mathbb{R}) & \longrightarrow & C([0,T],\mathbb{R}) \\ f & \longmapsto & t \mapsto \dfrac{1}{a}\left(\mathcal{D}_t^\gamma \psi(t) - \phi(t) - \displaystyle\int_0^t (t-s)^{\gamma-1} f(s) \rho(t-s) ds\right). \end{array} \tag{20}$$

Bielecki norm

- To prove that the operator B admits a fixed point, start by showing that B maps a certain closed convex set into itself, in the space $C([0,T],\mathbb{R})$ equipped with the Bielecki norm, which is equivalent to the uniform norm on $[0,T]$.
- For every $\delta > 0$, we introduce the Bielecki norm

$$\| u \|_\delta = \sup_{t\in[0,T]} \left(e^{-\delta\, t} | u(t) | \right). \tag{21}$$

- The space $(C([0,T],\mathbb{R})\,,\,\|\cdot\|_\delta)$ is a Banach space.
- The two norms $\|\cdot\|_\delta$ and $\|\cdot\|_\infty$ are equivalent.

Lemma 7

Under the above notations, there exists a positive constant $\delta_* > 0$ such that for any $\delta > \delta_*$ there is a radius $R_\delta > 0$ such that the closed convex ball

$$K = \{f \in C([0,T],\mathbb{R}) \;:\; \|f\|_\delta \leq R_\delta\}$$

is stable by the operator B, that is, $B(K) \subset K$.

Bielecki norm

Remark. Using the Bielecki norm allows to have no constraint on the maximum value that T can take. On the other hand, if we use the classical infinite norm, then T must be less than a finite quantity depending on the data of the problem.

Idea of the proof.

- For any $f \in C([0,T],\mathbb{R})$, $t \in [0,T]$ and $\delta > 0$, we have

$$\begin{aligned} e^{-\delta\, t}|B(f)(t)| &= \frac{e^{-\delta\, t}}{|a|}\left|D_t^\gamma \psi(t) - \phi(t) - \int_0^t (t-s)^{\gamma-1}\rho(t-s)\, f(s)\, ds\right|, \\ &= \frac{1}{|a|}\left|e^{-\delta\, t}\left(D_t^\gamma \psi(t) - \phi(t)\right) - \int_0^t e^{-\delta\,(t-s)}(t-s)^{\gamma-1}\rho(t-s)\, e^{-\delta\, s}\, f(s) ds\right|, \\ &\leq \frac{1}{|a|}\left(\|D_t^\gamma \psi\|_\delta + \|\phi\|_\delta + \|f\|_\delta \int_0^T e^{-\delta\, s} s^{\gamma-1}\, |\rho(s)|\, ds\right). \end{aligned}$$

- Then for $\delta > 0$ sufficiently small, we can find R_δ such that K_δ is stable by B.

Lemma 8

The family $(B(f))_{f\in K}$ *is equicontinuous, that is :* $\forall\, t \in [0,T],\ \forall\, \varepsilon > 0,$

$$\exists\, r > 0\ :\ \forall\, t' \in [0,T]\cap]t-r\,,\,t+r[,\ \forall\, f \in K,\ \ |B(f)(t) - B(f)(t')| \leq \varepsilon.$$

Idea of the proof : Classical estimates ...

By Arzela-Ascoli theorem, $B(K)$ is relatively compact. Now, we are able to use the Schauder fixed point theorem.

Theorem 9 (Schauder fixed point theorem)

Let $(E, \|.\|)$ *be a Banach space, let* $K \subset E$ *convex and closed. Let* $\mathbb{L} : K \longrightarrow K$ *be a continuous operator such that* $\mathbb{L}(K)$ *is relatively compact. Then* $\mathbb{L}$ *has a fixed point in* K*.*

[Proof of Theorem 6] We know, from Lemma 7 and Lemma 8 that B is a continuous operator and $B(K)$ is relatively compact, with K convex and closed. By Theorem 9, the operator B admits a fixed point in K, so the integral equation (16) has a solution in $[0,T]$.
Once the existence of the source term f is established, the existence of u follows in a similar way with the arguments developed above for the direct problem.

Theorem 10

Under the conditions of Theorem 6, every solution of the inverse problem (11) depends continuously on the given data φ, p and ψ.

Idea of the proof : Classical estimates ...

- The temporal source term f :

$$\left\| f-\widetilde{f}\right\|_{\delta} \leq C\left(\left\| \mathcal{D}_t^{\gamma}\psi - \mathcal{D}_t^{\gamma}\widetilde{\psi}\right\|_{L^{\infty}(0,T)} + \|\varphi-\widetilde{\varphi}\|_{L^{\infty}(\Omega)} + \|p-\widetilde{p}\|_{L^{\infty}(\Omega)} \right),$$

- The solution u : The continuity of the part u of the solution follows with classical arguments for linear problems with the norm

$$\|u\|_{\infty,\delta} = \sup_{x\in\overline{\Omega}} \sup_{t\in[0,T]} \left| e^{-\delta\, t} u(x,t)\right|.$$

Merci beaucoup pour votre attention !