Ecrire les lois de comportement sans (ab)user de l'hypothèse d'indifférence matérielle : cas des fluides visqueux et de l'hyperélasticité

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Outline

1. **Un point sur la controverse autour de** *Approximation of Material Frame Indifference* **(AMFI)**
2. **Fluides différentiels de degré 2**
3. **Solides hyperélastiques**
4. **Conclusions and recommandations**

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THE MATHEMATICAL THEORY OF NON-UNIFORM GASES

AN ACCOUNT OF THE KINETIC THEORY OF VISCOSITY, THERMAL CONDUCTION AND DIFFUSION IN GASES

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[Chapman and Cowling, 1970]

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CORRECTION OF TWO ERRORS IN THE KINETIC THEORY OF GASES WHICH HAVE BEEN USED TO CAST UNFOUNDED DOUBT UPON THE PRINCIPLE OF MATERIAL FRAME-INDIFFERENCE *

Clifford Truesdell**

He who regards the kinetic theory as providing the one and only right approach to gas flows should discard all of continuum mechanics, not just one or another part of it. Many physicists do so.

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MECCANICA [Truesdell, 1976]

CENTER FOR THE APPLICATION OF MATHEMATICS

LEHIGH UNIVERSITY

BETHLEHEM, PA.

RED HERRINGS AND SUNDRY UNIDENTIFIED FISH IN NON-LINEAR CONTINUUM MECHANICS

by

R. S. RIVLIN

1. Introduction

The title of this talk is evidently provocative and represents a radical departure from other papers I have written. The modern resurgence of interest in the fundamentals of non-linear continuum theories and in the study of their implications dates very largely from the end of the Second

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[Rivlin, 1969]

Chapman and Cowling's notations

$\overline{\boldsymbol{w}}$ is the *conjugate* / transpose of second order tensor $\boldsymbol{w}$ p. 15

$\overline{\overline{\boldsymbol{w}}} = \frac{1}{2}(\boldsymbol{w} + \overline{\boldsymbol{w}})$ is the symmetric part of $\boldsymbol{w}$ p. 15

$\overset{\circ}{\boldsymbol{w}} = \boldsymbol{w} - \dfrac{1}{3}(w_{xx} + w_{yy} + w_{zz})\mathbf{U}$ is *non–divergent* / deviatoric p. 15

The gradient of a vector field is[1] $\boldsymbol{\nabla c} = \dfrac{\partial c_j}{\partial x_i}\mathbf{e}_i \otimes \mathbf{e}_j$ p. 18

The *rate-of-strain* and *rate-of-shear* tensors are

$$\boldsymbol{e} = \overline{\overline{\boldsymbol{\nabla c_0}}}, \quad \overset{\circ}{\boldsymbol{e}} = \overset{\circ}{\overline{\overline{\boldsymbol{\nabla c_0}}}}$$

where $\boldsymbol{c_0}$ is the velocity vector p. 19

One of the vectors in a dyadic may be a vector differential operator such as $\partial/\partial \boldsymbol{r}$ ($\equiv \nabla$). If, for example, $\boldsymbol{c}$ is a vector with components u, v, w, the components of $\dfrac{\partial}{\partial \boldsymbol{r}}\boldsymbol{c}$ ($\equiv \nabla \boldsymbol{c}$) are

$$\left.\begin{matrix} \dfrac{\partial u}{\partial x}, & \dfrac{\partial v}{\partial x}, & \dfrac{\partial w}{\partial x}, \\ \dfrac{\partial u}{\partial y}, & \dfrac{\partial v}{\partial y}, & \dfrac{\partial w}{\partial y}, \\ \dfrac{\partial u}{\partial z}, & \dfrac{\partial v}{\partial z}, & \dfrac{\partial w}{\partial z}, \end{matrix}\right\}$$

(1.33, 1)

$$\begin{matrix} w_{xx}, & w_{xy}, & w_{xz}, \\ w_{yx}, & w_{yy}, & w_{yz}, \\ w_{zx}, & w_{zy}, & w_{zz}, \end{matrix}$$

1. Note that the transpose of this definition is frequently used.

Skew tensors

$\overset{\times}{\boldsymbol{w}} = \frac{1}{2}(\boldsymbol{w} - \overline{\boldsymbol{w}})$ is the skew–symmetric part of $\boldsymbol{w}$ p. 23

so that $\boldsymbol{w} = \overline{\overline{\boldsymbol{w}}} + \overset{\times}{\boldsymbol{w}}$ p. 23

The tensor $\overset{\times}{\boldsymbol{w}}$ has components [2]

$$\begin{matrix} 0, & \tfrac{1}{2}(w_{xy} - w_{yx}), & \tfrac{1}{2}(w_{xz} - w_{zx}), \\ \tfrac{1}{2}(w_{yx} - w_{xy}), & 0, & \tfrac{1}{2}(w_{yz} - w_{zy}), \\ \tfrac{1}{2}(w_{zx} - w_{xz}), & \tfrac{1}{2}(w_{zy} - w_{yz}), & 0. \end{matrix}$$

The vector $\overset{\times}{\boldsymbol{\omega}}$ is associated with tensor $\overset{\times}{\boldsymbol{w}}$ with components [3]

$$\tfrac{1}{2}(w_{yz} - w_{zy}), \quad \tfrac{1}{2}(w_{zx} - w_{xz}), \quad \tfrac{1}{2}(w_{xy} - w_{yx}).$$

and such that $\overset{\times}{\boldsymbol{w}} \cdot \boldsymbol{a} = \overline{\overline{\boldsymbol{w}}} \cdot \boldsymbol{a} + \boldsymbol{a} \wedge \overset{\times}{\boldsymbol{\omega}}$ p. 24

---

2. Note that this is the usual definition.
3. The opposite vector is often used.

The method of solution of Boltzmann's equation

- Expansion of the *pressure tensor* and heat flux vector

$$\mathsf{p} = \sum_0^\infty \mathsf{p}^{(r)}, \quad \boldsymbol{q} = \sum_0^\infty \boldsymbol{q}^{(r)},$$

$$\mathsf{p}^{(r)} = \int m\boldsymbol{C}\boldsymbol{C}f^{(r)}d\boldsymbol{c}, \quad \boldsymbol{q}^{(r)} = \int E\boldsymbol{C}f^{(r)}d\boldsymbol{c};$$

  p. 115

  $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{c}_0$ is the *peculiar velocity* p. 27

- Zero$^{\text{th}}$ order solution (first approximation)

  $\mathsf{p}^{(0)} = knT\mathsf{U} = \mathsf{U}p,$

  $\boldsymbol{q}^{(0)} = 0$ p. 115

  p is the hydrostatic pressure

- First order solution (second approximation)

  $\boldsymbol{p}^{(1)} = -2\mu \overset{\circ}{\boldsymbol{e}} = -2\mu \overset{\circ}{\overline{\overline{\boldsymbol{\nabla} \boldsymbol{c}_0}}}$ p. 126

  $\boldsymbol{q}^{(1)} = -\lambda \boldsymbol{\nabla} T$ p. 125

  the existence of *volume viscosity* can also be derived p. 408

- Viscosity and thermal conduction in a simple gas

  $\lambda = \frac{5}{2} C_v \mu$, mean free path time $\tau = \mu / p$ p. 160

The third approximation (Burnett)

$$\mathsf{p}^{(2)} = \varpi_1 \frac{\mu^2}{p} \Delta \mathring{\mathsf{e}} + \varpi_2 \frac{\mu^2}{p} \left(\frac{D_0}{Dt} (\mathring{\mathsf{e}}) - 2 \overset{\circ}{\overline{\overline{\nabla c_0 . \mathring{\mathsf{e}}}}} \right) + \varpi_3 \frac{\mu^2}{\rho T} \overset{\circ}{\overline{\overline{\nabla \nabla T}}}$$
$$+ \varpi_4 \frac{\mu^2}{\rho p T} \overset{\circ}{\overline{\overline{\nabla p \nabla T}}} + \varpi_5 \frac{\mu^2}{\rho T^2} \overset{\circ}{\overline{\overline{\nabla T \nabla T}}} + \varpi_6 \frac{\mu^2}{p} \overset{\circ}{\overline{\overline{\mathring{\mathsf{e}} . \mathring{\mathsf{e}}}}},$$

p. 286

$\boldsymbol{p}^{(2)}$ is *both symmetrical and non–divergent*[4].

The ϖ_i are pure numbers with approximate values

$\varpi_1 = \frac{4}{3}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right), \quad \varpi_2 = 2, \quad \varpi_3 = 3, \quad \varpi_4 = 0$

$\varpi_5 = \frac{3T}{\mu}\frac{d\mu}{dT}, \quad \varpi_6 = 8$

D/Dt is the *mobile operator, time–derivative following the motion* p. 48

$\frac{D_0}{Dt} = \frac{\partial_0}{\partial t} + \boldsymbol{c}_0 \cdot \frac{\partial}{\partial \boldsymbol{r}}$ denotes a first approximation to D/Dt p. 116

$\boldsymbol{c}_0$ is the *mean molecular velocity* p. 27

[Burnett, 1935]

4. The notation $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{c}_0 = \mathrm{Tr}\, \boldsymbol{e}$ is introduced on p.284

The third approximation (Burnett)

$$q^{(2)} = \theta_1 \frac{\mu^2}{\rho T} \Delta \nabla T + \theta_2 \frac{\mu^2}{\rho T} \left\{ \frac{D_0}{Dt} (\nabla T) - (\nabla c_0) . \nabla T \right\}$$
$$+ \theta_3 \frac{\mu^2}{\rho p} \nabla p . \overset{\circ}{\mathbf{e}} + \theta_4 \frac{\mu^2}{\rho} \nabla . (\overset{\circ}{\mathbf{e}}) + \theta_5 \frac{3\mu^2}{\rho T} \nabla T . \overset{\circ}{\mathbf{e}},$$

p. 285

The θ_i are pure numbers with approximate values

$$\theta_1 = \frac{15}{4} \left(\frac{7}{2} - \frac{T}{\mu} \frac{d\mu}{dT} \right), \quad \theta_2 = \frac{45}{8}, \quad \theta_3 = -3, \quad \theta_4 = 3$$

$$\theta_5 = \frac{35}{4} + \frac{T}{\mu} \frac{d\mu}{dT}$$

The fourth approximation (super–Burnett)

No exhaustive form of the super–Burnett terms is provided in the literature, apparently and unfortunately.

[Struchtrup, 2005]

Examen de la loi de comportement mécanique

Avec les notations désormais classiques [5],

$$
\begin{aligned}
\boldsymbol{\sigma} &= -p\mathbf{1} + 2\mu \boldsymbol{D}^{\mathrm{dev}} + \boldsymbol{\sigma}_2^{\mathrm{dev}} \\
\boldsymbol{\sigma}_2 &= -\varpi_1 \frac{\mu^2}{p} (\mathrm{trace}\, \boldsymbol{D})\, \boldsymbol{D} - 8\frac{\mu^2}{p} \boldsymbol{D}^{\mathrm{dev}} \cdot \boldsymbol{D}^{\mathrm{dev}} \\
&- 2\frac{\mu^2}{p} \left(\dot{\boldsymbol{D}} - \boldsymbol{L}^T \cdot \boldsymbol{D}^{\mathrm{dev}} - \boldsymbol{D}^{\mathrm{dev}} \cdot \boldsymbol{L} \right)
\end{aligned}
$$

où $L_{ij} = v_{i,j} = D_{ij} + W_{ij}$
La dernière contribution se développe de la manière suivante

$$
\dot{\boldsymbol{D}} - \boldsymbol{L}^T \cdot \boldsymbol{D}^{\mathrm{dev}} - \boldsymbol{D}^{\mathrm{dev}} \cdot \boldsymbol{L} = \dot{\boldsymbol{D}} - \boldsymbol{L}^T \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{L} + \frac{2}{3} (\mathrm{trace}\, \boldsymbol{D})\, \boldsymbol{D}
$$

Noter la dérivée temporelle similaire intervenant dans l'expression du flux :

$$
\dot{\boldsymbol{g}} - \boldsymbol{L}^T \cdot \boldsymbol{g}, \quad \text{avec} \quad \boldsymbol{g} = \boldsymbol{\nabla}\, T
$$

5. *C&C term* $\boldsymbol{L}^T$ *the velocity gradient* [Murdoch, 1983], note 5, p. 188

Examen de la loi de comportement mécanique

La contribution $\dot{\boldsymbol{D}} - \boldsymbol{L}^T \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{L}$ peut être comparée avec

- les *dérivées convectives*

$$
\begin{aligned}
\overset{\triangledown}{\boldsymbol{A}}_{\cdot\cdot} &= \dot{\boldsymbol{A}}_{\cdot\cdot} + \boldsymbol{L}^T \boldsymbol{A}_{\cdot\cdot} + \boldsymbol{A}_{\cdot\cdot} \boldsymbol{L} \quad \text{Cotter} - \text{Rivlin} \\
\overset{\triangledown}{\boldsymbol{A}}_{\cdot}^{\ \cdot} &= \dot{\boldsymbol{A}}_{\cdot}^{\ \cdot} + \boldsymbol{L}^T \boldsymbol{A}_{\cdot}^{\ \cdot} - \boldsymbol{A}_{\cdot}^{\ \cdot} \boldsymbol{L}^T \\
\overset{\triangledown}{\boldsymbol{A}}^{\cdot}_{\ \cdot} &= \dot{\boldsymbol{A}}^{\cdot}_{\ \cdot} - \boldsymbol{L} \boldsymbol{A}^{\cdot}_{\ \cdot} + \boldsymbol{A}^{\cdot}_{\ \cdot} \boldsymbol{L} \\
\overset{\triangledown}{\boldsymbol{A}}^{\cdot\cdot} &= \dot{\boldsymbol{A}}^{\cdot\cdot} - \boldsymbol{L} \boldsymbol{A}^{\cdot\cdot} - \boldsymbol{A}^{\cdot\cdot} \boldsymbol{L}^T \quad \text{Oldroyd ou Lie}
\end{aligned}
$$

Les dérivées convectives d'un champ de vecteur sont

$$
\overset{\triangledown}{\boldsymbol{g}}_{\cdot} = \dot{\boldsymbol{g}}_{\cdot} + \boldsymbol{L}^T \boldsymbol{g}_{\cdot}, \quad \overset{\triangledown}{\boldsymbol{g}}^{\cdot} = \dot{\boldsymbol{g}}^{\cdot} - \boldsymbol{L} \boldsymbol{g}^{\cdot}
$$

- Dérivée de Truesdell $\overset{\triangledown}{\boldsymbol{A}}^{\cdot\cdot} = \dot{\boldsymbol{A}}^{\cdot\cdot} - \boldsymbol{L}\boldsymbol{A}^{\cdot\cdot} - \boldsymbol{A}^{\cdot\cdot}\boldsymbol{L}^T + (\text{trace}\, \boldsymbol{L})\boldsymbol{A}^{\cdot\cdot}$
- les *tenseurs de Rivlin–Ericksen* [Truesdell, 1991] p. 123

$$
\boldsymbol{A}_1 = 2\boldsymbol{D}, \quad \boldsymbol{A}_{n+1} = \dot{\boldsymbol{A}}_n + \boldsymbol{A}_n \boldsymbol{L} + (\boldsymbol{A}_n \boldsymbol{L})^T
$$

En particulier, $\boldsymbol{A}_2/2 = \dot{\boldsymbol{D}} + \boldsymbol{L}^T \boldsymbol{D} + \boldsymbol{D}\boldsymbol{L}$

Examen de la loi de comportement mécanique

Le terme du second ordre de la théorie cinétique des gaz peut s'écrire sous la forme :

$$
\begin{aligned}
-\frac{p}{2\mu^2} &= \frac{\varpi_1}{2}(\mathrm{trace}\, \boldsymbol{D})\boldsymbol{D} + \dot{\boldsymbol{D}} - \boldsymbol{L}^T \boldsymbol{D}^{\mathrm{dev}} - \boldsymbol{D}^{\mathrm{dev}} \boldsymbol{L} + 4\boldsymbol{D}^{\mathrm{dev}} \boldsymbol{D}^{\mathrm{dev}} \\
&= (\frac{\varpi_1}{2} + \frac{2}{3} - \frac{8}{3})(\mathrm{trace}\, \boldsymbol{D})\boldsymbol{D} + 4\boldsymbol{D}\boldsymbol{D} + \dot{\boldsymbol{D}} - \boldsymbol{L}^T \boldsymbol{D} - \boldsymbol{D}\boldsymbol{L} \\
&= \frac{2}{3}(1 - s)(\mathrm{trace}\, \boldsymbol{D})\boldsymbol{D} + 4\boldsymbol{D}\boldsymbol{D} + \overset{\triangledown}{\boldsymbol{D}}^{Truesdell} + 2(\boldsymbol{W}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W})
\end{aligned}
$$

Examen de la loi de comportement mécanique

La loi de comportement issue de la théorie cinétique des gaz

- satisfait au *principe d'invariance galiléenne*, i.e. elle est invariante dans la transformation

$$\boldsymbol{\sigma} \longrightarrow \boldsymbol{Q}_0 \boldsymbol{\sigma} \boldsymbol{Q^T}_0, \quad \boldsymbol{L} \longrightarrow \boldsymbol{Q}_0 \boldsymbol{L} \boldsymbol{Q}_0^T, \quad \boldsymbol{D} \longrightarrow \boldsymbol{Q}_0 \boldsymbol{D} \boldsymbol{Q}_0^T$$

- ne satisfait pas au *principe d'indifférence matérielle*, i.e. elle n'est pas invariante dans la transformation euclienne

$$\boldsymbol{\sigma} \longrightarrow \boldsymbol{Q}(t) \boldsymbol{\sigma} \boldsymbol{Q^T}(t), \quad \boldsymbol{L} \longrightarrow \boldsymbol{Q}(t) \boldsymbol{L} \boldsymbol{Q^T}(t) + \dot{\boldsymbol{Q}} \boldsymbol{Q^T}, \quad \boldsymbol{D} \longrightarrow \boldsymbol{Q}(t) \boldsymbol{D} \boldsymbol{Q^T}(t)$$

  à cause de la contribution $\dot{\boldsymbol{D}} - \boldsymbol{L^T} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{L}$ qui se transforme en

$$\boldsymbol{Q}(\dot{\boldsymbol{D}} - \boldsymbol{L^T} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{L} + 2\boldsymbol{Q^T} \dot{\boldsymbol{Q}} \boldsymbol{D} + 2\boldsymbol{D} \dot{\boldsymbol{Q}}^{\boldsymbol{T}} \boldsymbol{Q}) \boldsymbol{Q^T}$$

Intrinsic spin as a constitutive variable by [Murdoch, 1983]

The appearance in the previous relations of the spin $\boldsymbol{W}$ has led a number of authors to conclude that these, regarded as constitutive equations, violate the principle of material frame–indifference. However, the above equations pertain to the stress and heat flux only in **inertial frames**. *This becomes clear when comparison is made with the analysis of [Müller, 1972], who derived approximative expressions for $\boldsymbol{\sigma}$ and $\boldsymbol{q}$ in a general frame.*

Considérons la loi de comportement $\boldsymbol{\sigma}^{gal} \propto \dot{\boldsymbol{D}}^{gal} + \boldsymbol{W}^{gal}\boldsymbol{D}^{gal} - \boldsymbol{D}^{gal}\boldsymbol{W}^{gal}$, formulée dans un référentiel *gal*iléen ϕ^{gal} *(and hence any)*.
Soit $\boldsymbol{Q}$ la transformation orthogonale permettant de passer de ϕ^{gal} à un référentiel quelconque ϕ, et $\boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T$, le taux de rotation associé :

$$\boldsymbol{D} = \boldsymbol{Q}\boldsymbol{D}^{gal}\boldsymbol{Q}^T, \quad \boldsymbol{W} = \boldsymbol{Q}\boldsymbol{W}^{gal}\boldsymbol{Q}^T + \boldsymbol{\Omega}$$

Loi de comportement dans ϕ s'écrit :

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{Q}\boldsymbol{\sigma}^{gal}\boldsymbol{Q}^T \propto \dot{\boldsymbol{D}} + \boldsymbol{W}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W} + 2\boldsymbol{\Omega}\boldsymbol{D} - 2\boldsymbol{D}\boldsymbol{\Omega} \\ &\propto \dot{\boldsymbol{D}} - \boldsymbol{W}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{W} + 2\boldsymbol{W}^{int}\boldsymbol{D} - 2\boldsymbol{D}\boldsymbol{W}^{int} \end{aligned}$$

où $\boldsymbol{W}^{int} = \boldsymbol{W} - \boldsymbol{\Omega}$ *is the intrinsic spin*, which is nothing but the Galilean spin tensor : $\boldsymbol{W}^{int} = \boldsymbol{Q}\boldsymbol{W}^{gal}\boldsymbol{Q}^T$.

Intrinsic spin as a constitutive variable by [Murdoch, 1983]

La quantité $\boldsymbol{W}^{int}$ est *objective*.
En effet, si l'on passe de ϕ^{gal} à un autre référentiel quelconque ϕ' par $\boldsymbol{Q'}$, et de ϕ à ϕ' par $\boldsymbol{Q''} = \boldsymbol{Q'}\boldsymbol{Q}^T$,

$$\begin{aligned}
\boldsymbol{W}^{int\prime} &= \boldsymbol{W'} - \dot{\boldsymbol{Q}}'\boldsymbol{Q'}^T \\
&= \boldsymbol{Q''}\boldsymbol{W}\boldsymbol{Q''}^T + \dot{\boldsymbol{Q}}''\boldsymbol{Q''}^T - \dot{\boldsymbol{Q}}'\boldsymbol{Q'}^T \\
&= \boldsymbol{Q''}(\boldsymbol{W} - \dot{\boldsymbol{Q}}\boldsymbol{Q}^T)\boldsymbol{Q''}^T = \boldsymbol{Q''}\boldsymbol{W}^{int}\boldsymbol{Q''}^T
\end{aligned}$$

Outline

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Objectivity vs. frame–indifference

- *On n'engendre pas de contraintes dans un corps en tournant autour !*[6]
  Cela va de soi mais la loi ci–dessous ne satisfait pas à cette exigence élémentaire :
  $$\boldsymbol{\sigma} = 2\mu \boldsymbol{D} + \alpha(\boldsymbol{W}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W})$$
  si elle est stipulée dans un référentiel quelconque.
  This relation is not objective.
- La loi précédente, formulée par rapport à un référentiel galiléen
  $$\boldsymbol{\sigma}^{gal} = 2\mu \boldsymbol{D} + \alpha(\boldsymbol{W}^{gal}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{W}^{gal})$$
  ne souffre pas du travers précédent.
  Elle prend la forme $\boldsymbol{\sigma} = 2\mu \boldsymbol{D} + \alpha((\boldsymbol{W} - \boldsymbol{\Omega})\boldsymbol{D} - \boldsymbol{D}(\boldsymbol{W} - \boldsymbol{\Omega}))$ par rapport à un référentiel quelconque
  This relation is objective but not frame–indifferent[7] dans la mesure où elle fait appel au référentiel galiléen.
  Un telle loi est exclue par MFI.

6. C'est à cette exigence que correspondent les conditions (1.5) dans [Woods, 1983], affirmant **l'objectivité des contraintes et du flux de chaleur**, propriétés admises (mais approchées !).
7. not Euclidean invariant

Lois de fluides incluant le taux de rotation

- Modèles de suspensions (+statut du principe : **AMFI**) [Ryskin et al., 1980]

Appendix. On the applicability of the approximation of material frame-indifference in suspension mechanics

By G. RYSKIN AND J. M. RALLISON

The approximation of material frame-indifference (AMFI; usually called 'principle') is frequently invoked in discussion of constitutive relations for non-Newtonian materials (Truesdell & Noll 1965; Astarita & Marrucci 1974; Bird, Armstrong & Hassager 1977; Truesdell 1977; Schowalter 1978). In simple terms, the approximation asserts that the constitutive behaviour of a local material is invariant to rigid-body motions. On the other hand, in at least two problems the AMFI has been shown not to

- Turbulence : motivation de [Razafindralandy and Hamdouni, 2006]

Lund and Novikov include the filtered vorticity tensor $\overline{\mathbf{W}} = (\nabla\overline{\boldsymbol{u}} - {}^T\nabla\overline{\boldsymbol{u}})$ in the expression of the subgrid model. Cayley–Hamilton theorem gives then the *Lund–Novikov model* (see [22]):

$$-\mathbf{T}_s^d = C_1\overline{\delta}^2|\overline{\mathbf{S}}|\overline{\mathbf{S}} + C_2\overline{\delta}^2(\overline{\mathbf{S}}^2)^d + C_3\overline{\delta}^2\left(\overline{\mathbf{W}}^2\right)^d + C_4\overline{\delta}^2(\overline{\mathbf{S}}\,\overline{\mathbf{W}} - \overline{\mathbf{W}}\,\overline{\mathbf{S}}) + C_5\overline{\delta}^2\frac{1}{|\overline{\mathbf{S}}|}(\overline{\mathbf{S}}^2\overline{\mathbf{W}} - \overline{\mathbf{S}}\,\overline{\mathbf{W}}^2), \quad (9)$$

where the coefficients C_i depend on the invariants obtained from $\overline{\mathbf{S}}$ and $\overline{\mathbf{W}}$. The expression of these coefficients are so complex that they are considered as constants and evaluated with statistic techniques.

Approche phénoménologique

formulation de la loi de comportement [8] en partant du **fluide homogène de degré** n selon [Truesdell and Euvrard, 1974], p. 217 (fluide de Rivlin–Ericksen), et en ajoutant $\boldsymbol{W}$.
application à $n = 2$

$$\boldsymbol{\sigma} = \lambda(\mathrm{trace}\,\boldsymbol{D})\mathbf{1} + 2\mu\boldsymbol{D} + \alpha_1\boldsymbol{D}^2 + \alpha_2(\boldsymbol{D}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{D}) + \alpha_3\boldsymbol{W}^2 + \alpha_4\dot{\boldsymbol{D}}$$

Les termes en rouge sont présents dans la loi d'ordre 2 de la théorie cinétique des gaz. Cette dernière a le mérite de fournir les expressions des paramètres $\mu, \alpha_1, \alpha_2, \alpha_4$.

8. par rapport au référentiel galiléen.

Approche phénoménologique

formulation de la loi de comportement [8] en partant du **fluide homogène de degré** n selon [Truesdell and Euvrard, 1974], p. 217 (fluide de Rivlin–Ericksen), et en ajoutant $\boldsymbol{W}$.
application à $n = 2$

$$\boldsymbol{\sigma} = \lambda(\mathrm{trace}\,\boldsymbol{D})\mathbf{1} + 2\mu\boldsymbol{D} + \alpha_1\boldsymbol{D}^2 + \alpha_2(\boldsymbol{D}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{D}) + \alpha_3\boldsymbol{W}^2 + \alpha_4\dot{\boldsymbol{D}}$$

Les termes en rouge sont présents dans la loi d'ordre 2 de la théorie cinétique des gaz. Cette dernière a le mérite de fournir les expressions des paramètres $\mu, \alpha_1, \alpha_2, \alpha_4$.
La positivité de la dissipation $\boldsymbol{\sigma} : \boldsymbol{D} \geq 0$ exige que $\alpha_3 = 0$.

8. par rapport au référentiel galiléen.

Approche phénoménologique

La formulation des lois de comportement doit respecter les 3 principes suivants :

- Covariance (tensorialité) des relations à exploiter grâce aux théorèmes disponibles selon les groupes de symétrie de la matière, notamment [Boehler, 1987, Zheng, 1994]
- Invariance galiléenne (faire fi de MFI !) (groupe galiléen étendu, au sens du principe d'équivalence d'Einstein)
- Second principe de la thermodynamique

Les lois satisfaisant à AMFI constituent une classe particulière de comportement de la matière, en général suffisante pour l'ingénieur, mais pas toujours (turbulence en particulier).

Outline

Les préjugés sur l'hyperélasticité

- Utilisation de AMFI pour réduire la loi

$$\psi_0(\boldsymbol{F}(t)) \Longrightarrow \psi_0(\boldsymbol{C}(t))$$

- Approche lagrangienne privilégiée

$$\boldsymbol{\Pi} = 2\rho_0 \frac{\partial \psi_0}{\partial \boldsymbol{C}}(\boldsymbol{C}(t))$$

avec la contrainte de Piola

$$\boldsymbol{\Pi} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

et le tenseur de Cauch-Green gauche

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$$

- Approche eulérienne dans le cas isotrope

$$\boldsymbol{\sigma} = 2\rho \frac{\partial \psi}{\partial \boldsymbol{B}}\boldsymbol{B}$$

avec le tenseur de Cauchy et le tenseur de Cauch-Green droit

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$$

Les préjugés sur l'hyperélasticité

- Utilisation de AMFI pour réduire la loi — Inutile !

$$\psi_0(\boldsymbol{F}(t)) \Longrightarrow \psi_0(\boldsymbol{C}(t))$$

- Approche lagrangienne privilégiée — Pas de raison !

$$\boldsymbol{\Pi} = 2\rho_0 \frac{\partial \psi_0}{\partial \boldsymbol{C}}(\boldsymbol{C}(t))$$

avec la contrainte de Piola

$$\boldsymbol{\Pi} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

et le tenseur de Cauch-Green gauche

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$$

- Approche eulérienne dans le cas isotrope — Pas seulement !

$$\boldsymbol{\sigma} = 2\rho \frac{\partial \psi}{\partial \boldsymbol{B}}\boldsymbol{B}$$

avec le tenseur de Cauchy et le tenseur de Cauch-Green droit

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$$

à condition de ne pas oublier de variables dans la modélisation !

Liste exhaustive des variables du modèle hyperélastique

- La contrainte à l'instant t dépend du gradient de la transformation à l'instant t

$$\boldsymbol{F}(t)$$

- Ne pas masquer les tenseurs *métriques*
  métrique spatiale $\boldsymbol{G}$ métrique matérielle $\boldsymbol{G}_0$
- Ne pas ignorer la variance des tenseurs

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{G} \boldsymbol{F}$$

  La déformation $\boldsymbol{C}$ est le *pull-back* de la métrique spatiale

$$\boldsymbol{B} = \boldsymbol{F} \boldsymbol{G}_0^{-1} \boldsymbol{F}^T$$

  La déformation $\boldsymbol{B}$ est le *push-forward* de la métrique matérielle
- Les *tenseurs de structure* pour caractériser l'anisotropie [Boehler, 1979]

Tenseurs de structure

- Les tenseurs de structures sont invariants par action du groupe de symétrie

$$\mathcal{G}_{\kappa_0} \leq \mathcal{U}(E)$$

- Exemple : tenseur de structure d'ordre 2

$$\boldsymbol{HAH}^T = \boldsymbol{A}, \quad \forall \boldsymbol{H} \in \mathcal{G}_{\kappa_0}$$

- Cas isotrope

$$\boldsymbol{HAH}^T = \boldsymbol{A}, \quad \forall \boldsymbol{H} \in SO(E)$$

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$$\Longrightarrow \boldsymbol{A} = \boldsymbol{G}_0$$

- La loi de comportement est une fonction tensorielle (covariante) de tous les arguments, y compris les tenseurs de structures

Solides hyperélastiques isotropes

- Liste exhaustive des variables

$$\{\boldsymbol{F}(t); \boldsymbol{G}, \boldsymbol{G}_0\}$$

- Potentiel eulérien obtenu par *push–forward*

$$\psi(\boldsymbol{G}, \boldsymbol{B}^{-1}(t))$$

- Potentiel lagrangien obtenu par *push-back*

$$\psi_0(\boldsymbol{C}(t), \boldsymbol{G}_0)$$

- Théorème de représentation de la fonction tensorielle (=isotrope) ψ

$$\rho\psi(\mathrm{trace}\,(\boldsymbol{BG}), \mathrm{trace}\,(\boldsymbol{BGBG}), \mathrm{trace}\,(\boldsymbol{BGBGBG}))$$

[Zheng, 1994]

Second principe de la thermodynamique

- Inégalité de la dissipation

$$\boldsymbol{\sigma} : \dot{\boldsymbol{G}}^{(c)} - 2\rho\dot{\psi}^{(c)} = \boldsymbol{\sigma} : \dot{\boldsymbol{G}}^{(c)} - 2\rho\frac{\partial\psi}{\partial\boldsymbol{B}} : \dot{\boldsymbol{B}}^{(c)} - 2\rho\frac{\partial\psi}{\partial\boldsymbol{A}} : \dot{\boldsymbol{A}}^{(c)} - 2\rho\frac{\partial\psi}{\partial\boldsymbol{G}} : \dot{\boldsymbol{G}}^{(c)} \geq 0$$

avec les dérivées convectives

$$\dot{\boldsymbol{A}}^{(c)} = \boldsymbol{F}\dot{\boldsymbol{A}}_0\boldsymbol{F}^T = 0 = \dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{L}^T$$

$$\dot{\boldsymbol{B}}^{(c)} = 0 = \dot{\boldsymbol{B}} - \boldsymbol{L}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{L}^T$$

$$\dot{\boldsymbol{G}}^{(c)} = 2\boldsymbol{D} = \dot{\boldsymbol{G}} + \boldsymbol{G}\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{G}$$

[Stumpf and Hoppe, 1997, Lu and Papadopoulos, 2000]

- Loi hyperélastique

$$\boldsymbol{\sigma} = 2\rho\frac{\partial\psi}{\partial\boldsymbol{G}}$$

C'est la forme due à
[Doyle and Ericksen, 1956, Marsden and Hughes, 1994] qui est très peu enseignée !

Hyperélasticité isotrope

$$\mathbf{\Pi} = 2\rho_0 \frac{\partial \psi_0}{\partial \boldsymbol{C}} \quad \Longleftrightarrow \quad \boldsymbol{\sigma} = 2\rho \frac{\partial \psi}{\partial \boldsymbol{G}}$$

$$\mathbf{\Pi} = 2\boldsymbol{C}^{-1} \rho_0 \frac{\partial \psi_0}{\partial \boldsymbol{G}_0^{-1}} \boldsymbol{G}_0^{-1} \quad \Longleftrightarrow \quad \boldsymbol{\sigma} = 2\boldsymbol{G}^{-1} \rho \frac{\partial \psi}{\partial \boldsymbol{B}} \boldsymbol{B}$$

$$\boldsymbol{\sigma} = \beta_1 \boldsymbol{B} + \beta_2 \boldsymbol{B}\boldsymbol{G}\boldsymbol{B} + \beta_3 \boldsymbol{B}\boldsymbol{G}\boldsymbol{B}\boldsymbol{G}\boldsymbol{B}$$

Hyperélasticité isotrope transverse

Tenseur de structure spécifique

$$\boldsymbol{A}_0 = \boldsymbol{a}_0 \otimes \boldsymbol{a}_0, \quad \boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a} = \boldsymbol{F}\boldsymbol{A}_0\boldsymbol{F}^T$$

[Boehler, 1987]

Hyperélasticité isotrope transverse

Tenseur de structure spécifique

$$\boldsymbol{A}_0 = \boldsymbol{a}_0 \otimes \boldsymbol{a}_0, \quad \boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a} = \boldsymbol{F}\boldsymbol{A}_0\boldsymbol{F}^T$$

[Boehler, 1987]

$$\boldsymbol{\Pi} = 2\rho_0 \frac{\partial \psi_0}{\partial \boldsymbol{C}} \quad \Longleftrightarrow \quad \boldsymbol{\sigma} = 2\rho \frac{\partial \psi}{\partial \boldsymbol{G}}$$

$$\boldsymbol{\Pi} = 2\rho_0 \boldsymbol{C}^{-1} \left(\frac{\partial \psi_0}{\partial \boldsymbol{G}_0^{-1}} \boldsymbol{G}_0^{-1} + \frac{\partial \psi_0}{\partial \boldsymbol{A}_0} \boldsymbol{A}_0 \right) \quad \Longleftrightarrow \quad \boldsymbol{\sigma} = 2\rho \boldsymbol{G}^{-1} \left(\frac{\partial \psi}{\partial \boldsymbol{B}} \boldsymbol{B} + \frac{\partial \psi}{\partial \boldsymbol{A}} \boldsymbol{A} \right)$$

$$\begin{aligned} \boldsymbol{\sigma} &= \beta_1 \boldsymbol{B} + \beta_2 \boldsymbol{BGB} + \beta_3 \boldsymbol{BGBGB} + \beta_4 (\boldsymbol{BGA} + \boldsymbol{ABG}) \\ &+ \beta_5 (\boldsymbol{BGAGB} + \boldsymbol{BGBGA} + \boldsymbol{AGBGB}) \end{aligned}$$

[Lu and Papadopoulos, 2000]

Outline

Conclusions et recommandations

- Fluides différentiels : utilisez le *spin intrinsèque* $\boldsymbol{W}^{int}$
- Solides élastiques : la loi est AMFI, c'est dans ce cas particulier une conséquence de la covariance
- Le rôle des tenseurs métriques est essentiel : la loi hyperélastique est obtenue en dérivant le potentiel par rapport à la métrique spatiale
- Insister plus dans le futur sur la variance des tenseurs en jeu

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