Hidden symmetries in nonholonomic mechanics

Paweł Nurowski

Centrum Fizyki Teoretycznej
Polska Akademia Nauk

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Kinematics and nonholonomic constraints

- Physicists are mainly concerned with *dynamics*. *Kinematics* is considered to be boring.
- What if we consider kinematics with nontrivial constraints?
- For the purpose of this talk nontrivial constraints are *nonholonomic*.
- A constraint $F(x, \dot{x}) = 0$ on positions x and velocities $\dot{x}$ of a mechanical system is *nonholonomic* if it can *not* be integrated to a constraint on positions only. Such constraints prevent a reduction of the *configuration space* of positions of a mechanical system to a submanifold, and without introducing any dynamics usually equip the configuration space with a *nontrivial geometry*. And geometry, especially in its flat model version, goes in pair with symmetry.

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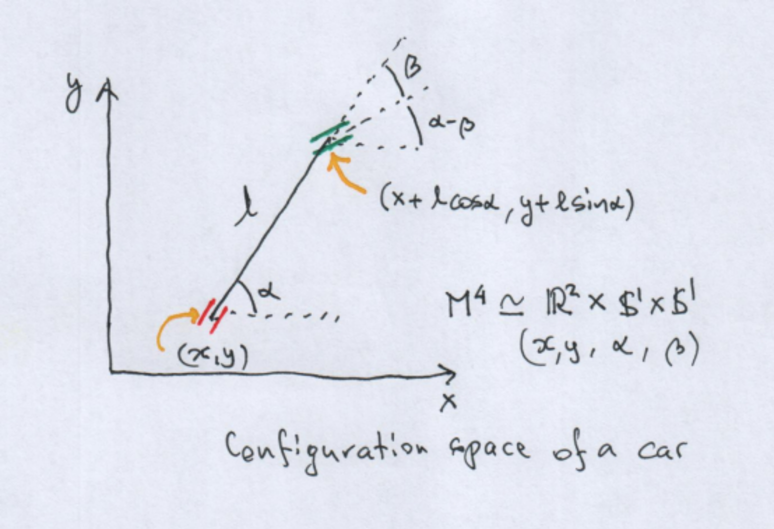
What is a car?



y
x
ℓ
α
β
$\alpha-\beta$
$(x+\ell\cos\alpha, y+\ell\sin\alpha)$
(x,y)
$M^4 \simeq \mathbb{R}^2 \times S^1 \times S^1$
(x, y, α, β)
Configuration space of a car

Configuration space and the movement

- Configuration space is locally $M = R^2 \times \mathbb{S}^1 \times \mathbb{S}^1$
- Convenient coordinates: (x, y) - position of the rear wheels, α - orientation of car's chasis, β - angle between the front wheels and the headlights
- When car is moving it traverses a curve
$$q(t) = (x(t), y(t), \alpha(t), \beta(t))$$
in M
- Car's velocity is $\dot{q} = (\dot{x}, \dot{y}, \dot{\alpha}, \dot{\beta})$.

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Role of the tires

Safe car has *tires*. Their role is to prevent car from *skidding*. Our car will have *infinitely good* tires. They impose *nonholonomic constraints*. These are constraints on positions *a*nd velocities, that can not be integreted to constraints on positions only.

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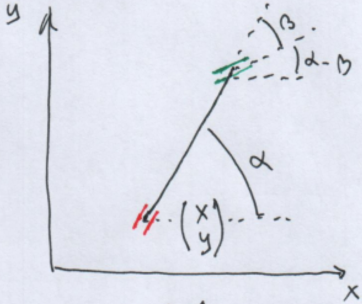

Nonholonomic constraints:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} \parallel \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \quad \text{rear wheels}$$

$$\frac{d}{dt}\begin{pmatrix} x + l\cos\alpha \\ y + l\sin\alpha \end{pmatrix} \parallel \begin{pmatrix} \cos(\alpha-\beta) \\ \sin(\alpha-\beta) \end{pmatrix} \quad \text{front wheels}$$

$$M^4 \ni q(t) = (x(t), y(t), \alpha(t), \beta(t)) - \text{trajectory}$$

Nonholonomic constraints

- Role of the tires:
  the curve $q(t) = (x(t), y(t), \alpha(t), \beta(t)) \in M^4$ at every moment of time t must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}(x, y) \quad || \quad (\cos\alpha, \sin\alpha) \qquad \&$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(x + \ell\cos\alpha, y + \ell\sin\alpha) \quad || \quad (\cos(\alpha-\beta), \sin(\alpha-\beta)),$$

  or, what is the same

$$\dot{x}\sin\alpha - \dot{y}\cos\alpha = 0 \qquad \&$$
$$(\dot{x} - \ell\dot{\alpha}\sin\alpha)\,\sin(\alpha-\beta) - (\dot{y} + \ell\dot{\alpha}\cos\alpha)\,\cos(\alpha-\beta) = 0.$$

- Note that these constraints are LINEAR in the velocity $(\dot{x}, \dot{y}, \dot{\alpha}, \dot{\beta})$.

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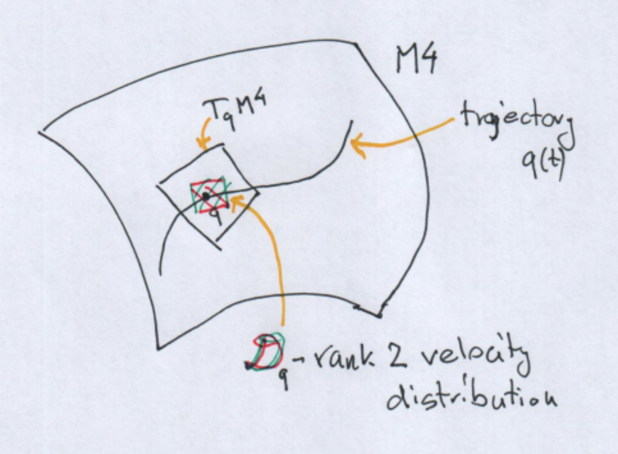
Velocity distribution



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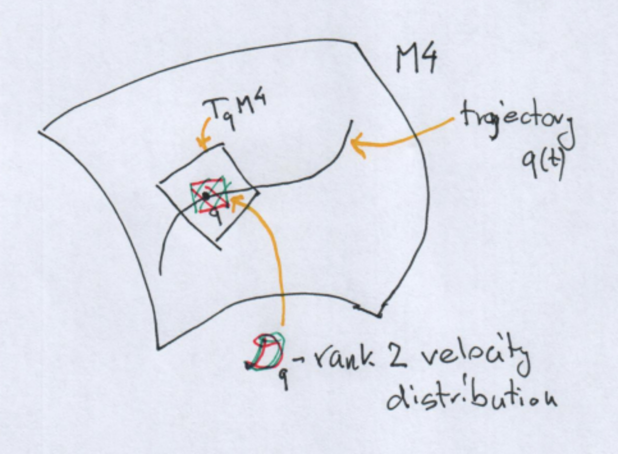

Car's structure

- Configuration space M is locally $M = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, with points q parameterized as $q = (x, y, \alpha, \beta)$
- There is a *rank 2* distribution $\mathcal{D}$ on M, describing the space of possible velocities, given by

$$\mathcal{D} = Span_{\mathcal{F}(M)}(X_3, X_4)$$

with

$$\begin{aligned} X_3 &= \partial_\beta \\ X_4 &= -\sin\beta\partial_\alpha + \ell\cos\beta(\cos\alpha\partial_x + \sin\alpha\partial_y) \end{aligned}.$$

- Therefore 'the structure of a car with perfect tires' is

$$(M, \mathcal{D})$$

- a 4-manifold M with a rank 2 distribution $(M, \mathcal{D})$.

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- Obviously NOT!
- the commutators

$$[X_3, X_4] = -\cos\beta\partial_\alpha - \ell\sin\beta(\sin\alpha\partial_y + \cos\alpha\partial_x) := X_2$$
$$[X_4, X_2] = \ell(\cos\alpha\partial_y - \sin\alpha\partial_x) := X_1 .$$

- It is easy to check that

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$$X_1 \wedge X_2 \wedge X_3 \wedge X_4 = \ell^2\partial_x \wedge \partial_y \wedge \partial_\alpha \wedge \partial_\beta \neq 0.$$

Is $\mathcal{D}$ integrable?

- Obviously NOT!
- the commutators

$$[X_3, X_4] = -\cos\beta\partial_\alpha - \ell\sin\beta(\sin\alpha\partial_y + \cos\alpha\partial_x) := X_2$$
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- Observe that:

$$\begin{array}{rcc} & & \text{rank} \\ \mathcal{D}_{-1} := \mathcal{D} & \mathrm{Span}(X_4, X_3) & 2 \\ \mathcal{D}_{-2} := [\mathcal{D}_{-1}, \mathcal{D}_{-1}] & \mathrm{Span}(X_4, X_3, X_2) & 3 \\ \mathcal{D}_{-3} := [\mathcal{D}_{-1}, \mathcal{D}_{-2}] & \mathrm{Span}(X_4, X_3, X_2, X_1) = \mathrm{T}M & 4 \end{array}$$

- We have a filtration $\mathcal{D}_{-1} \subset \mathcal{D}_{-2} \subset \mathcal{D}_{-3} = \mathrm{T}M$ of distributions of the *constant growth vector* $(2, 3, 4)$. By definition $\mathcal{D}$ is an *Engel distribution*.

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- Two distributions $\mathcal{D}$ and $\bar{\mathcal{D}}$ of the same rank on manifolds M and $\bar{M}$ of the same dimension are *(locally) equivalent* iff there exists a (local) diffeomorphism $\phi : M \to \bar{M}$ such that $\phi_* \mathcal{D} = \bar{\mathcal{D}}$.
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Another Engel distribution

- Take $\mathbb{R}^4$ with coordinates (x, y, p, q) and consider $X_3 = \partial_q$ and $X_4 = \partial_x + p\partial_y + q\partial_p$.
- We have $[X_3, X_4] = \partial_p = X_2$ and $[X_4, X_2] = -\partial_y = X_1$.
- Hence $\mathcal{D}_E = (\partial_q, \partial_x + p\partial_y + q\partial_p)$ is a $(2, 3, 4)$ distribution, therefore an *Engel distribution*.
- **Theorem** (Engel) Every Engel distribition is locally equivalent to the distribiution $\mathcal{D}_E$.
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- Look at the vector field: $X_4 = -\sin\beta\partial_\alpha + \ell\cos\beta(\cos\alpha\partial_x + \sin\alpha\partial_y)$.
- When $\beta = 0$ it is $X_4 = \ell(\cos\alpha\partial_x + \sin\alpha\partial_y)$, i.e. if the car chooses this direction of its velocity it goes along a straight line in the direction $(\cos\alpha, \sin\alpha)$ in the (x, y) plane.
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see the movie

The split!

- Car's structure is an Engel distribution $\mathcal{D}$ *with a split*!
- $\mathcal{D} = Span(X_3, X_4)$, with
  - $X_3 = \partial_\beta$ - rotation of the steering wheel by the angle β; this defines the STEERING WHEEL SPACE, $\mathcal{D} = Span(X_3)$,
  - $X_4 = -\sin\beta\partial_\alpha + \dots$ - this coresponds to an application of gas in the direction $(\cos\alpha, \sin\alpha)$ in the (x, y) plane, with a fixed position of the steereing wheel at an angle β ; this defines the GAS SPACE, $\mathcal{D} = Span(X_4)$.
- Thus, the car structure is $(M, \mathcal{D} = \mathcal{D} \oplus \mathcal{D})$, where $\mathcal{D}$ is an Engel distribution, and *the ranks of the summands in* $\mathcal{D}$ *are ONE*.

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Theorem

Consider the car structure $(M, \mathcal{D})$ *consisting of its velocity distribution* $\mathcal{D}$ *and the split of* $\mathcal{D}$ *onto rank 1 distributions* $\mathcal{D} = \mathcal{D} \oplus \mathcal{D}$ *with* $\mathcal{D} = Span(\partial_\beta)$, $\mathcal{D} = Span(-\sin\beta\partial_\alpha + \ell\cos\beta(\cos\alpha\partial_x + \sin\alpha\partial_y)$.
The Lie algebra of infinitesimal symmetries of this Engel structure with a split is 10-dimensional, with the following generators

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It is isomorphic to the simple real Lie algebra $\mathfrak{so}(2,3) = \mathfrak{sp}(2,\mathbb{R})$. *Moreover, there are plenty of locally nonequivalent Engel distributions with a split, but the split on the (Engel) car distribution used by car owners and provided by cars' producers is THE MOST SYMETRIC.*

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S_4 &= \ell(\sin\alpha\partial_x - \cos\alpha\partial_y) + \sin^2\beta\partial_\beta \\
S_5 &= x\partial_x + y\partial_y - \sin\beta\cos\beta\partial_\beta \\
S_6 &= (x^2 - y^2)\partial_x + 2xy\partial_y + 2y\partial_\alpha - 2\cos\beta\Big(\ell\cos\beta\sin\alpha + x\sin\beta\Big)\partial_\beta \\
S_7 &= \ell\Big(x(\sin\alpha\partial_x - \cos\alpha\partial_y) - \cos\alpha\partial_\alpha\Big) + \sin\beta\Big(\ell\cos\beta\sin\alpha + x\sin\beta\Big)\partial_\beta \\
S_8 &= \ell\Big(y(\sin\alpha\partial_x - \cos\alpha\partial_y) - \sin\alpha\partial_\alpha\Big) - \sin\beta\Big(\ell\cos\beta\cos\alpha - y\sin\beta\Big)\partial_\beta \\
S_9 &= 2xy\partial_x + (y^2 - x^2)\partial_y - 2x\partial_\alpha + 2\cos\beta\Big(\ell\cos\beta\cos\alpha - y\sin\beta\Big)\partial_\beta \\
S_{10} &= \ell(x^2 + y^2)\Big(\sin\alpha\partial_x - \cos\alpha\partial_y\Big) - 2\ell\Big(x\cos\alpha + y\sin\alpha\Big)\partial_\alpha + \\
&\quad \Big(2\ell\sin\beta\cos\beta\big(x\sin\alpha - y\cos\alpha\big) + \sin^2\beta(x^2 + y^2) + 2\ell^2\cos^2\beta\Big)\partial_\beta
\end{aligned}
$$

It is isomorphic to the simple real Lie algebra $\mathfrak{so}(2,3) = \mathfrak{sp}(2,\mathbb{R})$. Moreover, there are plenty of locally nonequivalent Engel distributions with a split, but the split on the (Engel) car distribution used by car owners and provided by cars' producers is THE MOST SYMETRIC.

Surprise!

Theorem

Consider the car structure $(M, \mathcal{D})$ *consisting of its velocity distribution* $\mathcal{D}$ *and the split of* $\mathcal{D}$ *onto rank 1 distributions* $\mathcal{D} = \mathcal{D} \oplus \mathcal{D}$ *with* $\mathcal{D} = \mathit{Span}(\partial_\beta)$, $\mathcal{D} = \mathit{Span}(-\sin\beta\,\partial_\alpha + \ell\cos\beta(\cos\alpha\,\partial_x + \sin\alpha\,\partial_y))$.
The Lie algebra of infinitesimal symmetries of this Engel structure with a split is 10-dimensional, with the following generators

$$
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S_1 &= \partial_x \\
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S_3 &= x\partial_y - y\partial_x + \partial_\alpha \\
S_4 &= \ell(\sin\alpha\,\partial_x - \cos\alpha\,\partial_y) + \sin^2\beta\,\partial_\beta \\
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&\quad \Big(2\ell\sin\beta\cos\beta\,(x\sin\alpha - y\cos\alpha) + \sin^2\beta(x^2 + y^2) + 2\ell^2\cos^2\beta\Big)\partial_\beta
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It is isomorphic to the simple real Lie algebra $\mathfrak{so}(2,3) = \mathfrak{sp}(2,\mathbb{R})$. *Moreover, there are plenty of locally nonequivalent Engel distributions with a split, but the split on the (Engel) car distribution used by car owners and provided by cars' producers is THE MOST SYMETRIC.*

WHY???

- Why the car structure $(M, \mathcal{D} = \mathcal{D} \oplus \mathcal{D})$ of an Engel distribution $\mathcal{D}$ with a particular (car's) split has the simple Lie algebra $\mathfrak{so}(2,3)$ as the Lie algebra of ininfinitesimal symmetries?
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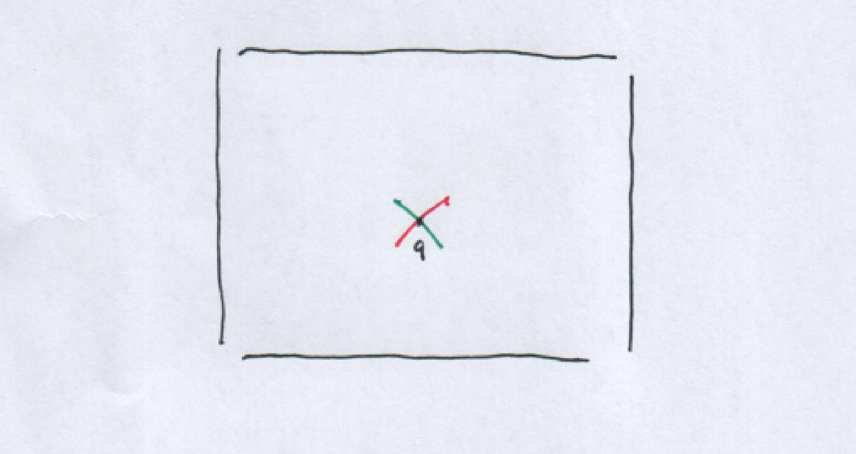
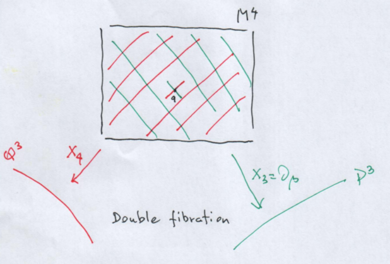
Two directions at each point of M



lead to a double fibration



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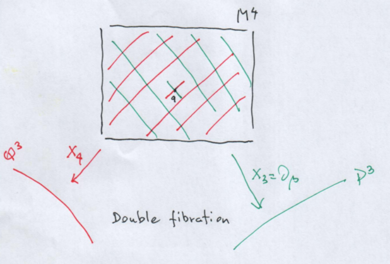

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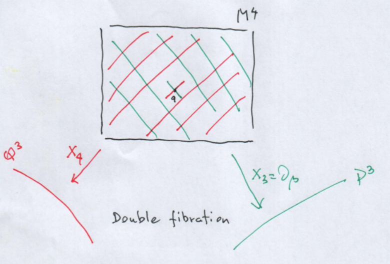

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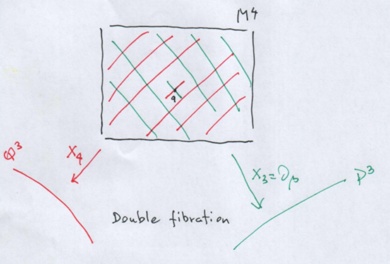

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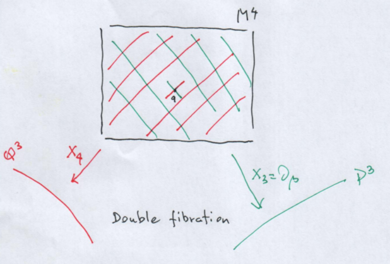

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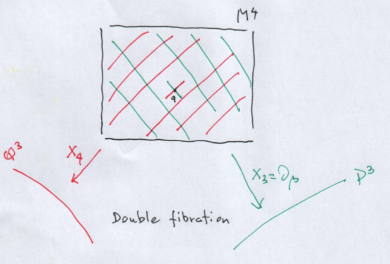

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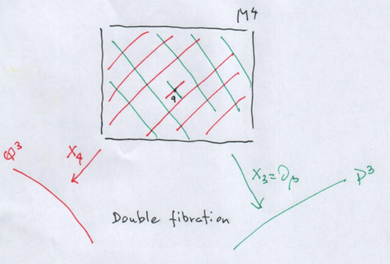

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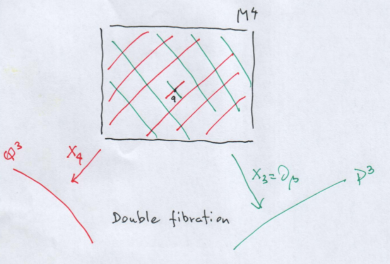

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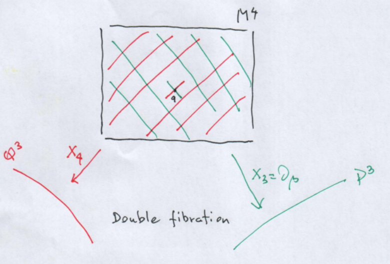

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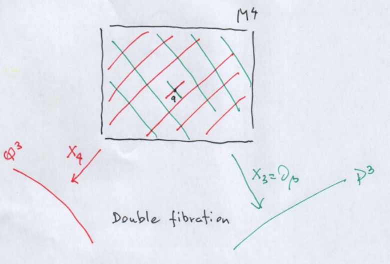

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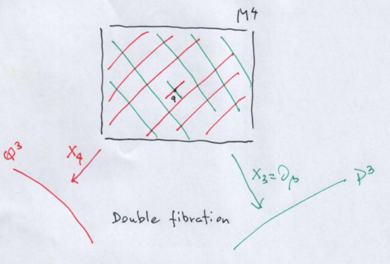

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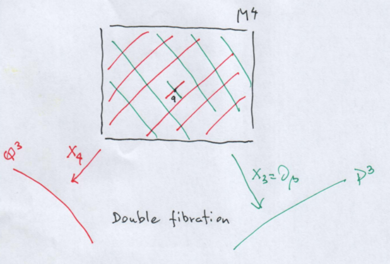

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- The red trajectories are *helices* in each slice $\beta =$const in M.
- The space of all of them is a 3D space Q^3 of all circles (including all lines and all points) in the plane.

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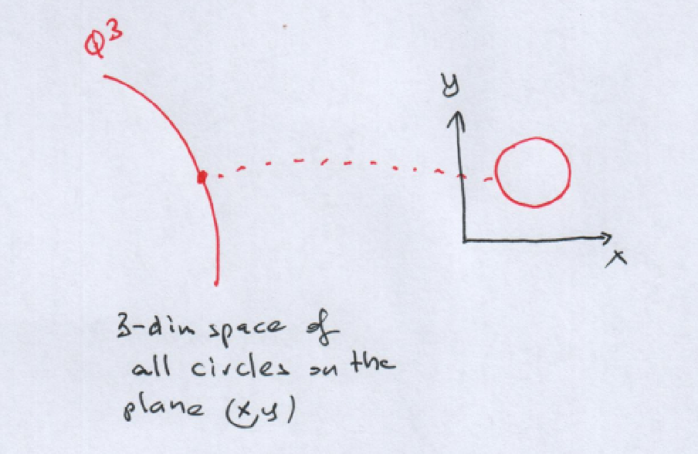
From the space of circles...



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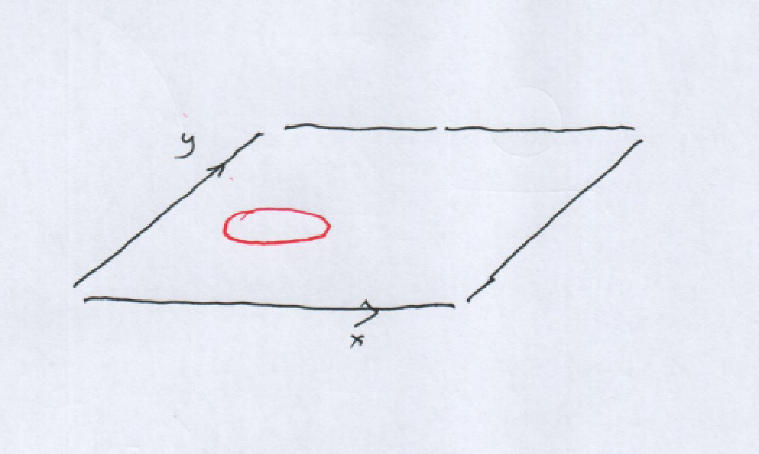

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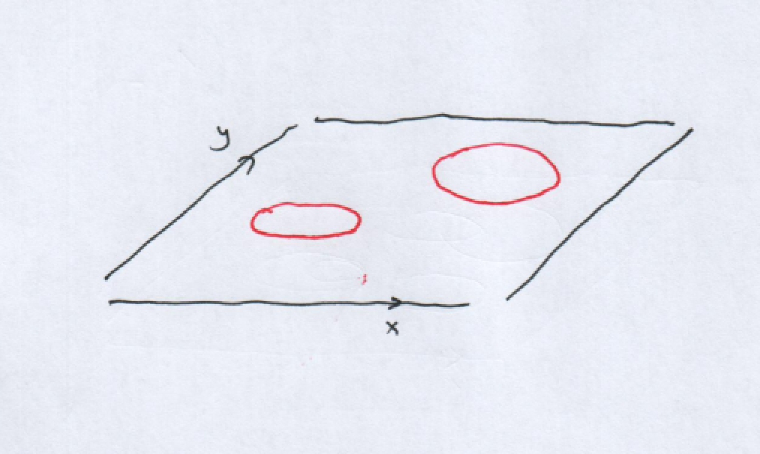

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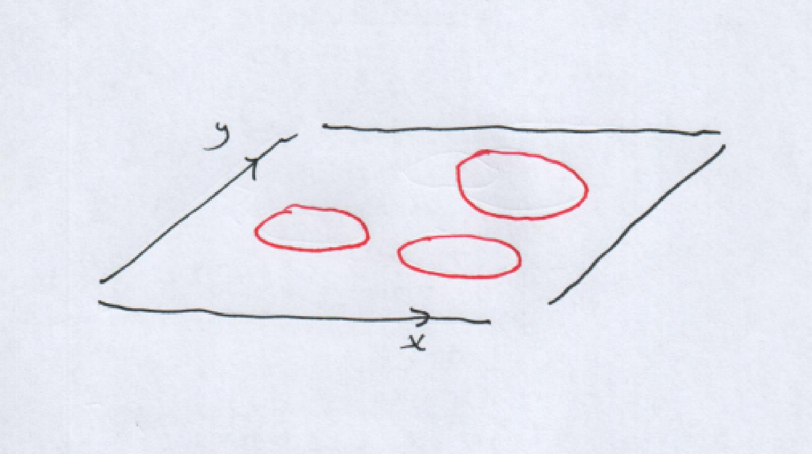

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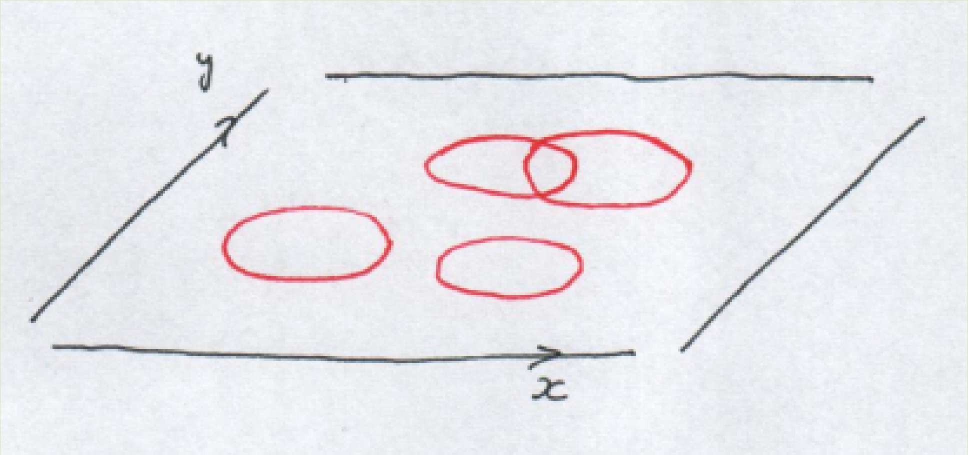
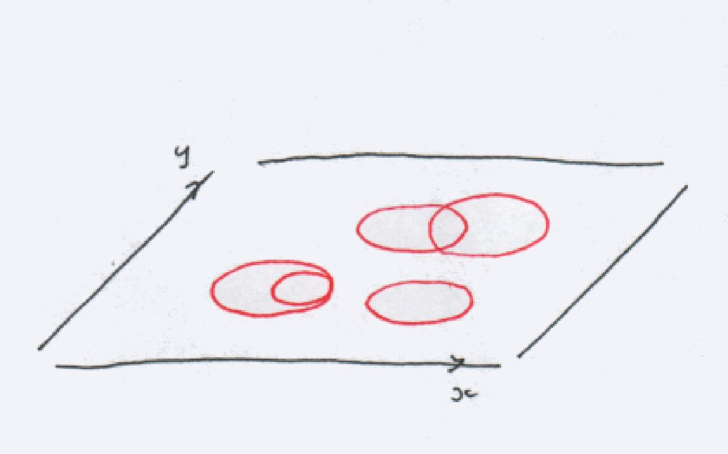
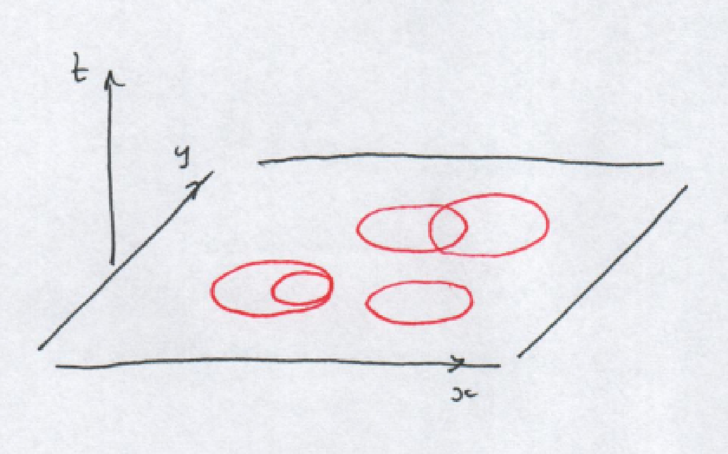


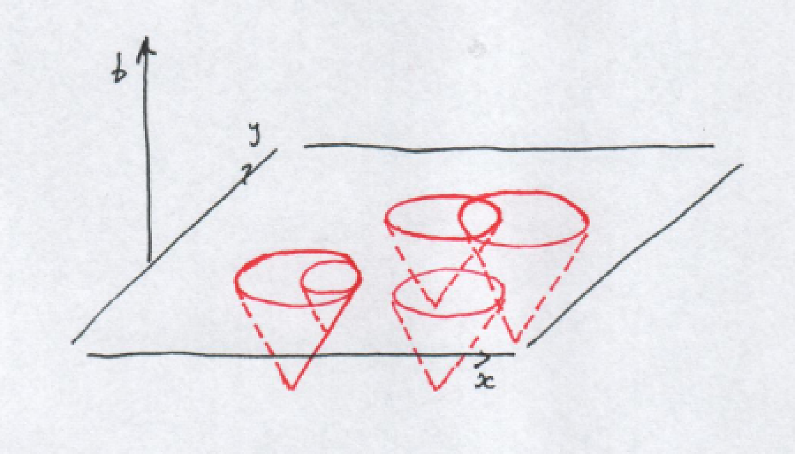
From circles to...



From circles to the light...



From circles to the light cones



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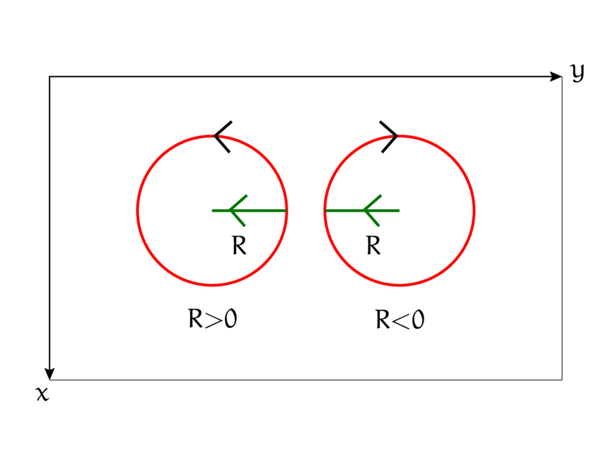

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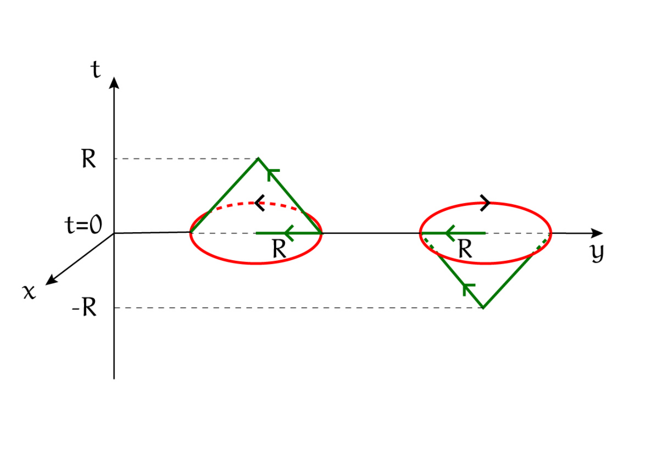

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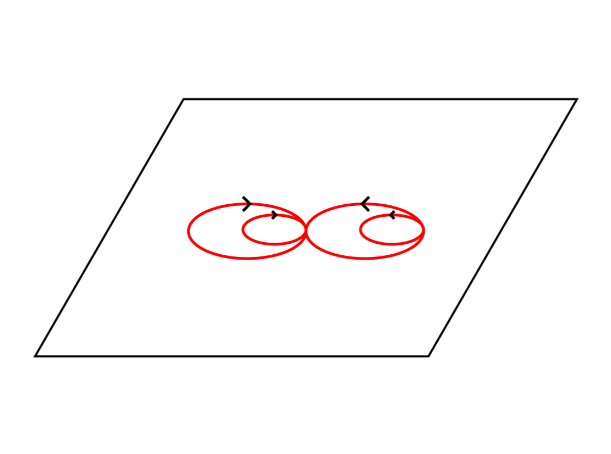

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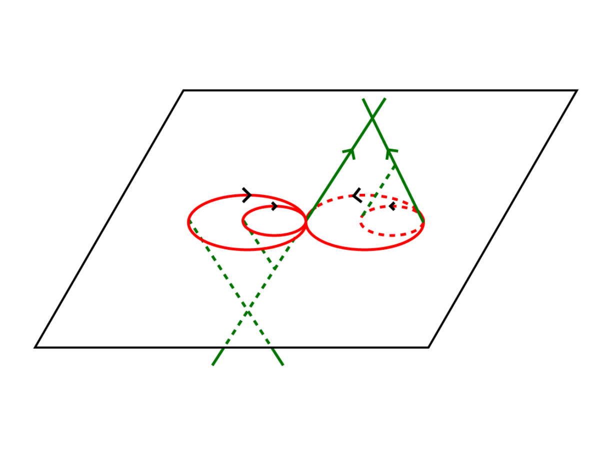

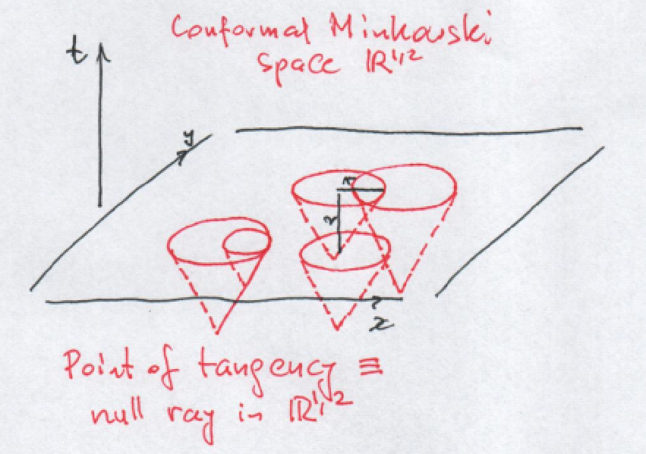
Conformal Minkowski space $\mathbb{R}^{1,2}$
t
y
r
z
Point of tangency ≡ null ray in $\mathbb{R}^{1,2}$

Geometry of oriented circles on the plane...

- is a geometry of light cones in 3D Minkowski space;
- 2 oriented circles are *null separated* if and only if they are *tangent* to each other and *their orientations match*.
- Therefore Q^3 - the quotient of M by the trajectories of X_4 is naturally equipped with a FLAT *conformal 3D Lorentzian structure*.
- OBVIOUSLY $SO(2,3)$ symmetric!

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Contact projective structure

A *contact projective structure* on a 3-dimensional manifold N is given by the following data.

- A contact distribution $\mathcal{C}$, that is the distribution annihilated by a 1-form ω on N, such that $\mathrm{d}\omega \wedge \omega \neq 0$.
- A family of unparameterized curves everywhere tangent to $\mathcal{C}$ and such that:
  - for a given point and a direction in $\mathcal{C}$ there is exactly one curve passing through that point and tangent to that direction,
  - curves of the family are among unparameterized geodesics for some linear connection on N.

I have no time to show that P^3 - the quotient of M by the trajectories of X_3 has a natural FLAT *contact projective structure in* 3D. It is known that such a structure is also naturally $SO(2,3)$ symmetric.

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  - for a given point and a direction in $\mathcal{C}$ there is exactly one curve passing through that point and tangent to that direction,
  - curves of the family are among unparameterized geodesics for some linear connection on N.

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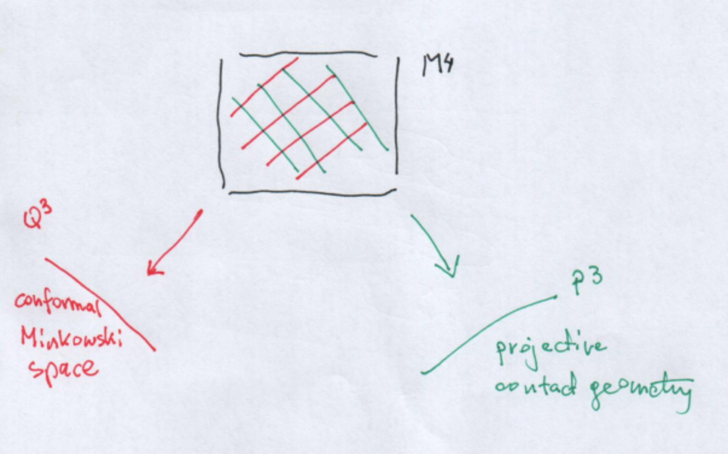
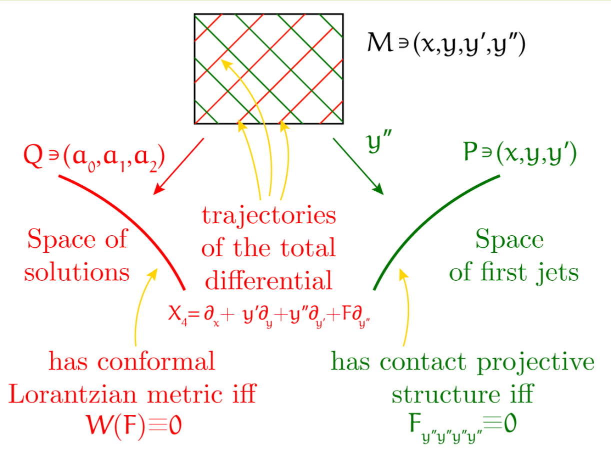
Has anyone seen such a fibration before?



Geometry of 3rd order ODEs

Chern in 1940 considered geomery of ODEs
$y''' = F(x, y, y', y'')$ up to contact transformations of variables.



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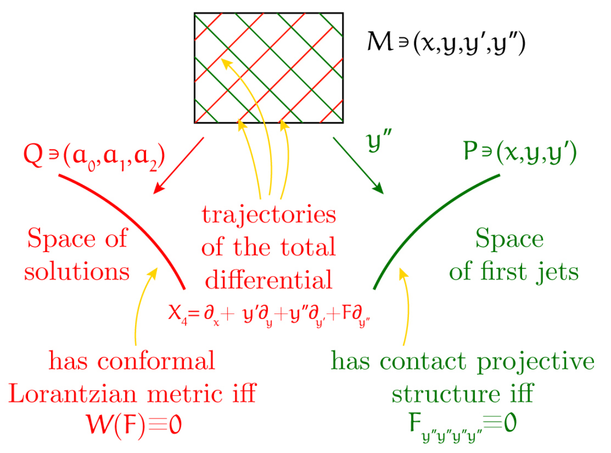

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Lie's description

- Sophus Lie considered vector space $\mathbb{R}^4$ equipped with a nondegenerate 2-form and the **Lagrangian** *vector subspaces* in $\mathbb{R}^4$.
- The space of all such subspaces $\mathcal{Q}$ is 3-dimensional, and there is an invertible map between the space Q^3 of all points and lines and circles in the plane (x, y) and the Lie space $\mathcal{Q}$.
- Lie established that the nonlinear condition of two circles *kissing each other* in Q^3 is, via this map, a linear condition on the coresponding two Lagrangian planes in $\mathcal{Q}$ to *intersect along a line*.
- this leads to...

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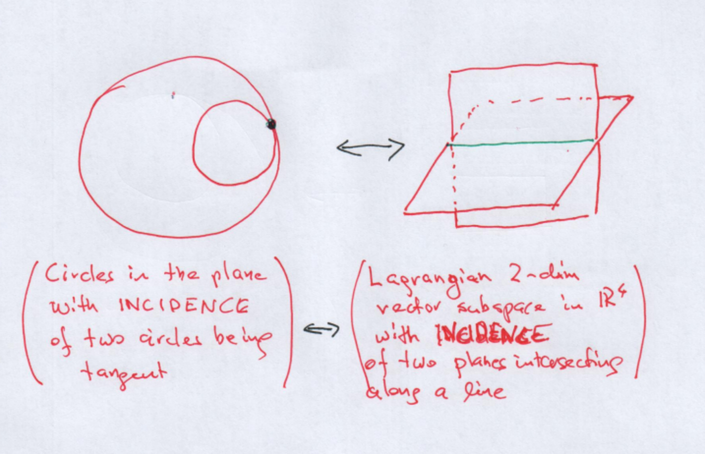

Lie's double fibration

- Let $V = (\mathbb{R}^4, \omega)$ be equipped with a nondegenerate 2-form ω. Let M be a space of all pairs (L, S) such that L is a 1-dim vector subspace in V and S is a Lagrangian 2-dim vector subspace in V, and such that $L \in S$,

$$M = \{(L, S) \mid L \in S\}.$$

- The manifold M is 4-dimensional and, using two natural projections $(L, S) \to S$ and $(L, S) \to L$, one can associate two 3-dimensional manifolds Q and P with M.
- This construction is naturally $Sp(2, \mathbb{R}) = SO(2, 3)$ symmetric, and this gives the Lie's fibration, isomorphic to car's fibration:

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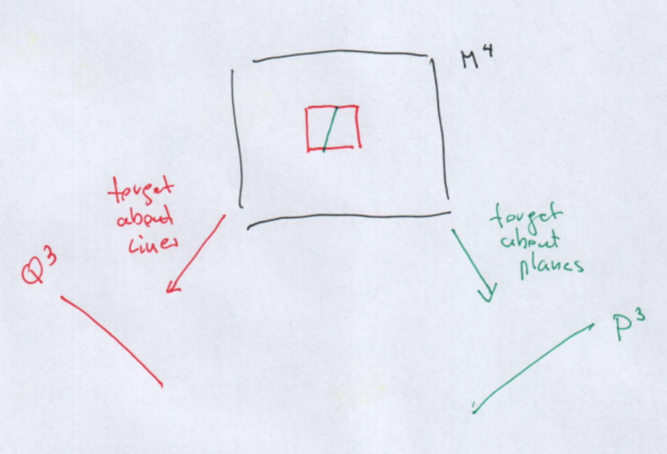
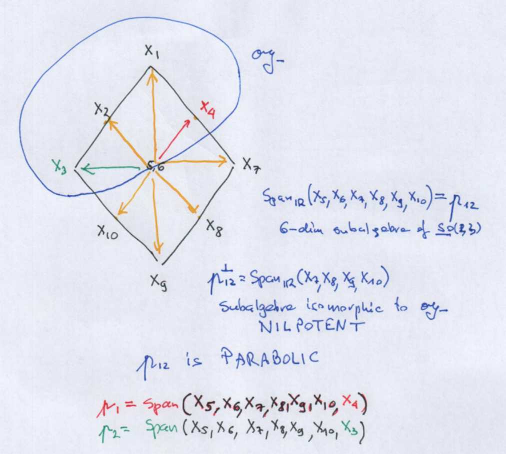
Lie's correspondence



Car and parabolics in $SO(2,3)$



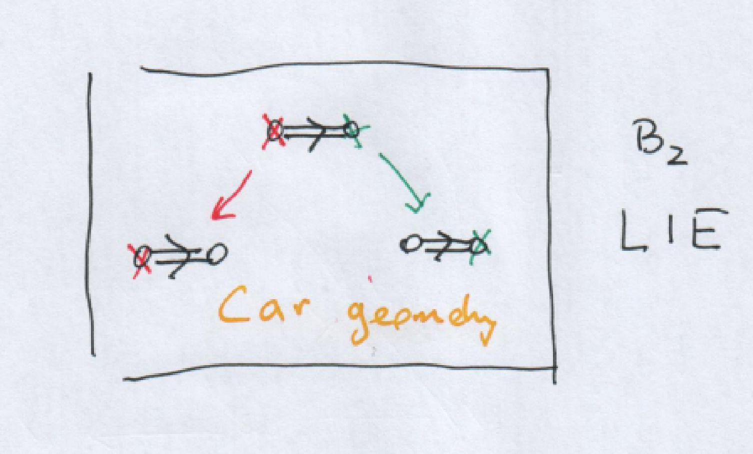
Car’s fibration and three flat parabolic geometries

$$\mathcal{D}_1 = \mathrm{Span}(X_5, X_6, X_7, X_8, X_9, X_{10}, X_4)$$

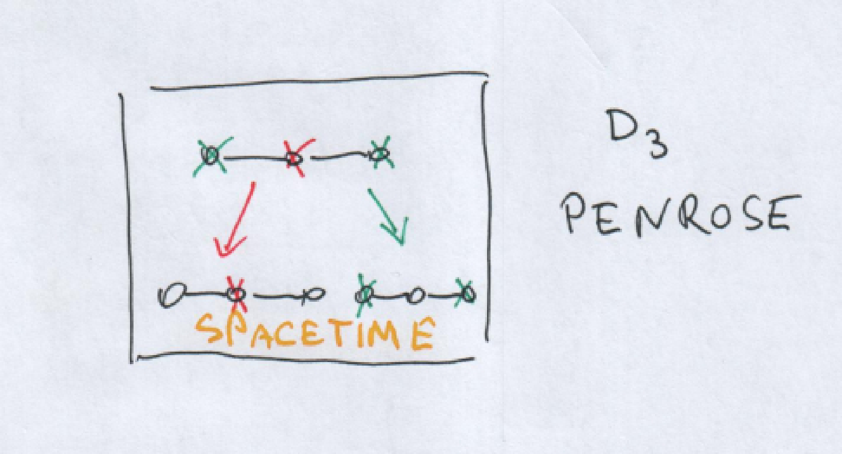
$$\mathcal{D}_2 = \mathrm{Span}(X_5, X_6, X_7, X_8, X_9, X_{10}, X_3)$$

$$\begin{tikzcd}
 & SO(2,3)/P_{12} \arrow[dl] \arrow[dr] & \\
SO(2,3)/P_1 \arrow[d, phantom, "\shortparallel"] & & SO(2,3)/P_2 \simeq \mathbb{P}^3 \\
\mathbb{Q}^3 & &
\end{tikzcd}$$

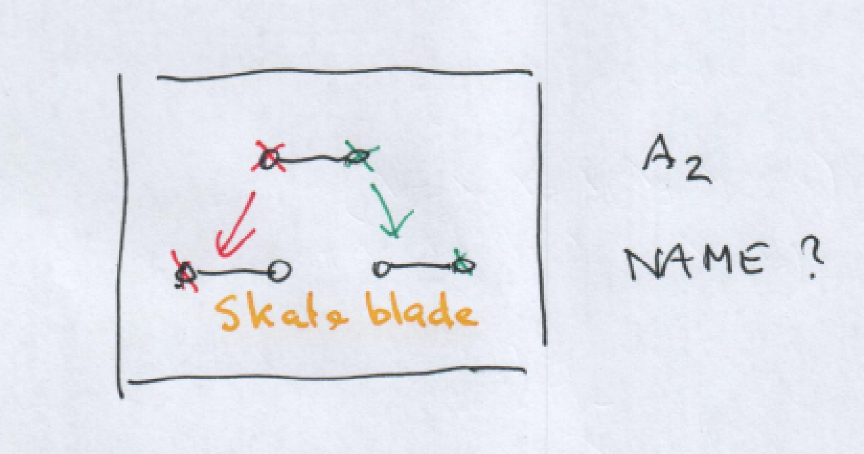
Geometry of a car



Geometry of spacetime



Geometry of a skate blade



Rolling balls and flying saucers