SIMO & Co Shell model

Ahmed Benallal & Jean-Pierre Cordebois

LMT

ENS Paris Saclay/CNRS/Université Paris Saclay

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ON A STRESS RESULTANT GEOMETRICALLY EXACT SHELL MODEL. PART I: FORMULATION AND OPTIMAL PARAMETRIZATION

J.C. SIMO and D.D. FOX
Division of Applied Mechanics, Department of Mechanical Engineering, Stanford University, Stanford, CA, U.S.A.

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OUTLINE

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KINEMATIC DESCRIPTION OF THE SHELL

$$\mathscr{C} := \{(\boldsymbol{\varphi}, \boldsymbol{t}):\ \mathscr{A} \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \times \mathrm{S}^2\}$$

$$\mathscr{S} := \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} = \boldsymbol{\varphi} + \xi \boldsymbol{t} \text{ where } (\boldsymbol{\varphi}, \boldsymbol{t}) \in \mathscr{C} \text{ and } \xi \in [h^-, h^+]\}$$

$$\mathscr{B} := \{\boldsymbol{x}_0 \in \mathbb{R}^3 \mid \boldsymbol{x}_0 = \boldsymbol{\varphi}_0 + \xi \boldsymbol{t}_0 \text{ with } (\boldsymbol{\varphi}_0, \boldsymbol{t}_0) \in \mathscr{C} \text{ and } \xi \in [h^-, h^+]\}$$

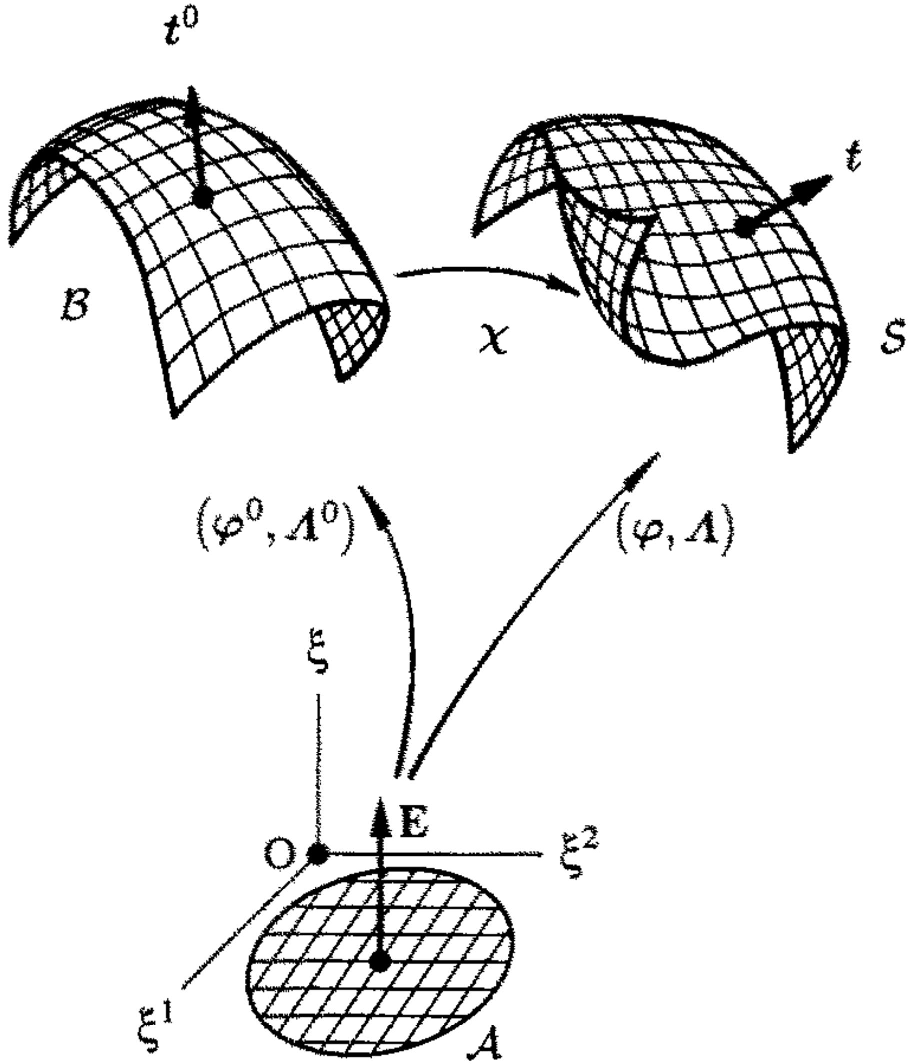
$$\boldsymbol{x} = \boldsymbol{\Phi}(\xi^1, \xi^2, \xi) := \boldsymbol{\varphi}(\xi^1, \xi^2) + \xi \boldsymbol{t}(\xi^1, \xi^2)$$

$$\boldsymbol{\Phi}:\ \mathscr{A} \times [h^-, h^+] \rightarrow \mathbb{R}^3$$

$$\boldsymbol{\chi}:\ \mathscr{B} \rightarrow \mathscr{S}$$

$$\boldsymbol{\chi} := \boldsymbol{\Phi} \circ \boldsymbol{\Phi}_0^{-1}$$

KINEMATIC DESCRIPTION OF THE SHELL



TANGENT MAP - DEFORMATION GRADIENT

Configuration $\boldsymbol{\Phi}: \mathscr{A} \times [h^{-}, h^{+}] \rightarrow \mathbb{R}^3$

Tangent map $$\nabla\boldsymbol{\Phi} := \frac{\partial \boldsymbol{\Phi}}{\partial \xi^I} \otimes \boldsymbol{E}^I \equiv \boldsymbol{g}_I \otimes \boldsymbol{E}^I$$

$$(\boldsymbol{\varphi}_{,\alpha} + \xi \boldsymbol{t}_{,\alpha}) \otimes \boldsymbol{E}^{\alpha} + \boldsymbol{t} \otimes \boldsymbol{E}$$

$\boldsymbol{g}_I = \partial \boldsymbol{\Phi} / \partial \xi^I$ **is the convected basis**

Deformation gradient

$$\boldsymbol{F} \equiv T\boldsymbol{\chi} := \nabla\boldsymbol{\Phi} \circ (\nabla\boldsymbol{\Phi}_0)^{-1}$$

SOME NOTATIONS

Convected basis $\quad g_I = \partial \Phi / \partial \xi^I \quad$ with $\quad g_3 \equiv t$

Reciprocal basis $\quad \{g^I\}_{I=1,2,3} \quad g_I \cdot g^J = \delta_I^J$

$$g_I = \nabla \Phi E_I \quad \text{and} \quad g^I = \nabla \Phi^{-t} E^I$$

Surface convected frame $\quad \{a_I\}_{I=1,2,3} \quad a_\alpha = \varphi_{,\alpha} \quad \text{and} \quad a_3 = g_3 = t$

Metric tensor $\quad a = \alpha_{\alpha\beta} a^\alpha \otimes a^\beta \,, \qquad a_{\alpha\beta} = \varphi_{,\alpha} \cdot \varphi_{,\beta}$

$\{a^I\}_{I=1,2,3}$ dual basis

Jacobians $\quad j := \det[\nabla \Phi] \,, \qquad j_0 := \det[\nabla \Phi_0] \,,$

$$J := \det F = j/j_0$$

$$\bar{j}_0 := \|a_{01} \times a_{02}\| \,, \qquad \bar{j} := \|a_1 \times a_2\|$$

STRESS AND COUPLE STRESS RESULTANTS

Given a motion $\chi_t\colon\ \mathscr{B} \times \mathbb{R}_+ \to \mathscr{S}$ $\quad \chi_t = \boldsymbol{\Phi}_t \circ \boldsymbol{\Phi}_0^-$

$$\mathscr{S}^\alpha := \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} = \boldsymbol{\Phi}|_{\xi^\alpha = \text{const.}}\}, \quad \alpha = 1, 2$$

$$\mathrm{d}\mathscr{S}^1 := j[\nabla\boldsymbol{\Phi}]^{-\mathrm{t}}\boldsymbol{E}^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi \equiv j\boldsymbol{g}^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi$$

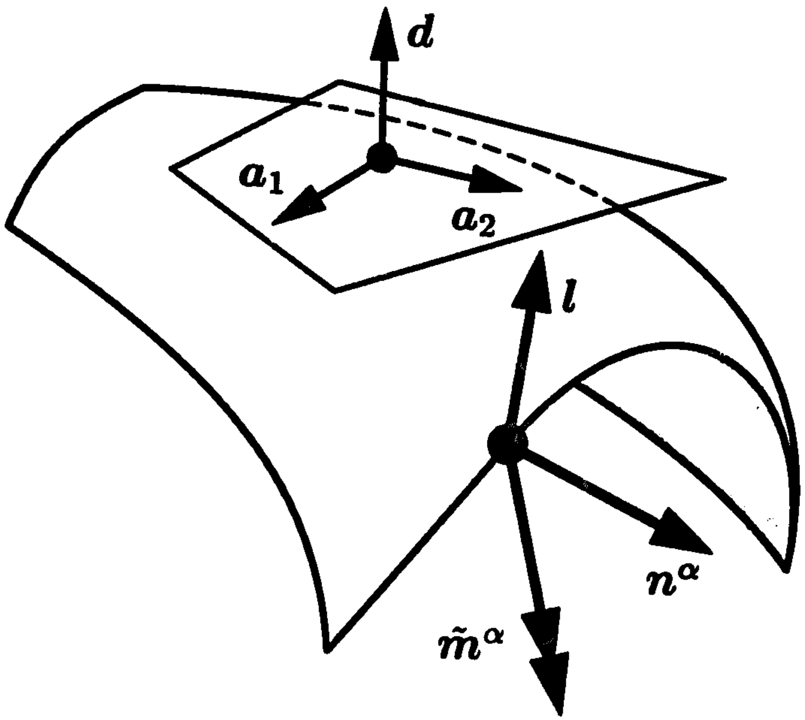
$$\boldsymbol{R}^1 := \int_{h^-}^{h^+} \boldsymbol{\sigma}\,\frac{\mathrm{d}\mathscr{S}^1}{\mathrm{d}\xi^2} = \int_{h^-}^{h^+} \boldsymbol{\sigma}\boldsymbol{g}^1 j\,\mathrm{d}\xi$$

$$\boldsymbol{n}^\alpha := \frac{1}{\bar{j}}\int_{h^-}^{h^+} \boldsymbol{\sigma}\boldsymbol{g}^\alpha j\,\mathrm{d}\xi$$

$$\boldsymbol{T}^1 := \int_{h^-}^{h^+} (\boldsymbol{x} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}\,\frac{\mathrm{d}\mathscr{S}^1}{\mathrm{d}\xi^2} = \int_{h^-}^{h^+} (\boldsymbol{x} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}\boldsymbol{g}^1 j\,\mathrm{d}\xi$$

$$\boldsymbol{m}^\alpha := \frac{1}{\bar{j}}\int_{h^-}^{h^+} (\boldsymbol{x} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}\boldsymbol{g}^\alpha j\,\mathrm{d}\xi$$

$$\boldsymbol{l} = \frac{1}{\bar{j}}\int_{h^-}^{h^+} \boldsymbol{\sigma}\boldsymbol{g}^3 j\,\mathrm{d}\xi$$


d
a1
a2
l
n^α
m̃^α

STRESS AND COUPLE STRESS RESULTANTS

Remarks:

1) Resultants can be also expressed in term of the first Piola Kirchhoff stress

$$\boldsymbol{n}^{\alpha} = \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{\sigma} \boldsymbol{g}^{\alpha} j \, \mathrm{d}\xi \equiv \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{P} \boldsymbol{g}_0^{\alpha} j_0 \, \mathrm{d}\xi$$

2) We have through the kinematic assumption $(\boldsymbol{x} - \boldsymbol{\varphi}) = \xi \boldsymbol{t}$

$$\boldsymbol{m}^{\alpha} = \boldsymbol{t} \times \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi \boldsymbol{\sigma} \boldsymbol{g}^{\alpha} j \, \mathrm{d}\xi \equiv \boldsymbol{t} \times \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi \boldsymbol{P} \boldsymbol{g}_0^{\alpha} j_0 \, \mathrm{d}\xi$$

3) We therefore have

$$\boldsymbol{m}^{\alpha} \cdot \boldsymbol{t} = 0$$

and there is no component of the stress couple resultant along the director

BALANCE EQUATIONS

Equilibrium is written solely in terms of the resultants quantities

$$\frac{1}{\bar{j}}(\bar{j}\boldsymbol{n}^{\alpha})_{,\alpha} + \bar{\boldsymbol{n}} = 0$$

$$\frac{1}{\bar{j}}(\bar{j}\boldsymbol{m}^{\alpha})_{,\alpha} + \boldsymbol{\varphi}_{,\alpha} \times \boldsymbol{n}^{\alpha} + \bar{\boldsymbol{m}} = 0$$

where $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{m}}$ are the applied stress resultant and the applied stress couple resultant

Remark: These relations are all it is needed for the formulation of the boundary value problem or its weak formulation, allowing to avoid the introduction of Christoffel symbols

BALANCE EQUATIONS

Balance equations can be further reduced using the symmetry of the Cauchy stress

$$\boldsymbol{\varphi}_{,\alpha} \times \boldsymbol{n}^{\alpha} + \boldsymbol{t}_{,\alpha} \times \tilde{\boldsymbol{m}}^{\alpha} + \boldsymbol{t} \times \boldsymbol{l} = \boldsymbol{0}$$

$$\boldsymbol{m}^{\alpha} = \boldsymbol{t} \times \tilde{\boldsymbol{m}}^{\alpha}$$

which leads

$$\frac{1}{\bar{j}} (\bar{j}\boldsymbol{n}^{\alpha})_{,\alpha} + \bar{\boldsymbol{n}} = \mathbf{0}$$

$$\frac{1}{\bar{j}} (\bar{j}\tilde{\boldsymbol{m}}^{\alpha})_{,\alpha} - \bar{\boldsymbol{l}} + \bar{\tilde{\boldsymbol{m}}} = \mathbf{0}$$

where $\tilde{\bar{\boldsymbol{m}}} = \bar{\boldsymbol{m}} \times \boldsymbol{t}$ **and**

$$\bar{\boldsymbol{l}} = \boldsymbol{l} + \bar{\lambda}\boldsymbol{t}$$

EFFECTIVE RESULTANTS

$$\boldsymbol{t}_{,\alpha} = \lambda_{\alpha}^{\mu}\boldsymbol{\varphi}_{,\mu} + \lambda_{\alpha}^{3}\boldsymbol{t}$$

$$\boldsymbol{n}^{\alpha} = n^{\beta\alpha}\boldsymbol{\varphi}_{,\beta} + q^{\alpha}\boldsymbol{t}$$

$$\tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\beta\alpha}\boldsymbol{\varphi}_{,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}$$

$$\tilde{n}^{\alpha\beta} = n^{\beta\alpha} - \lambda_{\mu}^{\beta}\tilde{m}^{\alpha\mu}$$

$$\tilde{q}^{\alpha} := q^{\alpha} - \lambda_{\mu}^{3}\tilde{m}^{\alpha\mu} \equiv q^{\alpha} + \lambda_{\mu}^{\beta}\gamma_{\beta}\tilde{m}^{\alpha\mu}$$

Effective membrane and effective shear Stress resultants

KINEMATIC VARIABLES

$$a_{\alpha\beta} := \boldsymbol{\varphi}_{,\alpha} \cdot \boldsymbol{\varphi}_{,\beta} \,, \qquad a_{0\alpha\beta} = \boldsymbol{\varphi}_{0,\alpha} \cdot \boldsymbol{\varphi}_{0,\beta}$$

$$\gamma_{\alpha} := \boldsymbol{\varphi}_{,\alpha} \cdot \boldsymbol{t} \,, \qquad \gamma_{0\alpha} = \boldsymbol{\varphi}_{0,\alpha} \cdot \boldsymbol{t}_{0} \,,$$

$$\kappa_{\alpha\beta} := \boldsymbol{\varphi}_{,\alpha} \cdot \boldsymbol{t}_{,\beta} \,, \qquad \kappa_{0\alpha\beta} = \boldsymbol{\varphi}_{0,\alpha} \cdot \boldsymbol{t}_{0,\beta} \,.$$

$$\boldsymbol{\varepsilon} := \tfrac{1}{2}(a_{\alpha\beta} - a_{0\alpha\beta})\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta}$$ **membrane**

$$\boldsymbol{\delta} := (\gamma_{\alpha} - \gamma_{0\alpha})\boldsymbol{a}^{\alpha} \,,$$ **tranverse shear**

$$\boldsymbol{\rho} := (\kappa_{\alpha\beta} - \kappa_{0\alpha\beta})\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} \,.$$ **bending**

STRESS POWER

$$\mathscr{W} := \int_{\mathscr{B}} \boldsymbol{P} : \dot{\boldsymbol{F}} \, \mathrm{d}\mathscr{V} = \int_{\bar{\mathscr{A}}} [\boldsymbol{n}^{\alpha} \cdot \dot{\boldsymbol{\varphi}}_{,\alpha} + \tilde{\boldsymbol{m}}^{\alpha} \cdot \dot{\boldsymbol{t}}_{,\alpha} + \boldsymbol{l} \cdot \dot{\boldsymbol{t}}] \; \bar{j} \, \mathrm{d}\xi^{1} \, \mathrm{d}\xi^{2}$$

$$\mathscr{W} := \int_{\mathscr{B}} \boldsymbol{P} : \dot{\boldsymbol{F}} \, \mathrm{d}\mathscr{V} = \int_{\bar{\mathscr{A}}} [\tilde{n}^{\beta\alpha} \tfrac{1}{2} \dot{a}_{\beta\alpha} + \tilde{q}^{\alpha} \dot{\gamma}_{\alpha} + \tilde{m}^{\beta\alpha} \dot{\kappa}_{\beta\alpha}] \, \mathrm{d}\mu$$

$$\boldsymbol{\varepsilon} := \tfrac{1}{2}(a_{\alpha\beta} - a_{0\alpha\beta})\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} \qquad L_{v}\boldsymbol{\varepsilon} = \tfrac{1}{2}\dot{a}_{\alpha\beta}\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta}$$

$$\boldsymbol{\delta} := (\gamma_{\alpha} - \gamma_{0\alpha})\boldsymbol{a}^{\alpha} \,, \qquad L_{v}\boldsymbol{\delta} = \dot{\gamma}_{\alpha}\boldsymbol{a}^{\alpha} \,,$$

$$\boldsymbol{\rho} := (\kappa_{\alpha\beta} - \kappa_{0\alpha\beta})\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} \,. \qquad L_{v}\boldsymbol{\rho} = \dot{\kappa}_{\alpha\beta}\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} \,.$$

CONSTITUTIVE RELATIONS

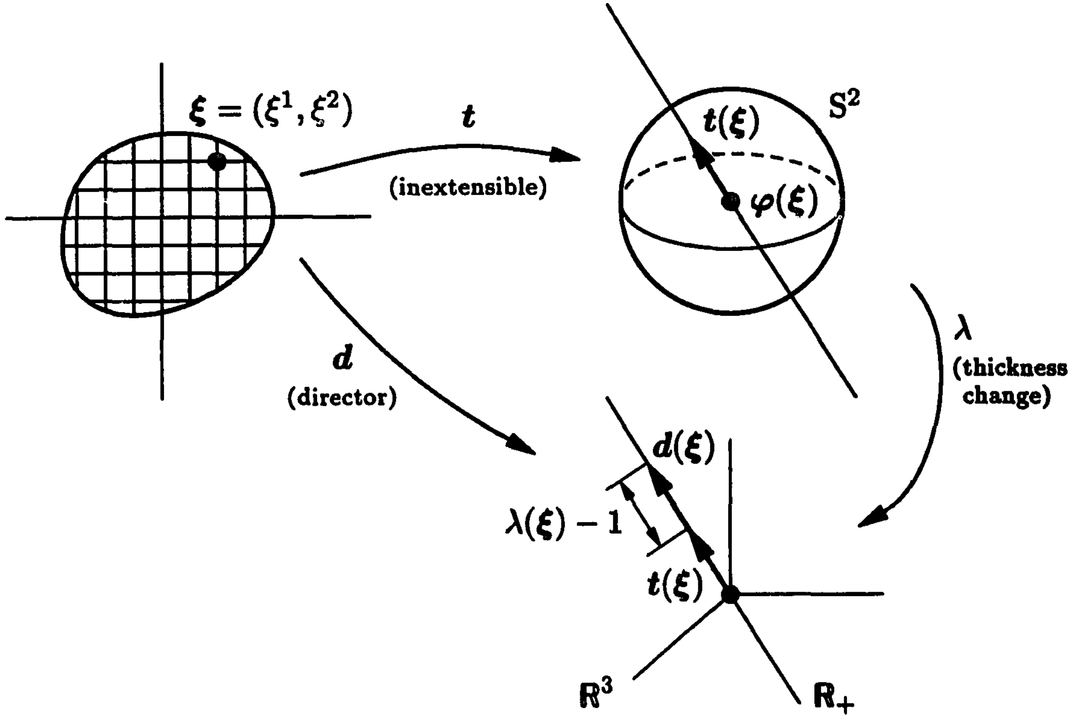
Hyperelasticy: Stored energy function + Standard arguments +…

Plasticity: Strain partition
Yield surface
Flow rule
Hardening rules

EXTENSIBLE DIRECTOR THEORY
Multiplicative decomposition of the director field

$$\boldsymbol{x} = \hat{\boldsymbol{\Phi}}(\xi^1, \xi^2, \xi) := \boldsymbol{\varphi}(\xi^1, \xi^2) + \xi \boldsymbol{d}(\xi^1, \xi^2)$$

$$\boldsymbol{d}(\xi^1, \xi^2) = \lambda(\xi^1, \xi^2)\boldsymbol{t}(\xi^1, \xi^2)$$



GEOMETRICAL ASPECTS

Basic idea:

The nonlinear response of the shell in bending is intimately related to the rotation of the director field

From a geometric point of view, it is convenient to view the rotation of the director field as the motion of a point in the unit sphere

Use of the links between the unit sphere and the rotation group