Monge-Ampère Equations: Geometry, Invariants and Applications in 3D Meteorological Models

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Réunion du GDR - GDM
"Géométrie Différentielle et Mécanique"
Université de La Rochelle, du 4 au 7 juin 2019

June 3, 2019

Plan

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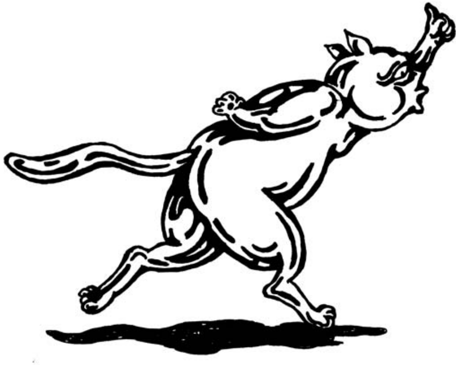


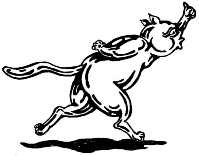
Basic object

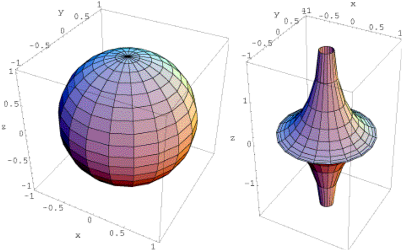
 

Monge and Ampère

$$A\frac{\partial^2 f}{\partial q_1^2} + 2B\frac{\partial^2 f}{\partial q_1 \partial q_2} + C\frac{\partial^2 f}{\partial q_2^2} + D\left(\frac{\partial^2 f}{\partial q_1^2} \cdot \frac{\partial^2 f}{\partial q_2^2} - \left(\frac{\partial f}{\partial q_1 \partial q_2}\right)^2\right) + E = 0$$

Global Solutions: Monge





sphere and pseudosphere

An example: curvature of a surface in $\mathbb{R}^3$

$$\frac{u_{q_1 q_1} \cdot u_{q_2 q_2} - u_{q_1 q_2}^2}{(1 + u_{q_1}^2 + u_{q_2}^2)^2} = \mathcal{K}(u)$$

Monge-Ampère structure

Definition

A Monge-Ampère structure on a $2n$-dimensional manifold X is a pair of differential form $(\Omega, \omega) \in \Omega^2(X) \times \Omega^n(X)$ such that Ω is symplectic and ω is Ω-effective i.e. $\Omega \wedge \omega = 0$.

Main idea

- Let $F : \mathbb{R}^n \to (i)\mathbb{R}^n$ be a vector-function and its graph is a subspace in $T^*(\mathbb{R}^n) = \mathbb{R}^n \oplus (i)\mathbb{R}^n$.

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- This graph is a Lagrangian subspace in $T^*(\mathbb{R}^n)$ iff $(dF)_x$ is a symmetric endomorphism . The matrix $|| \frac{\partial F_i}{\partial x_j} ||$ is symmetric $\forall x$ iff the differential form $\sum_i F_i dx_i \in \Lambda^1(\mathbb{R}^n)$ is closed or, equivalently, exact:

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- The projection of the graph of ∇f on $(\mathbb{R}^n)_x$ is given in coordinates by $\nabla^2(f) = \det || \frac{\partial^2 f_i}{\partial x_j^2} ||$.

Correspondence: Forms -Symplectic MAO

Let M be a smooth $n-$dimensional manifold and ω is a differential n-form on T^*M. A (symplectic) Monge-Ampère operator $\Delta_\omega : C^\infty(M) \to \Omega^n(M)$ is the differential operator defined by

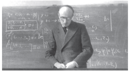
$$\Delta_\omega(f) = (df)^*(\omega),$$

where $df : M \to T^*M$ is the natural section associated to f.

Examples

ω	$\boldsymbol{\Delta}_\omega = 0$
$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	$\Delta f = 0$
$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	$\Box f = 0$
$dp_1 \wedge dp_2 \wedge dp_3 - dq_1 \wedge dq_2 \wedge dq_3$	$Hess(f) = 1$
$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3$ $+dp_3 \wedge dq_1 \wedge dq_2 - dp_1 \wedge dp_2 \wedge dp_3$	$\Delta f - Hess(f) = 0$

Hodge-Lepage-Lychagin theorem

Hodge, Lepage and Lychagin

Theorem (Hodge-Lepage-Lychagin)

- *Every form* $\omega \in \Lambda^k(V^*)$ *can be uniquely decomposed into the finite sum*

$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 + \dots,$$

where all ω_i *are effective forms.*

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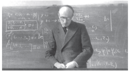


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where all ω_i *are effective forms.*

- *If two effective* k*-forms vanish on the same* k*-dimensional isotropic vector subspaces in* (V, Ω)*, they are proportional.*

Symplectic Monge-Ampère Equations: Solutions

- A generalised solution of a MAE $\Delta_\omega = 0$ is a lagrangian submanifold of (T^*M, Ω) which is an integral manifold for the MA differential form ω:

$$\omega|_L = 0.$$

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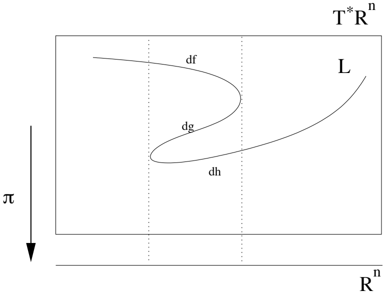
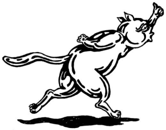
- A generalised solution (generically) locally is the graph of an 1-form df for a regular solution f.

Generalized solution

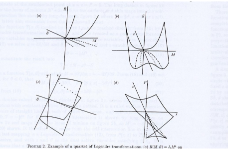




Generalised solution of a MAE

Generic types of singularities for Generalized solutions of MAE

Specific property of the graph-like Lagrangian submanifolds: their projection on the "configuration space" $\mathbb{R}^n$ is a diffeomorphism. Our generalised solutions are general Lagrangian immersions and they have Arnold's lagrangian singularities.



Lagrangian singularities (Wave fronts, foldings etc.) This singularities describe the formation of fronts (Chynoweth, Porter, Sewell 1988)

Symplectic Equivalence-1

- Two SMAE $\Delta_{\omega_1} = 0$ and $\Delta_{\omega_2} = 0$ are locally equivalent iff there is exist a local symplectomorphism $F : (T^*M, \Omega) \to (T^*M, \Omega)$ such that

$$F^*\omega_1 = \omega_2.$$

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- L is a generalised solution of $\Delta_{F^*\omega_1} = 0$ iff $F(L)$ is a generalised solution of $\Delta_\omega = 0$.

Legendre partial transformation

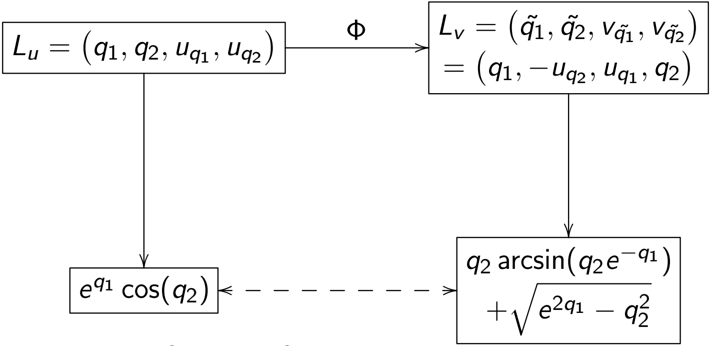
Legendre

$$u_{q_1 q_1} + u_{q_2 q_2} = 0 \quad \longleftarrow - - - \longrightarrow \quad v_{q_1 q_1} v_{q_2 q_2} - v_{q_1 q_2}^2 = 1$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

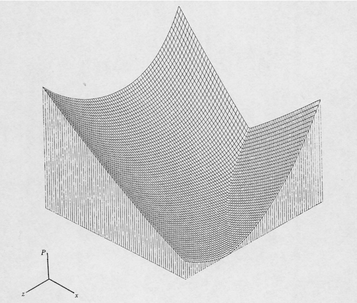
$$\omega = dq_1 \wedge dp_2 - dq_2 \wedge dp_1 \quad \xleftarrow{\Phi^*} \quad \tilde{\omega} = dp_1 \wedge dp_2 - dq_1 \wedge dq_2$$

Legendre partial transformation-2



with $\Phi : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$, $(q_1, q_2, p_1, p_2) \mapsto (q_1, -p_2, p_1, q_2)$.

Sewell-Chynoweth SG- equation



Numerical Solution of the semi-geostrophic $3D$ equation (Cullen, Sewell-Chynoweth...)

$$hess_{x,y}(u) + \frac{\partial^2 u}{\partial z^2} = hess(u) \tag{1}$$

Sewell-Chynoweth MAO form and its equivalence

- The effective form of (??):

$$\omega = dp \wedge dq \wedge dz + dx \wedge dy \wedge dr - \gamma dx \wedge dy \wedge dz,$$

$(x, y, z, p, q, r)-$ canonical coordinates system of $T^*\mathbb{R}^3$.

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- This form is a sum of two decomposable 3-forms:

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- $\phi^*(\omega) = dp \wedge dq \wedge dr - dx \wedge dy \wedge dz$ where ϕ is the symplectomorphism

$$\phi(x, y, z, p, q, r) = (x, y, r, p, q, \gamma r - z).$$

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- The equation (??) is symplectically equivalent to the equation

$$hess(u) = 1. \tag{2}$$

An exact solution of the SG 3D equation

$$f(x,y,z) = \int_a^{\sqrt{xy+yz+zx}} (b+4\xi^3)^{1/3} d\xi$$

is a regular solution of (??). Therefore,

$$L = \Big\{(x, y, (x+y)\alpha, (y+z)\alpha, (z+x)\alpha, \gamma(x+y)\alpha - z)\Big\}$$

is a generalised solution of (??) with

$$\alpha = \frac{1}{2}\Big(\frac{b}{(xy+yz+zx)^{\frac{3}{2}}} + 4\Big)^{\frac{1}{3}}.$$

Hoskins geostrophic coordinate transformation

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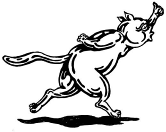
- B. Hoskins (1975) had proposed a remarkable coordinate transformation (a passage to geostrophic coordinates in $x - y$ directions such that the geostrophic velocity and potential temperature may be represented in terms of one function both in the transformed coordinates as in physical ones



$$\begin{cases} X := x + \frac{v_g}{f} = x + \frac{1}{f^2}\frac{\partial \phi}{\partial x} \\ Y := y - \frac{u_g}{f} = y + \frac{1}{f^2}\frac{\partial \phi}{\partial y} \\ Z := z; \quad T := t. \end{cases}$$

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- if the potential vorticity is uniform ($q_g = \frac{f\theta_0}{g} N^2$) then one have in the interior of the fluid for any time $T = t$

$$\frac{1}{f^2}(\Phi_{XX} + \Phi_{YY}) - \frac{1}{f^4}(\Phi_{XX}\Phi_{YY} - \Phi_{XY}^2) + \frac{1}{N^2}\Phi_{ZZ} = 1. \quad (3)$$

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- Here (and in what follows) f is the *Coriolis parameter* taking as a constant and N is the *Brunt - Väisälä* frequency:

$$N = \sqrt{\frac{q_g g}{f\theta_0}},$$

for the uniform potential vorticity q_g and the constant potential temperature θ_0.

Hoskins geostrophic MA effective form

- This is a 3D Monge-Ampére equation with the effective form

$$\omega = \frac{1}{f^2}(dp \wedge dy \wedge dz + dx \wedge dq \wedge dz) + \frac{1}{N^2} dx \wedge dy \wedge dr -$$

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- This form is the sum of two decomposable forms:

$$\omega = \frac{1}{N^2} dx \wedge dy \wedge dr - (dx - \frac{1}{f^2} dp) \wedge (dy - \frac{1}{f^2} dq) \wedge dz.$$

Hoskins geostrophic MA effective form : equivalence

- Consider the symplectomorphism

$$F(x, y, z, p, q, r) = (p, q, z, -x + f^2 p, -y + f^2 q, r). \qquad (4)$$

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- The new canonical coordinate system $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r})$

$$\begin{cases} \tilde{p} := -x + f^2 p; & \tilde{x} := p; \\ \tilde{q} := -y + f^2 q; & \tilde{y} := q; \\ \tilde{r} := r; \quad \tilde{z} := z \end{cases}$$

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- The Hoskins SG (**??**) is equivalent to the (**??**):

$$\text{hess}(u) = \frac{N^2}{f^4} = \frac{(q_g g)^2}{f^6 (\theta_0)^2} \qquad (5)$$

  by the symplectomorphism (**??**).

Table 1. Effective forms with constant coefficients in $2D$

$\mathbf{\Delta_\omega = 0}$	ω	$\mathbf{pf(\omega)}$
$\Delta f = 0$	$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	1
$\Box f = 0$	$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	-1
$\frac{\partial^2 f}{\partial q_1^2} = 0$	$dq_1 \wedge dp_2$	0

Invariants for effective 3-forms

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- The Hitchin pfaffian defined by

$$pf(\omega) = \frac{1}{6} tr A_\omega^2.$$

	$\boldsymbol{\Delta_\omega = 0}$	ω	$\varepsilon(q_\omega)$	$pf(\omega)$
1	$\nu\,\mathrm{hess}(f) = 1$	$-dq_1 \wedge dq_2 \wedge dq_3 + \nu \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(3,3)$	ν^2
2	$\Delta f - \nu\,\mathrm{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 - \nu \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(0,6)$	$-\nu^2$
3	$\Box f + \nu\,\mathrm{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 + \nu \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(4,2)$	$-\nu^2$
4	$\Delta f = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2$	$(0,3)$	0
5	$\Box f = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2$	$(2,1)$	0
6	$\Delta_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 - dp_2 \wedge dq_1 \wedge dq_3$	$(0,1)$	0
7	$\Box_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 + dp_2 \wedge dq_1 \wedge dq_3$	$(1,0)$	0
8	$\frac{\partial^2 f}{\partial q_1^2} = 0$	$dp_1 \wedge dq_2 \wedge dq_3$	$(0,0)$	0
9		0	$(0,0)$	0

Table: Classification of effective 3-formes in dimension 6

HyperKäler triple of MAE

The conservation law (the Ertel's theorem) of the potential vorticity obtains (using the Hamiltonian representation of the system):

$$\frac{d}{dt}\left(\frac{\partial(q_1, q_2)}{\partial(a, b)}\right) =$$

$$\frac{d}{dt}(1 + \phi_{q_1 q_1} + \phi_{q_2 q_2} + \det \mathrm{Hess}\, \phi\ ) = 0,$$

This equation is a part of the HyperKähler triple of MAEs (R. and Roulstone 1997, 2001):

$$\begin{cases} \omega_I = \left[1 + a(p_{11} + p_{22}) + (a^2 - c^2)(p_{11}p_{22} - p_{12}^2)dq_1\right] \wedge dq_2 & , \\ \omega_J = \left[2cp_{12} + ac(p_{11}p_{22} - p_{12}^2)\right] dq_1 \wedge dq_2 & , \\ \omega_K = -c\Omega \end{cases}$$

$2D$ balanced model MAE

- The general family of (elliptic) MAE with constant coefficients carries all flat balanced models:

$$1 + \phi_{q_1 q_1} + a\phi_{q_2 q_2} + (a^2 - c^2)\,\det \mathrm{Hess}\,\phi = \zeta^{\mathbf{C}}/f, \qquad (6)$$

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  Among them are:

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  Among them are:

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- *The L_1 Salmon dynamics* with $a = c = 1$;

$2D$ balanced model MAE

- The general family of (elliptic) MAE with constant coefficients carries all flat balanced models:

$$1 + \phi_{q_1 q_1} + a\phi_{q_2 q_2} + (a^2 - c^2)\det \operatorname{Hess} \phi = \zeta^{\mathbf{C}}/f, \qquad (6)$$

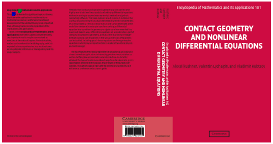
  Among them are:
- *The semi-geostrophic model*($a = 1,\ c = 0$ with $\zeta^{\mathbf{C}}/f$ positive);
- *The L_1 Salmon dynamics* with $a = c = 1$;
- *The $\sqrt{3}$ dynamics of McIntyre - Roulstone* for $a = 1, c = \sqrt{3}$ and $\zeta^{\mathbf{C}}/f < 3/2$;

  Our classification theorem in $2D$ gives a classification of all "almost-balanced"($0 < c < \sqrt{3}$) models with a uniform potential vorticity.

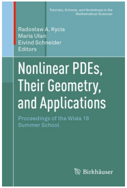
The subjects which I had no time to describe:

- Symmetries, conservation laws and Noether theorem for MAO and MAE
- Self-similar solutions, shock waves and Hugoniot-Rankin conditions
- Variational MAE, divergent MAE and Euler-Lagrange operators
- Jacobi 2D non-linear 1st order systems and Genralised Complex Geometry of Hitchin
- Generalised Calabi-Yau 3D structures
- Linearisation of Dritchell-Viudez coupled MAE in 2D and 3D
- Many-many other interesting things...

Bibliography:



Cambridge University Press,2007



Springer-Birkhäuser, 2019

Thank you for your attention!