Kinematics of defected material: a geometrical point of view

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Abstract

In the framework of defected medium

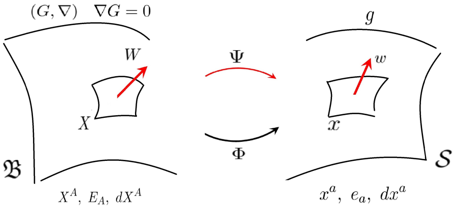
- T is identified with dislocation
- R is associated with disclination

Let (G, ∇) defined on $\mathfrak{B}$ st $\nabla G = 0$. Undergoing transformation, $(\overline{g}, \overline{\nabla})$ is defined to describe defects state in $\Phi(\mathfrak{B})$.

Material transformation

$$\Xi \quad : \quad \begin{array}{ccc} T\mathfrak{B} & \longrightarrow & \mathcal{S} \\ (X, W) & \longmapsto & (\Phi(X), \Psi(X)W) \end{array}$$

- $\Phi \in C^1$-regular is the map of point.
- $\Psi \in C^1$ with $\det \Psi \neq 0$ is the map of vector.



Transformation is

- holonomic if $\Psi = \mathbf{F}$,
- nonholonomic if $\Psi \neq \mathbf{F}$.

- Recall $\mathbf{F}$ (or Φ_*) is the deformation gradient of Φ.

$$\begin{array}{rcccc} \mathbf{F}(X) & : & T_X\mathfrak{B} & \longrightarrow & T_x\mathcal{S} \\ & & V & \longmapsto & \mathbf{F}(X)V \end{array}$$

In components,

$$\mathbf{F} = \mathbf{F}_A^{\ a} e_a \otimes dX^A, \quad \text{with} \quad \mathbf{F}_A^{\ a} = \partial_A \Phi^a.$$

- The linear map

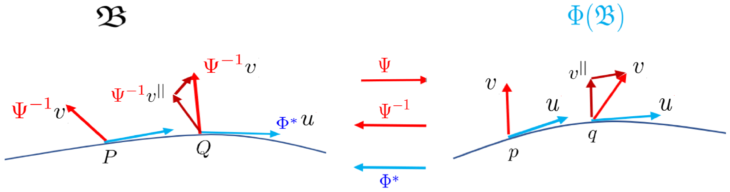
$$\begin{array}{rcccc} \Psi(X) & : & T_X\mathfrak{B} & \longrightarrow & T_x\mathcal{S} \\ & & V & \longmapsto & \Psi(X)V \end{array}$$

In components,

$$\Psi = \Psi_A^{\ a} e_a \otimes dX^A.$$

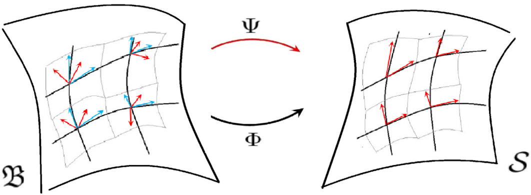
The induced $(\bar{g}, \overline{\nabla})$ on $\Phi(\mathfrak{B})$

Let u, v be vector fields on $\Phi(\mathfrak{B})$.



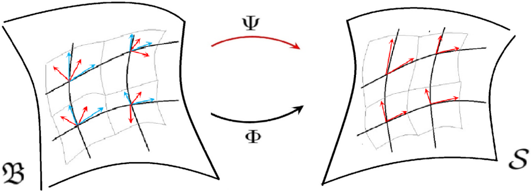
By the construction, the induced connection is natural defined by

$$\overline{\nabla}_u v = \Psi \nabla_{\Phi^* u} \Psi^{-1} v.$$



The induced metric and connection are defined, respectively, by

$$\overline{g}(u, v) = G(\Psi^{-1}u, \Psi^{-1}v),$$



The induced metric and connection are defined, respectively, by

$$\overline{g}(u,v) = G(\Psi^{-1}u, \Psi^{-1}v),$$

$$\overline{\nabla}_u v = \Psi \nabla_{\Phi^*u} \Psi^{-1} v.$$

The metric compatibility $\overline{\nabla}\overline{g} = 0$ is respected.

$$\overline{g}_{ab} = (\Psi^{-1})^A_{\ a} (\Psi^{-1})^B_{\ b} G_{AB},$$

$$\overline{\Gamma}^c_{\ ba} = \Psi^c_C (F^{-1})^A_{\ b} \Big((\Psi^{-1})^B_{\ a} \Gamma^C_{AB} + \partial_A (\Psi^{-1})^C_{\ a} \Big).$$

Connection, Torsion and Curvature

Connection $\overline{\Gamma}$:

$$\overline{\Gamma}^c{}_{ba} = \Psi_C^c (F^{-1})_b^A \Big((\Psi^{-1})_a^B \Gamma_{AB}^C + \partial_A (\Psi^{-1})_a^C \Big).$$

Curvature $\overline{R}$: $\overline{R}(u,v)w = \overline{\nabla}_u \overline{\nabla}_v w - \overline{\nabla}_v \overline{\nabla}_u w - \overline{\nabla}_{[u,v]} w$,

$$\overline{R}(u,v)w = \Psi R(\Phi^* u, \Phi^* v) \Psi^{-1} w.$$

Torsion $\overline{T}$:

$$\overline{T}^a{}_{bc} = \overline{\Gamma}^a{}_{bc} - \overline{\Gamma}^a{}_{cb}.$$

$$\text{If } \Psi = \mathbf{F}, \quad \overline{T}(u,v) = \Phi_* T(\Phi^* u, \Phi^* v).$$

Connection

$$\overline{\Gamma}^{c}{}_{ba} = \Psi_{C}^{c}(F^{-1})^{A}_{b}\Big((\Psi^{-1})^{B}_{a}\Gamma^{C}_{AB} + \partial_{A}(\Psi^{-1})^{C}_{a}\Big).$$

Consider $\Gamma^{C}_{AB} = 0$. The induced connection is given by

$$\overline{\Gamma}^{c}{}_{ba} = \Psi_{C}^{c}\partial_{x^{b}}(\Psi^{-1})^{C}_{a}.$$

Connection

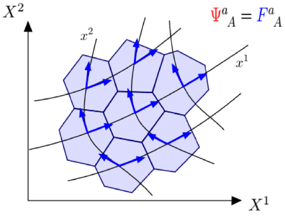
$$\overline{\Gamma}^{c}{}_{ba} = \Psi^{c}_{C}(F^{-1})^{A}_{b}\Big((\Psi^{-1})^{B}_{a}\Gamma^{C}_{AB} + \partial_{A}(\Psi^{-1})^{C}_{a}\Big).$$

Consider $\Gamma^{C}_{AB} = 0$. The induced connection is given by

$$\overline{\Gamma}^{c}{}_{ba} = \Psi^{c}_{C}\partial_{x^b}(\Psi^{-1})^{C}_{a}.$$

If $\Psi = \mathbf{F}$, $\Rightarrow \overline{T} = \Phi_{*}T$, $\overline{R} = \Psi R$

$$\Rightarrow \overline{T} = 0,\ \overline{R} = 0.$$



Connection

$$\overline{\Gamma}^{c}{}_{ba} = \Psi^{c}_{C}(F^{-1})^{A}_{b}\left((\Psi^{-1})^{B}_{a}\Gamma^{C}_{AB} + \partial_A(\Psi^{-1})^{C}_{a}\right).$$

Consider $\Gamma^{C}_{AB} = 0$. The induced connection is given by

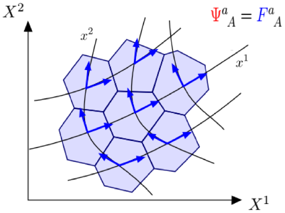
$$\overline{\Gamma}^{c}{}_{ba} = \Psi^{c}_{C}\partial_{x^b}(\Psi^{-1})^{C}_{a}.$$

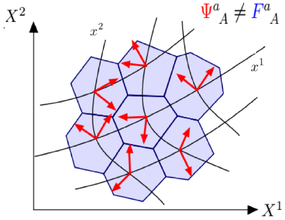
If $\Psi = \mathbf{F}$, $\Rightarrow \overline{T} = \Phi_* T$, $\overline{R} = \Psi R$

$\Rightarrow \overline{T} = 0$, $\overline{R} = 0$.

If $\Psi \neq \mathbf{F}$, $\Rightarrow \overline{R} = \Psi R$

$\Rightarrow \overline{T} \neq 0$, $\overline{R} = 0$.





Incompatibility law

Given the reference manifold with $\Gamma = 0$,

which are the conditions for Ψ such that $\overline{T}$ vanishes?

The torsion $\overline{T} = 0$ if and only if

$$\overline{\Gamma}^c{}_{ba} = \overline{\Gamma}^c{}_{ab} \quad \text{or} \quad \Psi^c_C \partial_{x^b} (\Psi^{-1})^C_{\ a} = \Psi^c_C \partial_{x^a} (\Psi^{-1})^C_{\ b}.$$

It is equivalent to

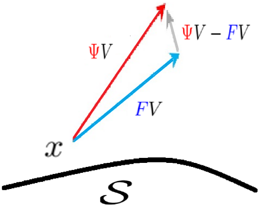
$$\mathrm{curl}(\Psi^{-1}) = 0.$$

Notice: $\Psi = \mathbf{F} \Rightarrow \mathrm{curl}(\Psi^{-1}) = 0$, but $\mathrm{curl}(\Psi^{-1}) = 0 \not\Rightarrow \Psi = \mathbf{F}$.

Strain measures (Mindlin)

(1) Classical elastic strain:

$$\mathbf{E} = \frac{1}{2}(\Phi^* g - G).$$



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(2) Nonholonomic strain measures the difference between the motion of point and vector:

$$\mathcal{E} = \frac{1}{2}(\Psi - \mathbf{F})^{\bigstar} g.$$

$$\mathcal{E}(U, V) = \tfrac{1}{2} g\Big((\Psi - \mathbf{F})U, (\Psi - \mathbf{F})V\Big).$$

$\mathcal{E} = 0 \Leftrightarrow \Psi = \mathbf{F}$. Moreover, $\mathcal{E}$ is invariant under body rotation.

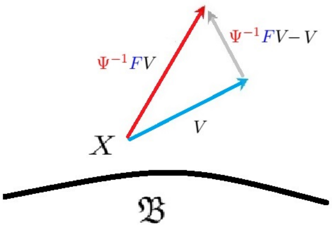
Other strain measures (Eringen)

(1) Microscopic strain:

$$\underline{\mathbf{E}} = \frac{1}{2}(\Psi^* g - G),$$

$\Psi^* g(U, V) = g(\Psi U, \Psi V).$

If $\Psi = \mathbf{F}\mathcal{Q}$ with $\mathcal{Q}$ a rotation, $\underline{\mathbf{E}} = \mathbf{E}$.



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(2) Relative strain:

$$\underline{\mathcal{E}} = \frac{1}{2}(\Phi^* \overline{g} - G),$$

$$\underline{\mathcal{E}}(V, U) = \tfrac{1}{2} G\Big(\Psi^{-1}\mathbf{F}V - V, \Psi^{-1}\mathbf{F}U - U\Big).$$

Moreover, $\underline{\mathcal{E}} = 0 \Leftrightarrow \Psi = \mathbf{F}$. It is invariant under body rotation.

Example

Flat reference configuration is defined by $\Gamma = 0$, $g = \delta$.

Transformation: Φ is an identity and Ψ is given as

$$\Psi_B^{\ a} = \begin{pmatrix} 1 & -\psi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\psi : \mathfrak{B} \to \mathbb{R}$ is C^∞.

The material transformation

$$\mathbf{F} = \mathbb{I} \text{ and } \Psi_B^{\ a} = \begin{pmatrix} 1 & -\psi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The strain measures (Mindlin)

- $\mathbf{E} = 0$
- $\mathcal{E}_{22} = \frac{1}{2}(\Psi - \mathbf{F})_2{}^1(\Psi - \mathbf{F})_2{}^1 = \frac{1}{2}\psi^2.$

The strain measures (Eringen)

- $\underline{\mathbf{E}}_{12} = -\frac{1}{2}\psi$
- $\underline{\mathcal{E}}_{22} = \frac{1}{2}(\Psi^{-1}\mathbf{F} - \mathbb{I})^1{}_2(\Psi^{-1}\mathbf{F} - \mathbb{I})^1{}_2 = \frac{1}{2}\psi^2.$

The material transformation $\mathbf{F} = \mathbb{I}$ and $\Psi_B{}^a = \begin{pmatrix} 1 & -\psi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Metric $\bar{g}_{ab} = \begin{pmatrix} 1 & \psi & 0 \\ \psi & 1+\psi^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Connection

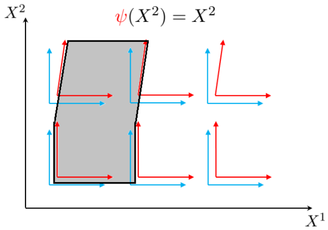
$$\overline{\Gamma}^1{}_{b2} = \Psi_1{}^1 \partial_{X^b} (\Psi^{-1})^1{}_2 = \partial_{X^b} \psi.$$

Torsion

$$\overline{T}^1{}_{\alpha 2} = \partial_{X^\alpha} \psi, \text{ with } \alpha = \{1, 3\},$$

$\overline{T}^1{}_{32}$ screw-type dislocation density. $\overline{T}^1{}_{12}$ edge-type dislocation density.

For ψ is a function of X^2, the current state is defect free.



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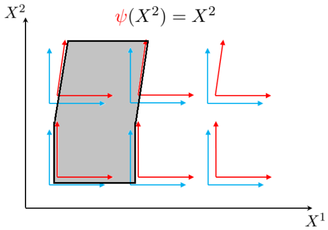
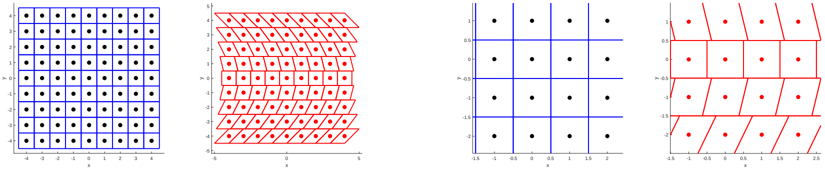
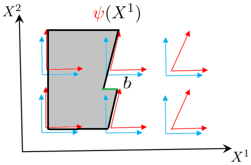
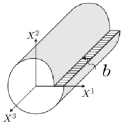




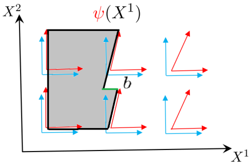
If ψ depends on X^1, there is only edge-type dislocation density.

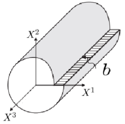




$\overline{T}^1{}_{12}$ edge dislocation

If ψ depends on X^1, there is only edge-type dislocation density.





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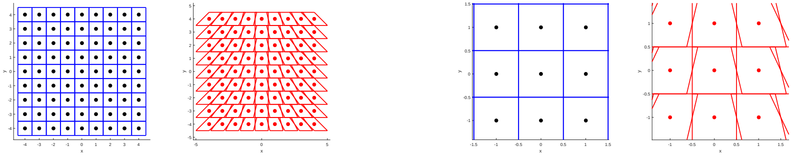
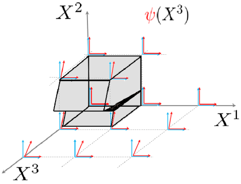
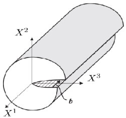


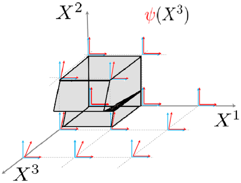
If ψ depends on X^3, there is only screw-type dislocation density.

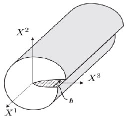




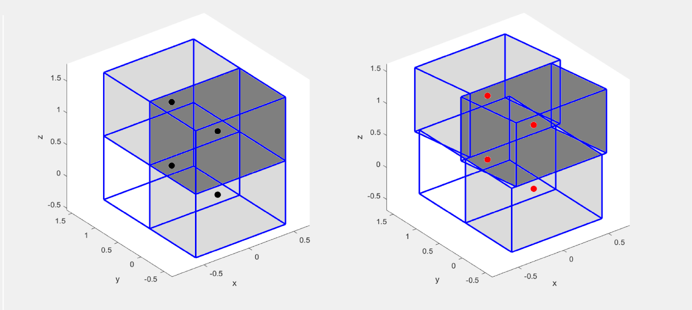
$\overline{T}^1{}_{32}$ screw dislocation

If ψ depends on X^3, there is only screw-type dislocation density.





$\overline{T}^1{}_{32}$ screw dislocation



THANK FOR YOUR ATTENTION!