Some geometric integrators

A road to multisymplectic integrators

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Cartan's lesson (summary)

- Principle of least action with $\mathcal{A} = \int_{t_0}^{t_1} L\,\mathrm{d}t$ leads to

$$\delta\mathcal{A} = \mathrm{d}\mathcal{A}(Z) = [\Theta]_{t_0}^{t_1} - \int_{t_0}^{t_1} (E.L.)\,\delta q \mathrm{d}t, \quad \Theta = p\,\mathrm{d}q - \mathcal{H}dt \tag{1}$$

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- Hamilton formalism with $\Omega = -d\Theta = \underbrace{\mathrm{d}q \wedge \mathrm{d}p}_{\tilde{\Omega}} + \mathcal{H} \wedge \mathrm{d}t$

$$X_{\mathcal{H}} \lrcorner \Omega = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{q} = \frac{\partial \mathcal{H}}{\partial p} \\ \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \end{cases}$$

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- Poisson formalism $\dot{F} = \{F, \mathcal{H}\} = \tilde{\Omega}(X_F, X_{\mathcal{H}})$

Moment map

Theorem (Moment map)

Let X_S be (inifinitesimal) vector field of symmetry. The quantity $J = X_S \lrcorner \Theta$ is conserved along the solutions of the variational problem.

Proof

Since the Lagrangian $\mathcal{L}$ is invariant under X_S, we also have the invariance of the Poincaré-Cartan form

$$0 = \mathcal{L}_{X_S}\Theta = d\left(X_S \lrcorner \Theta\right) + X_S \lrcorner d\Theta.$$

Therefore, along the solutions (vector field X_H), we have

$$d\left(X_S \lrcorner \Theta\right)(X_H) = -X_S \lrcorner d\Theta(X_H) = \Omega(X_H, X_S) = 0 = dJ(X_H),$$

according to the variation theorem and the result follows.

Symmetry example: conservative systems

Invariance by time translation

$$X_S = \partial_t$$

Computation of the moment map

$$J = X_S \lrcorner \Theta = \partial_t \lrcorner \left(p\mathrm{d}q - \mathcal{H}\mathrm{d}t\right) = p\mathrm{d}q(\partial_t) - \mathcal{H}\underbrace{\mathrm{d}t(\partial_t)}_{=1} = -\mathcal{H}$$

Hamiltonian $\mathcal{H}$ is conserved

A road to multisymplectic numerical methods

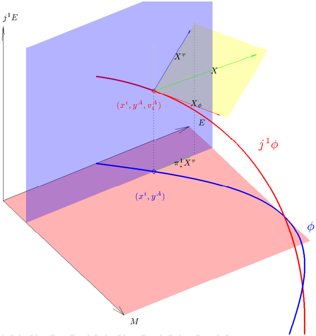


Figure: Sketch of a one-jet fiber-bundle J^1E: the section $j^1\phi$ is called the canonical lifting or the canonical prolongation of ϕ to J^1E. A section of $\jmath\pi$ which is the canonical extension of a section of π is called a **holonomic section**. Any vector is a sum of a tangent vector to the section $j^1\phi$ and a vertical vector $X = X_\phi + X^v$.

Curved manifolds

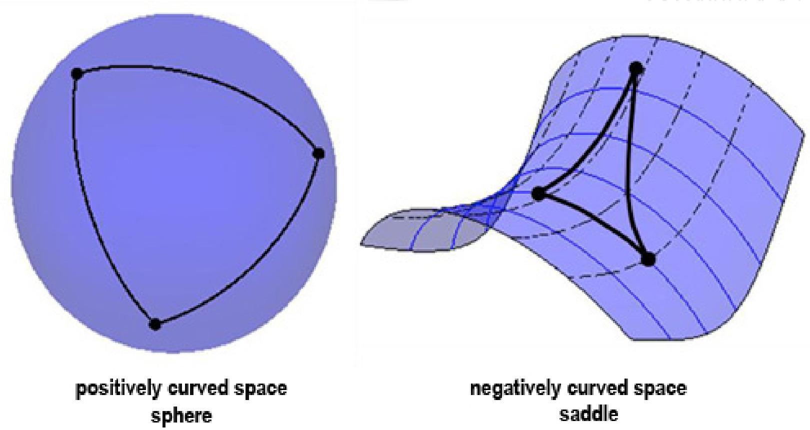


Figure: Idea: Insure that numerical solutions stay on the configuration space

The Runge-Kutta Munthe-Kaas methods (RKMK)

RKMK are examples of Lie group methods [9, 10, 11]. They can be used for a given initial value problem

$$\dot{Y} = A(t, Y)\, Y, \qquad Y(0) = Y_0 \in \mathcal{M} \tag{2}$$

Homogeneous space

$Y \in \mathcal{M}$ on which a Lie group G acts $\rightarrow$ $Y(t) = g(t)Y_0$

The exponential map

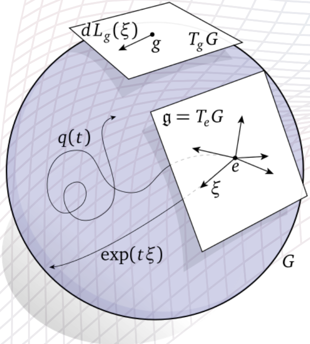


Figure: The exponential map is used to guess a solution of (2) on the form $Y(t) = g(t)Y_0$ with $g(t) = \exp(\xi(t))$

The exponential map...

The exponential map

$$\begin{aligned} exp : \mathfrak{g} &\rightarrow G \\ \xi &\mapsto g = exp(\xi) \end{aligned}$$

and its differential (tangential map)

$$\begin{aligned} Texp : T\mathfrak{g} \simeq \mathfrak{g} &\rightarrow T_g G \\ \dot{\xi} &\mapsto \dot{g} = Texp(\dot{\xi}) \end{aligned}$$

The exponential map...

Computation of the time derivative of $Y(t) = g(t)Y_0$

$$\dot{Y} = \dot{g}Y_0 = \dot{g}g^{-1}gY_0 = \dot{g}g^{-1}Y = TR_{g^{-1}}(\dot{g})Y = TR_{g^{-1}}\left(T\exp(\dot{\xi})\right)Y,$$

shows clearly that (2) may be written as

$$\dot{Y} = \mathrm{d}^{\mathrm{R}}\exp(\dot{\xi})Y = AY$$

$\mathrm{d}^{\mathrm{R}}\exp = TR_{g^{-1}} \circ T\exp$ is the right trivialized derivative.

Lie Group structure preserving ODE

Inverting $\mathrm{d}^{\mathrm{R}}\exp$, a differential equation on the variable $\xi \in \mathfrak{g}$ is then obtained

$$\dot{\xi} = \mathrm{d}^{R}\exp^{-1}(A)\,, \quad \xi(0) = 0 \tag{3}$$

The solution $\xi(t)$ of this equation is then finally used to compute $Y(t)$ via the exponential map. Doing so ensures that the **structure of the Lie group is preserved** — namely that the solution lies on G.

Lie Group structure preserving ODE....

This is a general initial value problem

$$\dot{\xi} = f(t, \xi), \quad \xi(t_0) = \xi_0$$

if the function f is given by $f = dR\exp_{\xi}^{-1} = \sum_{k=0}^{\infty}(B_k/k!)\,\mathrm{ad}_{\xi}^{k}$, where $(B_k)_{k\geq 0}$ are the Bernouilli numbers. A classical RK methods can now be used for $\mathfrak{g}$ is a linear vector space.

Example: RKMK method of order 4

The RKMK4, based on the order 4 classical RK4 method, is obtained by truncation of the sum up to the term of order $q = 2$, yielding

$$\dot{\xi} := f(t,\xi) = A(t,\ Y) - \frac{1}{2}\,\mathrm{ad}_{\xi}\left(A(t,\ Y)\right) + \frac{1}{12}\,\mathrm{ad}^2_{\xi}\left(A(t,\ Y)\right)$$

Classical RK4 method given by the Butcher table

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	2/6	2/6	1/6

$$k_1 = f(t_n, 0)$$
$$k_2 = f(t_n + h/2, \tfrac{h}{2} k_1)$$
$$k_3 = f(t_n + h/2, \tfrac{h}{2} k_2)$$
$$k_4 = f(t_n + h, h\, k_3)$$

leads to the numerical algorithm

$$\widetilde{\xi} = \frac{h}{6}\,\left(k_1 + 2\,k_2 + 2\,k_3 + k_4\right), \quad Y_{n+1} = \exp\left(\widetilde{\xi}\right)\,Y_n.$$

Free rigid body dynamics

For the free rigid body, the Lie group $G = \mathrm{SO}(3)$ acts transitively on the homogeneous space $\mathcal{M} = S^2$. Equation (2) yields in this case

$$\dot{\boldsymbol{\pi}} = -\begin{pmatrix} 0 & \frac{\pi_3}{I_3} & -\frac{\pi_2}{I_2} \\ -\frac{\pi_3}{I_3} & 0 & \frac{\pi_1}{I_1} \\ \frac{\pi_2}{I_2} & -\frac{\pi_1}{I_1} & 0 \end{pmatrix} \boldsymbol{\pi}, \qquad \boldsymbol{\pi}(0) = \boldsymbol{\pi}_0 \tag{4}$$

with $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)^T$ and $\mathbb{I} = diag(I_1, I_2, I_3)$ is the inertia tensor.

Free rigid body dynamics...

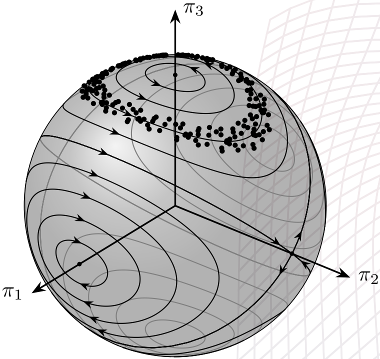


Figure: Angular momentum. RKMK4 methods (left) compared to a variational integrator of order 1 (right) for time step $h = 0.9$, $\boldsymbol{\pi}_0 = \begin{pmatrix}\cos(\pi/3) & 0 & \sin(\pi/3)\end{pmatrix}^T$, $\mathbb{I} = diag(2/3, 1, 2)$.

Relative energy error

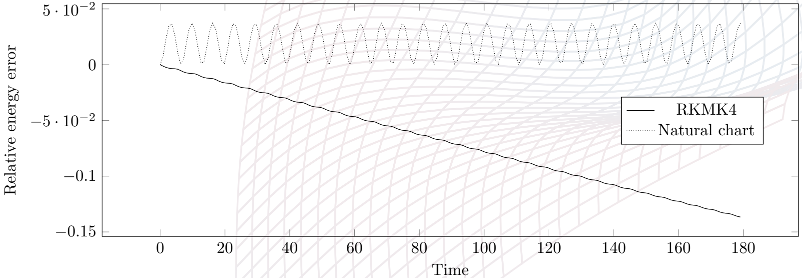
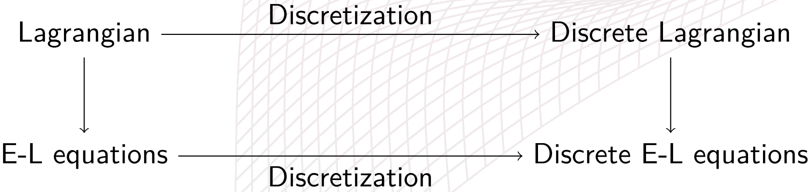
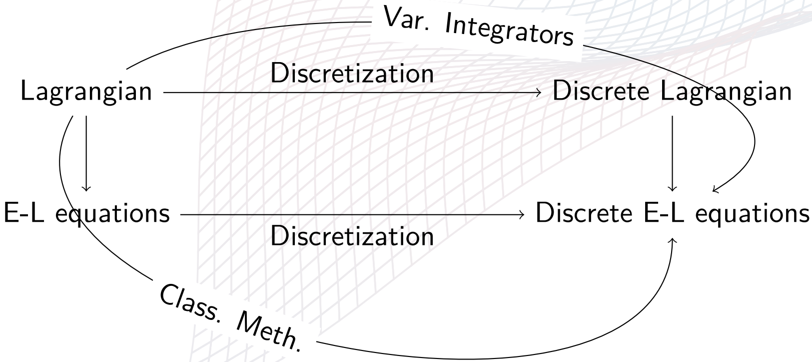


Figure: Relative energy error for $h = 0.9$.The RKMK4 method generates numerical errors that result over the long term in energy dissipation. For variational integrator the energy is not exactly preserved but remains in a bounded interval.

Variational integrators



Variational integrators



Covariant variational methods, Lie groups

Consider a reduced Lagrangian ℓ, for each interval $[t_i, t_{i+1}]$, the discrete action is a sum of approximated integral $\ell_d(\xi_i) \approx \int_{t_i}^{t_{i+1}} \ell(\xi)\, \mathrm{d}\, t$ given by

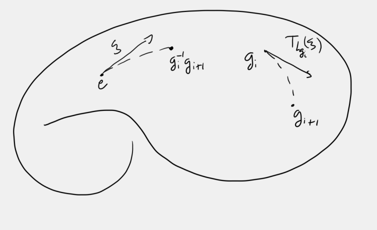
$$S_d(g_d) = \sum_{i=0}^{N-1} \ell_d(\xi_i).$$

Local diffeomorphism

A local diffeomorphism $\tau : \mathfrak{g} \to G$ is used to move from point g_i to

$$g_{i+1} = g_i \tau(h\xi_i) \quad (5)$$

according to a velocity vector ξ_i given in the Lie-algebra $TG_e \equiv \mathfrak{g}$. Inverting the formula for a time step h, we obtain $\xi_i := \frac{1}{h}\tau^{-1}(g_i^{-1}g_{i+1})$.



Variational calculus

Since the variation $\delta S_d(g_d) = \sum_{i=0}^{N-1} \left\langle \frac{\partial \ell_d}{\partial \xi}(\xi_i), \delta\xi_i \right\rangle$, to apply the Hamilton principle $\delta\xi_i$ has to be computed. Knowing

$$\delta\left(g_i^{-1} g_{i+1}\right)$$

$$= \delta g_i^{-1} g_{i+1} + g_i^{-1}\delta g_{i+1} = -\underbrace{g_i^{-1}\delta g_i}_{\delta\zeta_i}\underbrace{g_i^{-1} g_{i+1}}_{\tau(h\xi_i)} + \underbrace{g_i^{-1} g_{i+1}}_{\tau(h\xi_i)}\underbrace{g_{i+1}^{-1}\delta g_{i+1}}_{\delta\zeta_{i+1}}$$

$$= \left(-\delta\zeta_i + \tau(h\xi_i)\,\delta\zeta_{i+1}\,\tau^{-1}(h\xi_i)\right)\tau(h\xi_i) = \left(-\delta\zeta_i + \mathrm{Ad}_{\tau(h\xi_i)}\,\delta\zeta_{i+1}\right)\tau(h\xi_i$$

we obtain, $\delta\xi_i = \frac{1}{h}\,\mathrm{d}\,\tau^{-1}_{\tau(h\xi_i)}\left[\left(-\delta\zeta_i + \mathrm{Ad}_{\tau(h\xi_i)}\,\delta\zeta_{i+1}\right)\tau(h\xi_i)\right]$
Hence the right trivialized differential $\mathrm{d}^{\mathrm{R}}\,\tau^{-1} : \mathfrak{g} \to \mathfrak{g}$ defined by $\mathrm{d}^{\mathrm{R}}\,\tau_\xi^{-1} := T_{\tau(\xi)}\tau^{-1} \circ TR_{\tau(\xi)}$ is introduced, to write

$$\delta\xi_i = \frac{1}{h}\,\mathrm{d}^{\mathrm{R}}\,\tau^{-1}_{h\xi_i}\left(-\delta\zeta_i + \mathrm{Ad}_{\tau(h\xi_i)}\,\delta\zeta_{i+1}\right), \quad \delta\zeta_i = g_i^{-1}\delta g_i$$

Variational calculus...

Using the definition of the adjoint $\langle \pi, A\xi \rangle = \langle A^*\pi, \xi \rangle$ where $\pi \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$, the variation of the action functional now reads

$$\delta S_d(g_d) = \sum_{i=0}^{N-1} \left\langle \frac{1}{h} \left(\mathrm{d}^{\mathrm{R}} \tau_{h\xi_i}^{-1} \right)^* \frac{\partial \ell_d}{\partial \xi}(\xi_i), \mathrm{Ad}_{\tau(h\xi_i)} \delta\zeta_{i+1} - \delta\zeta_i \right\rangle .$$

Introducting the momentum μ_i associated to ξ_i via the formula

$$\mu_i := \left(\mathrm{d}^{\mathrm{R}} \tau_{h\xi_i}^{-1} \right)^* \frac{\partial \ell_d}{\partial \xi}(\xi_i) \tag{6}$$

and changing the indexes in the sum (discrete integration by part), we finally get, by the independence of $\delta\zeta_i$ for all $i \in \{1, \ldots, N-1\}$, the discrete Euler-Poincaré equations

$$\mu_i - \mathrm{Ad}^*_{\tau(h\xi_{i-1})} \mu_{i-1} = 0. \tag{7}$$

Numerical algorithm

Algorithm 1: General implementation of the covariant variational method.

Data: g_0, ξ_0

$g_1 = g_0 \tau\left(h\xi_0\right), \quad \mu_0 = h\left(\mathrm{d}^{\mathrm{R}}\, \tau_{h\xi_0}^{-1}\right)^* \frac{\partial \ell_d}{\partial \xi}(\xi_0)$

for $i = 1$ **to** $N-1$ **do**

- **Compute** $\mu_i = \mathrm{Ad}^*_{\tau(h\xi_{i-1})}\, \mu_{i-1}$, (eq. (7))
- **Solve** $\left(\mathrm{d}^{\mathrm{R}}\, \tau_{h\xi_i}^{-1}\right)^* \frac{\partial \ell_d}{\partial \xi}(\xi_i) - h\mu_i = 0$ **to find** ξ_i, (eq. (6))
- **Update** $g_{i+1} = g_i \tau\left(h\xi_i\right)$, (eq. (5))

end

This implicit algorithm (eq. (6)) is solved using a numerical solver such as a Newton method.

A road to multisymplectic numerical methods

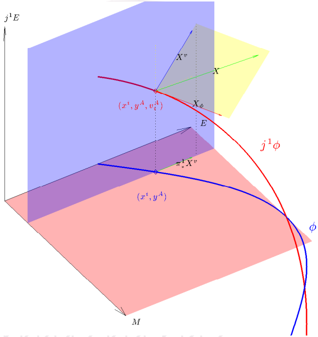


Figure: Sketch of a one-jet fiber-bundle J^1E: the section $j^1\phi$ is called the canonical lifting or the canonical prolongation of ϕ to J^1E. A section of $\jmath\pi$ which is the canonical extension of a section of π is called a **holonomic section**. Any vector is a sum of a tangent vector to the section $j^1\phi$ and a vertical vector $X = X_\phi + X^v$.

Methods based on generating functions

Main idea

- A numerical method can be viewed as a canonical transformation at each time step
- It generates a structure preserving method since canonical transformations preserve the (pre)-symplectic 2-form ω
- Generating functions are used to construct canonical transformations
- Each approximation a generating function gives rise to a numerical method (to a certain order)

Canonical transformations

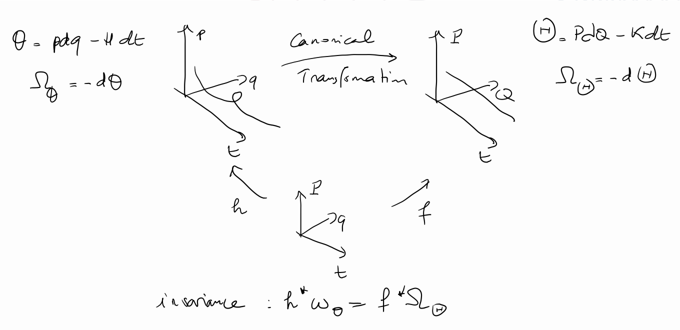


Figure: A canonical transformation is a map $(t, q, p) \mapsto (t, Q, P)$ between coordinates of extended phase space considered as a manifold M. Independent variables (q, P) are used to construct the second kind of generating function $G(t, q, P)$.

The Poincaré-Cartan form

The Poincaré-Cartan form θ is a differential 1-form on M for which $H(t, q, p)$ is a Hamiltonian function. The (pre)-symplectic form ω_θ is obtained by differentiation

$$\theta = p\mathrm{d}q - H\mathrm{d}t \mapsto \omega_\theta = -\mathrm{d}\theta$$

The coordinates (t, Q, P) can be considered as giving another chart on M associated to the 1-form Θ and 2-form Ω_Θ with a corresponding Hamiltonian function $K(t, Q, P)$

$$\Theta = P\mathrm{d}Q - K\mathrm{d}t \mapsto \Omega_\Theta = -\mathrm{d}\Theta$$

Generating function of the second kind

As it is well-know, it is possible to find four[1] generating functions depending of all mixes of old and new variables: (q, Q), (q, P), (p, Q), or (p, P). It appears that the second kind (q, P) of generating function is easily used to generate an infinitesimal transformation closed to the identity. And in turn, defines, by construction, a structure preserving numerical method. The mixed coordinates system (t, q, P) may be related to the previous ones through two mappings h and f : such that

$$h : (t, q, P) \mapsto p(t, q, P) \quad \text{and} \quad f : (t, q, P) \mapsto Q(t, q, P)$$

[1] at least 4, since other possibilities exist

Invariance of the symplectic map

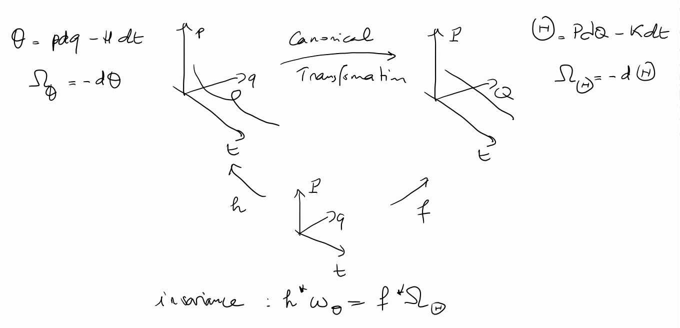


Figure: If each of the (pre-)symplectic forms $\omega_\theta = -\mathrm{d}\theta$ and $\Omega_\Theta = -\mathrm{d}\Theta$ are invariantly associated to one another, their pull-back should agree

Invariance of the symplectic map...

Since the operator $(\mathrm{d}\,)$ and $(\,^*)$ commute, that means $\mathrm{d}(h^*\theta) = \mathrm{d}(f^*\Theta)$. Consequently, $h^*\theta$ and $f^*\Theta$ differ from a closed form

$$dS(t,q,P) = h^*\theta - f^*\Theta = h^*\left(p\mathrm{d}q - H\mathrm{d}t\right) - f^*\left(P\mathrm{d}Q - K\mathrm{d}T\right)$$

Introducing[2] $G = (f^*QP) + S$, one computes

$$\frac{\partial G}{\partial t}\mathrm{d}t + \frac{\partial G}{\partial q}\mathrm{d}q + \frac{\partial G}{\partial P}\mathrm{d}P = h^*\left(p\mathrm{d}q - H\mathrm{d}t\right) - f^*\left(Q\mathrm{d}P - K\mathrm{d}t\right)$$

---

[2] $f^*(P\mathrm{d}Q) = f^*\mathrm{d}(QP) - f^*Q\mathrm{d}P$

Hamilton-Jacobi equation

$$\left(f^*K - h^*H - \frac{\partial G}{\partial t}\right)\mathrm{d}t - \left(f^*Q - \frac{\partial G}{\partial P}\right)\mathrm{d}P + \left(h^*p - \frac{\partial G}{\partial q}\right)\mathrm{d}q = 0$$

i.e.

$$\begin{cases} K(t, Q(t,q,P), P) = H(t, q, p(t,q,P)) + \frac{\partial G}{\partial t} \\ Q(t,q,P) = \frac{\partial G}{\partial P} \\ p(t,q,P) = \frac{\partial G}{\partial q} \end{cases}$$

Hamilton-Jacobi equation

Tacking $K \equiv 0$ yields the so-called Hamilton-Jacobi equation

$$H(t, q, \frac{\partial G}{\partial q}) + \frac{\partial G}{\partial t} = 0. \tag{8}$$

Any solution $G(t, q, P)$ generates a canonical transformation ψ that transforms the Hamiltonian vector fields X_H to equilibrium: $\psi_* X_H = X_{K=0} = 0$.

$$\begin{cases} f^* Q = Q(t, q, P) = \frac{\partial G}{\partial P} \\ h^* p = p(t, q, P) = \frac{\partial G}{\partial q} \end{cases} \tag{9}$$

Integrable system

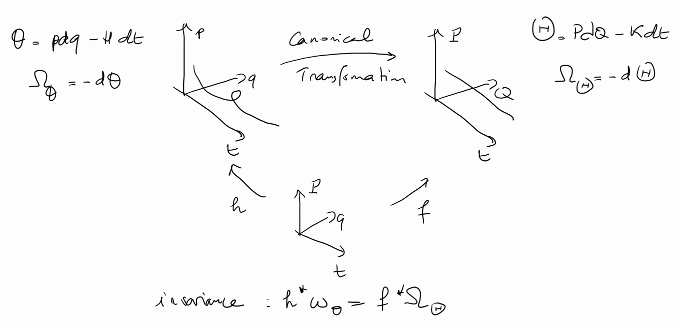


Figure: The canonical transformation ψ transforms the Hamiltonian vector fields X_H to equilibrium: $\psi_* X_H = X_{K=0} = 0$.The integral cuves of X_K are represented by straight lines in the image space. The vector field has been "integrated" by the transformation

Example: the identity transformation

The choice of the second kind of generating function is convenient to easily generate the identity (canonical) transformation.
Choosing $G = qP$ in (9b) and (9c) reads

$$\begin{cases} Q(t,q,P) = \frac{\partial G}{\partial P} = q \\ p(t,q,P) = \frac{\partial G}{\partial q} = P \end{cases}$$

with $H = \frac{\partial G}{\partial t} = 0$ eq. (8)

Infinitesimal transformation

So, a canonical (infinitesimal) transformation is obtained by plugging the ansatz

$$G(t,q,P) = qP + \sum_{m=1}^{\infty} \frac{t^m}{m!} G_m(q,P) = qP + tG_1(q,P) + \frac{t^2}{2} G_2(q,P) + \ldots \tag{10}$$

into the Hamilton-Jacobi equation (8). Equating coefficients

$$G_1 = -H(q,P), \quad G_2 = -\frac{\partial H}{\partial p}\frac{\partial G_1}{\partial q}, \quad G_3 = -\frac{\partial H}{\partial p}\frac{\partial G_2}{\partial q} - \frac{\partial^2 H}{\partial p^2}\frac{\partial G_1}{\partial q} \ldots$$

Structure preserving numerical method

A numerical method of the order k is obtained by truncating the serie (10) to a certain order k (see also [?]). The remaining variables (p, Q) are computed using the generating function G in (9b) and (9c): $Q = \frac{\partial G}{\partial P}$ and $p = \frac{\partial G}{\partial q}$. Putting (q, p) in the left-hand size, the numerical algorithm is finally

$$\begin{cases} q = Q - \sum_{m=1}^{k} \frac{t^m}{m!} \frac{\partial G_m}{\partial P}(q, P) \\ p = P + \sum_{m=1}^{k} \frac{t^m}{m!} \frac{\partial G_m}{\partial q}(q, P) \end{cases}$$

As it can be seen, the first step may be implicit for the variable q. But when it is solved, the second step is explicit for p.

The symplectic Euler method

The symplectic Euler method is an example of such methods of order 1 with $G_1 = -H(q, P)$.

$$\begin{cases} q = Q + t\frac{\partial H}{\partial P}(q, P) \\ p = P - t\frac{\partial H}{\partial q}(q, P) \end{cases}$$

Backward analysis question

G chosen, what is the approximative hamiltonian system that is exactly solved by the numerical methods?

Initial Hamiltonian system H → Exact solution S

→ Numerical solution $\tilde{S}$

→ Approximative Hamiltonian $\tilde{H} = -\frac{\partial G}{\partial t}$

$$\tilde{H}(t, q, P + \sum_{m=1}^{k} \frac{t^m}{m!} \frac{\partial G_m}{\partial q}) = -\frac{\partial G}{\partial t} = -\sum_{m=1}^{k} \frac{t^{m-1}}{(m-1)!} G_m(q, P)$$

Poincaré-Cartan form for Lie reducion

Following the same approach as the preceding section, the Hamilton-Jacobi theory is reduced from T^*G to $\mathfrak{g}^*$, the dual Lie algebra. Let (t, q_0, π_0) be coordinate functions in some chart of extended phase space considered as a manifold $M = \mathbb{R} \times G \times \mathfrak{g}^*$. The 1-form Poincaré-Cartan is

$$\theta = \pi_0 \lambda_{q_0} - H \mathrm{d}t$$

where $\lambda_{q_0}(v) = (L_{q_0^{-1}})_*(v)$ is the Maurer-Cartan form.
The coordinates (t, q_1, π_1) the 1-form is $\Theta = \pi_1 \lambda_{q_1} - K \mathrm{d}t$ with $\lambda_{q_1}(v) = (L_{q_1^{-1}})_*(v)$.

Lie-Poisson Hamilton-Jacobi integrators

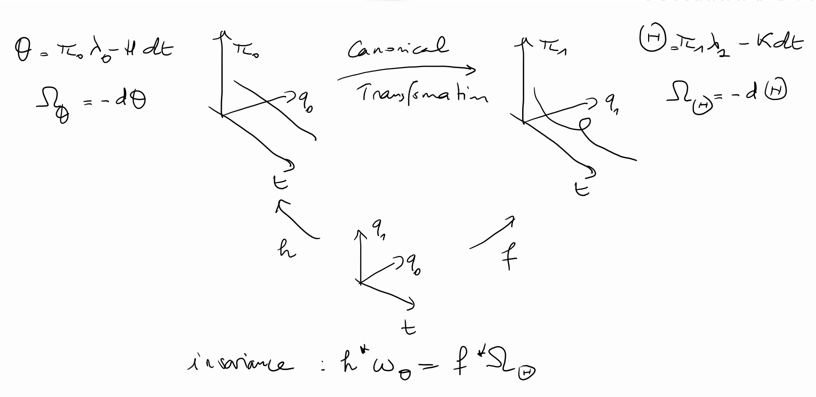


Figure: The mixed coordinates system (t, q_0, q_1) may be related to the previous ones through two mappings $h : (t, q_0, q_1) \mapsto \pi_0(t, q_0, q_1)$ and $f(t, q_0, q_1) \mapsto \pi_1(t, q_0, q_1)$.

The generating function of the first kind

For the left invariant system, the Hamiltonian function is left invariant. It is then natural to seek for left invariant generating functions $S_t(q_0, q_1) = S_t(qq_0, qq_1)$, $\forall q \in G$. Choosing $q = q_0^{-1}$ we can construct a left invariant function $\bar{S}_t$ given by

$$S_t(q_0, q_1) = S_t(e, q_0^{-1} q_1) = S_t(e, g) = \bar{S}_t(g), \quad g = q_0^{-1} q_1.$$

The invariance of the (pre-)symplectic forms $\omega_\theta = -\mathrm{d}\theta$ and $\Omega_\Theta = -\mathrm{d}\Theta$ gives now rise to a function $\bar{S}_t(g)$ such that

$$\mathrm{d}\bar{S}_t = f^*\Theta - h^*\theta = f^*\left(\pi\lambda_{q_1} - K\mathrm{d}t\right) - h^*\left(\pi_0\lambda_{q_0} - H\mathrm{d}t\right) \quad (11)$$

So computing $\mathrm{d}\bar{S}_t = \frac{\partial \bar{S}_t}{\partial t}\mathrm{d}t + \frac{\partial \bar{S}_t}{\partial g}\mathrm{d}g$, it appears that $\mathrm{d}g$ must also be computed in term of λ_{q_0} and λ_q,

$$\begin{aligned}
\mathrm{d}g &= \mathrm{d}(q_0^{-1}q_1) = \mathrm{d}q_0^{-1}q_1 + q_0^{-1}\mathrm{d}q_1 \\
&= -\underbrace{q_0^{-1}\mathrm{d}q_0}_{\lambda_{q_0}}\underbrace{q_0^{-1}q_1}_{g} + \underbrace{q_0^{-1}q_1}_{g}\underbrace{q_1^{-1}\mathrm{d}q_1}_{\lambda_{q_1}} \\
&= -\lambda_{q_0}g + g\lambda_{q_1} = -(R_g)_*\lambda_{q_0} + (L_g)_*\lambda_{q_1}.
\end{aligned}$$

So, comparing the expression
$\mathrm{d}\bar{S}_t = \frac{\partial \bar{S}_t}{\partial t}\mathrm{d}t - \frac{\partial \bar{S}_t}{\partial g}(R_g)_* \lambda_{q_0} + \frac{\partial \bar{S}_t}{\partial g}(L_g)_* \lambda_{q_1}$ with (11), one obtains

$$\begin{cases} h^* H = f^* K + \frac{\partial \bar{S}_t}{\partial t} \\ f^* \pi_1 = (L_g)^* \frac{\partial \bar{S}_t}{\partial g} \\ h^* \pi_0 = (R_g)^* \frac{\partial \bar{S}_t}{\partial g} \end{cases} \mapsto \begin{cases} H(t, \pi_0(t,g)) = K(t, \pi_1(t,g)) + \frac{\partial \bar{S}_t}{\partial t} \\ \pi_1(t,g) = (L_g)^* \frac{\partial \bar{S}_t}{\partial g} \\ \pi_0(t,g) = (R_g)^* \frac{\partial \bar{S}_t}{\partial g} \end{cases} \tag{12}$$

For $H \equiv 0$, this yields the Lie-Poisson Hamilton-Jacobi equation

$$K\left(t, (L_g)^* \frac{\partial \bar{S}_t}{\partial g}\right) + \frac{\partial \bar{S}_t}{\partial t} = 0, \quad g = q_0^{-1} q_1 \tag{13}$$

So equation (12c)

$$\pi_0(t, g) = (R_g)^* \frac{\partial \bar{S}_t}{\partial g} \tag{14}$$

plugged into equation (12b) gives

$$\pi_1(t, g) = Ad_g^* \pi_0(t, g) \tag{15}$$

Lie-Poisson integrator is obtained by approximately solving the Lie-Poisson Hamilton-Jacobi equation (13) and then using (14) and (15) to generate the algorithm. This last equation (15) manifestly preserves the co-adjoint orbit $\mathcal{O}_{\pi_0} = \{\pi \in \mathfrak{g}^* | \pi = Ad_g^* \pi_0, \forall g \in G\}$. .

As in the classical case, one can generate algorithms of arbitrary accuracy by approximating the generative function by an ansatz such as the one given by (10), i.e

$$\bar{S}_t(g) = S_0(g) + \sum_{m=1}^{\infty} \frac{t^m}{m!} S_m(g) = S_0 + tS_1(g) + \frac{t^2}{2} S_2(g) + \dots \quad (16)$$

Li [**?**] propose to reformulate the above theory of a generating function on $TG*$ by the exponential mapping in terms of algebra variable. For $g \in G$, choose $\xi \in \mathfrak{g}$ so that $g = \exp \xi$. He use Channel and Scovel's [**?**] results for which $S_0 = (\xi, \xi)/2$.

Perspectives

In our case, our perspective is to relate the Lie-Poisson Hamilton-Jacobi algorithm to the Euler-Poincaré algorithm developed in section 3 based on the Cayley map. In particular, since equations (15) and (7) are the same in both algorithm, it will be instructive to compare the approximation of the Lie-Poisson Hamilton-Jacobi equation (13) to the relationship between μ and ξ given by equation (6).

Thank you...

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