Role of continuous symmetries in analytic mechanics & field theories

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Henri Poincaré :

« le concept général de groupe préexiste dans notre esprit. Il s'est imposé à nous non pas comme une forme de notre sensibilité, mais comme une forme de notre entendement »

Outline

- From analytic mechanics to field theories: role of symmetries

- Symmetries in continuum mechanics

- Mathematical approach: symmetries, conservation Laws, Noether's theorem, the direct construction method of cons. laws

- Case of elastodynamics: examples of symmetries, cons. laws & equivalence transformations

- Outlook

Condensed form of Noether's theorem in classical mechanics

Noether's theorem: on the real trajectory of a dynamical system, a quantity is conserved for each symmetry (discrete or continuous).

Measurement of observable physical quantities implies their invariance by a change of experimental conditions: *relativity principle* entails conservation laws.

Non observable quantities are then not measurable (extends to quantum mechanics).

Non observable	Symmetry	Conservation law
Absolute origin of time	Temporal translation	Energy
Absolute origin of space	Spatial translation	Linear Momentum
Privileged direction	Rotation	Angular momentum

Corollary: incompatibility between different physical quantities.

Ex.: energy conservation associated with non observable nature of absolute time formulated as classical limit when Planck constant vanishes (Heisenberg inequality):

$$\Delta E.\Delta t = h \to 0$$

Lagrangian formulation in classical mechanics

Lagrangian formulation of the laws of physics trace back to about 1790.

frequently used in classical mechanics to write laws of motion from a least action principle.

Generalization: many physical laws derived from a Lagrangian formulation
-> Allows non mechanistic vision of classical mechanics
-> Highlight symmetry properties.

Allows description of elementary phenomena (set of interacting particles) and provides linkage with quantum mechanics thanks to Hamiltonian formalism.

Basic idea: represent a system depending on N DOF's by a point or vector with N generalized coordinates $\{q_\alpha\}$

Phase space: add velocities $\{q_\alpha, \dot{q}_\alpha\}$ (two sets of DOF's considered independent a priori).

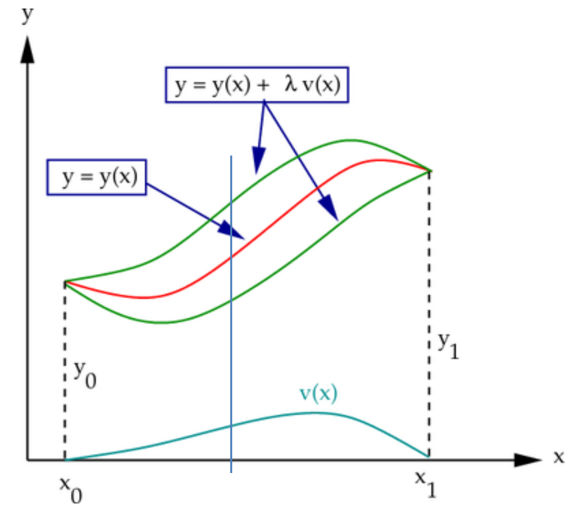
System characterized by Lagrangian function: $L[q_\alpha, \dot{q}_\alpha, t]$

-> Hamilton-Jacobi action $S[q_\alpha] := \int_{t_1}^{t_2} L[q_\alpha, \dot{q}_\alpha, t] dt$

Lagrangian & Hamiltonian formulation in classical mechanics (2)

Isochronal variation (at fixed time) of the action:

$$\delta S[q_\alpha] = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q_\alpha}\delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha}\delta \dot{q}_\alpha \right\} dt \equiv \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) \right\} \delta q_\alpha dt + \left[\frac{\partial L}{\partial \dot{q}_\alpha}\delta q_\alpha \right]_{t_1}^{t_2}$$



-> Euler-Lagrange equations (necessary conditions):

$$\forall \alpha \in \{1,2,\ldots,N\},\quad \delta S[q_\alpha] = 0 \Rightarrow \forall \alpha \in \{1,2,\ldots,N\},\ \frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0$$

E.L. equ. invariant by adding total derivative of a function: $L \to L + \dfrac{dF(\{q_i\},t)}{dt}$

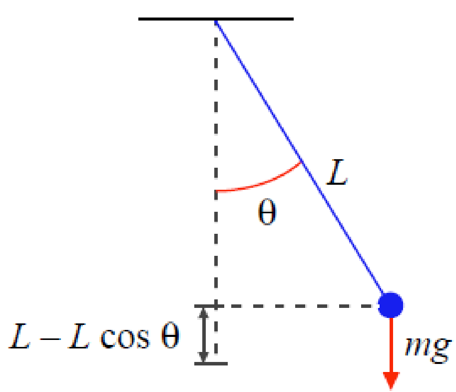
Ex. (pendulum): $L(\theta,\dot{\theta}) = \frac{1}{2} m L^2 \dot{\theta}^2 - mgL(1-\cos\theta) \to \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$ free oscillations

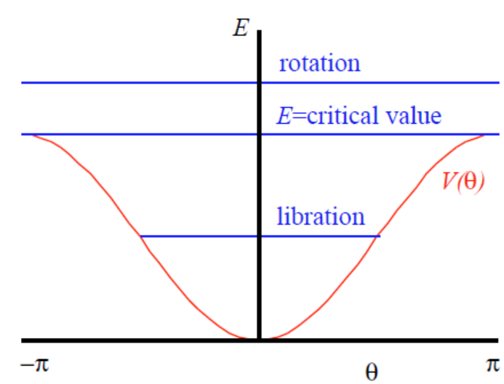
<u>Hamiltonian formulation</u>:

$$p = mv \Rightarrow H(q=\theta, p, t) = \frac{p^2}{2m} + V \equiv E,$$

$$E = Cte \Rightarrow p = \pm\sqrt{2m(E - V(\theta))}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \to \frac{d}{dt}\begin{pmatrix}\theta \\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\dot{\theta} \\ -g\sin\theta / l\end{pmatrix}$$





Symmetries and conservation laws

Symmetries and conservation laws:

Def.: first integral = scalar quantity $f\left(q_i,\dot{q}_i,t\right)=Cte$

Ex.: cyclic coordinate q_i s.t. $\dfrac{\partial L}{\partial q_i}=0 \rightarrow p_i := \dfrac{\partial L}{\partial \dot{q}_i}=Cte$ by E.L. equations, since $\dfrac{dp_i}{dt}=0$

Ex.: conservation of energy for a time-independent Lagrangian:

Energy conservation results from absence of absolute origin of time: time translation invariance leads to

$$\frac{\partial L\left(q_i,\dot{q}_i,t\right)}{\partial t}=0 \Rightarrow \frac{dL}{dt}=\left(\frac{\partial L}{\partial q_i}\frac{dq_i}{dt}+\frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right)=\dot{p}_i\dot{q}_i+p_i\ddot{q}_i=\frac{d}{dt}\left(p_i\dot{q}_i\right)$$

$$\longrightarrow \frac{d\left(L-p_i\dot{q}_i\right)}{dt}=0 \Rightarrow \frac{dH}{dt}=0 \Rightarrow \boxed{E := L-p_i\dot{q}_i=Cte}$$

Construction of Lagrangian function based on symmetries

Galilean referential (class of referentials in relative motion at uniform velocity): assume <u>uniform time</u> (Newtonian absolute time), <u>homogeneous space</u> (same properties whatever position), <u>isotropic space</u> (same properties in all directions).

$$\frac{\partial L}{\partial t} = 0 = \frac{\partial L}{\partial q} \Rightarrow L\left(q,\dot{q},t\right) = \alpha\dot{q}^2 + \beta$$

E.L. equ. gives for this **free particle Lagrangian** (no external forces -> no potential energy):

$$\frac{\partial L}{\partial q} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}} \equiv \frac{\partial K}{\partial \dot{q}} = Cte \Rightarrow \dot{q} = Cte$$

-> Law of inertia: a free particle moves at constant velocity in any Galilean frame.

<u>Remark</u>: rest state nothing but a particular case of a motion at nil velocity.

Adopt kinetic energy of free particle: $L(q,\dot{q},t) = \frac{1}{2}m\dot{q}^2$

Euler equations invariant by rescaling Lagrangian by a multiplicative factor

Noether's theorem in classical (analytic) mechanics

Conserved quantities play an important role for the analysis of dynamical systems:

- Highlight invariant properties.
- Allow to solve dynamical equ. more easily.

For isolated Newtonian systems: 10 conserved quantities due to invariance of laws of non relativistic physics w.r. Galilean symmetry transformations:

translations in time & space, spatial rotation, proper Galilean transformations = boosts.

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t$$

<u>Poincaré group</u> : Lie group of Minskowski space-time isometries (special relativity)

Consider point transformations of generalized coordinates in Lagrangian mechanics = canonical transformations leaving action invariant:

$$t \to \bar{t} = \bar{t}(t),\ q \to \bar{q} = \bar{q}(q(t), t)$$

Non isochronal transformations

$$S := \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt \to \bar{S} = S$$

Requires following law of transformation of the Lagrangian for S to be invariant:

$$\bar{L}\left(\bar{q}, \dot{\bar{q}}, \bar{t}\right) = \frac{\partial t}{\partial \bar{t}} L(q, \dot{q}, t), \quad \frac{\partial t}{\partial \bar{t}} = J^{-1}$$ inverse of Jacobean of transformation $t \to \bar{t}(t)$

Noether's theorem in classical (analytic) mechanics

Infinitesimal change of L under infinitesimal changes $\Delta t, \Delta q$ shall satisfy:

$$\frac{\partial(\Delta t)}{\partial t} + \Delta L = -\frac{d(\Delta F)}{dt} \Rightarrow \Delta S = -\int_{t_1}^{t_2} dt \frac{d(\Delta F)}{dt} \qquad (1)$$

Consider infinitesimal variation (**non** isochronal):

$$t \to \bar{t}(t) = t + \delta t(t); \quad q \to \bar{q}(\bar{t}) = q(t) + \delta q(t)$$

$$\Rightarrow \delta S[q] = \left[p_j \delta q_j - H \delta t \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} EL^j \left(\delta q_j - \dot{q}_j \delta t \right) dt \qquad (2)$$

$$EL^j := \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right)$$ Euler operator

Evaluate variation of action:

$$t \to \bar{t} = \bar{t}(t), \; q \to \bar{q} = \bar{q}(q(t), t) \Rightarrow \delta S[q] = \int_{t_1}^{t_2} L\left(\bar{q}, \dot{\bar{q}}, \bar{t}\right) d\bar{t} - \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt$$

$\delta(.)$ variation associated to a symmetry $\neq$ $\Delta(.)$ more general variation

Identify (1) and (2) -> $\frac{d}{dt}\left[p_j \delta q_j - H \delta t + \Delta F \right]_{t_1}^{t_2} + EL^j \left(\Delta q_j - \dot{q}_j \Delta t \right) = 0$ (3)

$$H\left\{ p_j, \dot{q}_j, t \right\} := p_j \dot{q}_j - L$$

holds on virtual paths

Noether's theorem in classical (analytic) mechanics

Remark: more specific case of an invariant Lagrangian leads to $\Delta F = 0$

Assume now system admits Lie group of transformations depending on finite number of parameters $\Delta\mu_i$ not depending on time:

$$\Delta t = \frac{\partial \Delta t(t)}{\partial \Delta\mu_i}\Delta\mu_i, \quad \Delta q_j = \frac{\partial \Delta q_j(t)}{\partial \Delta\mu_i}\Delta\mu_i$$

Equ. (3) becomes: $$\left\{\frac{dQ_i}{dt} + EL^j\left(\frac{\partial q_j(t)}{\partial \mu_i} - \dot{q}_j \frac{\partial \Delta t(t)}{\partial \Delta\mu_i}\right)\right\}\Delta\mu_i = 0$$

with <u>Noether's charge</u>:

$$Q_i := p^j \frac{\partial q_j(t)}{\partial \Delta\mu_i} - H\frac{\partial \Delta t(t)}{\partial \Delta\mu_i} + \frac{\partial \Delta F}{\partial \Delta\mu_i} \rightarrow \frac{dQ_i}{dt} = -EL^j\left(\frac{\partial q_j(t)}{\partial \Delta\mu_i} - \dot{q}_j \frac{\partial \Delta t(t)}{\partial \Delta\mu_i}\right)$$

-> Noether's th. in classical mechanics: on the path of motion, $EL^j \equiv 0$, thus $Q_i = Cte$

-> **Charge Q is conserved**.

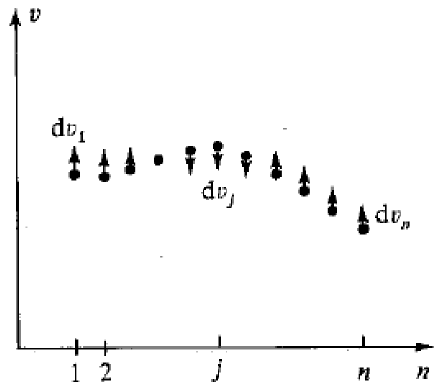
From discrete systems (analytic mechanics) to a field description

$$L = L(\psi_k, \partial_i \psi_k) \rightarrow I = \int_V \int_{t_1}^{t_2} L d^4X \equiv \int_X L d^4X$$

Lagrangian of the field (time-space density)

Transition from analytical mechanics to field theory = discrete description to a continuum

Field continuous in space and time, present enverywhere.



Chain of N equidistant material points aligned along x-axis

$$T_n = \frac{1}{2} m v_n^2 \rightarrow K = \sum_{n=1}^{N} T_n$$ Kinetic energy -> $$dK = \sum_{n=1}^{N} m v_n dv_n$$

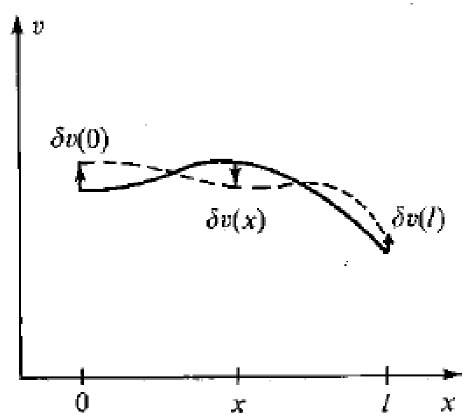
Increase particle number and their mutual distance, at constant density of particles per unit length $\mu = m/a$ and constant total length $l = Na$

$$T(x) = \frac{1}{2} \mu v(x)^2 \rightarrow K = \int_0^l T(x) dx$$

Discrete index ‘n’ replaced by continuous variable x

Differential of kinetic energy involves **functional derivative** :

$$\delta K = \int_0^l dx \frac{\delta K}{\delta v(x)} \delta v(x) = \int_0^l dx \frac{dK}{dv(x)} \delta v(x) = \int_0^l dx \mu v(x) \delta v(x)$$



[C. Cohen Tannoundji]

From discrete systems (analytic mechanics) to a field description (2)

Def.: $$\left.\frac{dF\left[u(x)+\varepsilon v(x)\right]}{d\varepsilon}\right|_{\varepsilon=0} = \int dx \frac{\delta F}{\delta u}\delta v(x) \to \frac{\delta F}{\delta u}$$ functional derivative w.r. function u

Ex.: $$F\left[g\right] := g(x) \equiv \int g(x')\delta(x'-x)dx' \Rightarrow \frac{\delta F}{\delta g\left(x'\right)} = \delta(x'-x)$$ Dirac

Euler equations: $$\delta S = \int_{t_1}^{t_2} dt \frac{\delta S}{\delta x_j(t)} \delta x_j(t) \equiv 0, \quad \forall \delta x_j(t) \Rightarrow \frac{\delta S}{\delta x_j(t)} = 0$$

Euler equation relative to x_j

Variation of S: $$\delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j} \right) \right\} \delta x_j dt + \left[\frac{\partial L}{\partial \dot{x}_j} \delta x_j \right]_{t_1}^{t_2} \equiv \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j} \right) \right\} \delta x_j dt$$

Identify both variations -> functional derivative of action:

$$\frac{\delta S}{\delta x_j(t)} = \left\{ \frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j} \right) \right\} = 0$$ Euler equ.

From discrete systems (analytic mechanics) to a field description (3)

Action for a continuous system functional of the dynamical DOF $A_j(\vec{x},t)$

$$S=\int_{t_1}^{t_2} dt L(t)=\int_{t_1}^{t_2} dt\int d^3X\ \tilde{L}\left(A_j,\dot{A}_j,\partial_i A_j\right)\rightarrow \delta S=\int_{t_1}^{t_2} dt\int d^3X\ \frac{\delta S}{\delta A_j}\delta A_j$$

Euler equ. : $$\delta S=\int_{t_1}^{t_2} dt\,\delta L(t)=\int_{t_1}^{t_2} dt\int d^3X\left\{\frac{\delta L}{\delta A_j}-\frac{d}{dt}\frac{\delta L}{\delta \dot{A}_j}\right\}\delta A_j \Rightarrow \frac{\delta S}{\delta A_j}=\frac{\delta L}{\delta A_j}-\frac{d}{dt}\frac{\delta L}{\delta \dot{A}_j}=0$$

Writes using functional derivatives: $$\frac{\delta L}{\delta A_j}\equiv\frac{\partial\tilde{L}}{\partial A_j}-\partial_i\left(\frac{\partial\tilde{L}}{\partial\left(\partial_i A_j\right)}\right),\quad \frac{\delta L}{\delta \dot{A}_j}\equiv\frac{\partial\tilde{L}}{\partial \dot{A}_j}$$

$A_j,\dot{A}_j$ are independent, but not A_j and $\partial_i A_j$

$$\longrightarrow\quad \boxed{\forall j,\quad \frac{\partial\tilde{L}}{\partial A_j}-\frac{\partial}{\partial t}\left(\frac{\partial\tilde{L}}{\partial\dot{A}_j}\right)-\mathrm{Div}\left(\frac{\partial\tilde{L}}{\partial\nabla_X A_j}\right)=0}$$

<u>Field equations formally identical to the discrete case using functional derivative</u>

Hamilton equations of motion take a similar form: define momentum & Hamitonian

$$\Pi_j:=\frac{\partial L}{\partial\dot{A}_j}\rightarrow H:=\int d^3X\left(\Pi_j\dot{A}_j-L\right)\quad\longrightarrow\quad\boxed{\dot{\Pi}_j:=\frac{\delta H}{\delta A_j},\quad \dot{A}_j:=\frac{\delta H}{\delta \Pi_j}}$$

Noether's theorem & conservation laws in field theory

Imposed continuous symmetries (spatial translation, temporal translation & rotation) reflected in the form of the Lagrangian

-> conserved quantities (linear & angular momentum, energy, …).

Similarly, require invariance of physics of the field, in terms of action integral,

w.r. same continuous transformations: leaves action integral invariant.

-> identify conserved quantities like **energy, linear & angular momentum of the field**.

Infinitesimal transformations (change of referential & variation of the field):

Restrict here presentation to <u>scalar fields</u>

Variation compares fields at <u>two different points</u> ('nonlocal' variation)

$$X_i \rightarrow X_i + \delta X_i$$
$$\psi(\mathbf{X}) \rightarrow \psi'(\mathbf{X}') = \psi(\mathbf{X}) + \delta\psi(\mathbf{X})$$

Noether's theorem & conservation laws in field theory (2)

Def.: *proper variation of the field* (local variation) difference of the field **at the same point**

$$\bar{\delta}\psi(\mathbf{X}) := \psi'(\mathbf{X}) - \psi(\mathbf{X})$$

Taylor series expansion: $$\psi'(\mathbf{X}') = \psi'(\mathbf{X}) + \frac{\partial \psi'(\mathbf{X})}{\partial X_i}\delta X_i$$

$$\longrightarrow \quad \psi'(\mathbf{X}') = \psi(\mathbf{X}) + \bar{\delta}\psi + \frac{\partial \psi(\mathbf{X})}{\partial X_i}\delta X_i + \frac{\partial\left(\bar{\delta}\psi\right)(\mathbf{X})}{\partial X_i}\delta X_i \equiv \psi(\mathbf{X}) + \delta\psi(\mathbf{X})$$

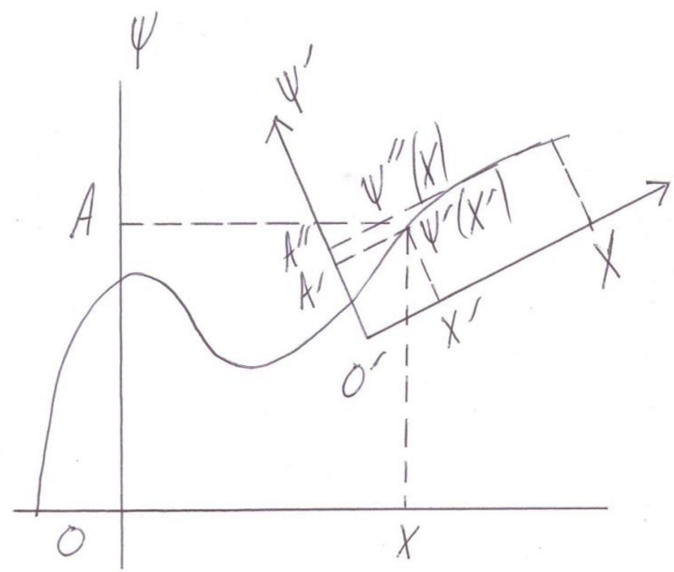
$$\longrightarrow \quad \bar{\delta}\psi = \delta\psi - \frac{\partial \psi(\mathbf{X})}{\partial X_i}\delta X_i$$

Invariance of action integral under change of referential leads to:

$$0 = \delta I = \int_V \int_{t_1}^{t_2} L\left(\psi + \bar{\delta}\psi, \partial_i\psi + \bar{\delta}\partial_i\psi\right) d^4X' - \int_V \int_{t_1}^{t_2} L\left(\psi, \partial_i\psi\right) d^4X$$

involves proper variations of the field &

its first order spatial derivatives



Noether's theorem & conservation laws in field theory (3)

Expand Jacobean & Lagrangian density: $J = 1 + \partial_i \delta X_i + o(\delta X_i)$ i=1...3, space - i=4: time

$$L\left(\psi + \bar{\delta}\psi, \partial_i\psi + \bar{\delta}\partial_i\psi\right) \cong L\left(\psi, \partial_i\psi\right) + \frac{\partial L}{\partial\psi}.\bar{\delta}\psi + \frac{\partial L}{\partial\left(\partial_i\psi\right)}.\bar{\delta}\partial_i\psi$$

Account for relations:

$$\bar{\delta}\partial_i\psi = \partial_i\bar{\delta}\psi, \quad \frac{\partial L}{\partial\left(\partial_i\psi\right)}.\partial_i\left(\bar{\delta}\psi\right) = \partial_i\left(\frac{\partial L}{\partial\left(\partial_i\psi\right)}\bar{\delta}\psi\right) - \partial_i\frac{\partial L}{\partial\left(\partial_i\psi\right)}\bar{\delta}\psi$$

Green formula

$$\longrightarrow \quad \delta I = \iint_V\int_{t_1}^{t_2}\sum_i \partial_i\left\{\sum_j\left(L\delta_{ij} - \frac{\partial L}{\partial\left(\partial_i\psi\right)}\partial_j\psi\right)\delta X_j + \frac{\partial L}{\partial\left(\partial_i\psi\right)}.\delta\psi\right\}d^4X \equiv 0$$

More compact writing using quadrivergence or d'Alembertien:

$$\partial_i f_i = 0 \leftrightarrow \quad \Box\mathbf{f} = 0,$$

$$f_i := \sum_j\left(L\delta_{ij} - \frac{\partial L}{\partial\left(\partial_i\psi\right)}\partial_j\psi\right)\delta X_j + \frac{\partial L}{\partial\left(\partial_i\psi\right)}.\delta\psi \equiv \sum_j T_{ij}\delta X_j + \frac{\partial L}{\partial\left(\partial_i\psi\right)}.\delta\psi$$

Conservation of force-like quadrivector highlights energy-momentum tensor

$$T_{ij} = L\delta_{ij} - \frac{\partial L}{\partial\left(\partial_i\psi\right)}\partial_j\psi$$

Conserved for a purely horizontal variation (field is fixed)

Similar to Eshelby tensor in context of configurational mechanics

Noether's theorem & conservation laws in field theory (4)

Integrate previous conservation law in infinite 3D-volume;
isolate spatial and time-like force components leads to:

$$\mathbf{f}=\left(\vec{f},f_4\right),\ \partial_i f_i=0\rightarrow \operatorname{div}\vec{f}+\frac{\partial f_4}{\partial t}=0\Rightarrow \int_{V_3}\operatorname{div}\vec{f}\,dV+\frac{d}{dt}\int_{V_3} f_4 dV=0$$

First integral vanishes (Green's formula over infinite volume) -> it remains

$$F:=\int_{V_3} f_4 dV\equiv\int_{V_3}\left\{\sum_j T_{4j}\delta X_j+\frac{\partial L}{\partial\left(\partial_4\psi\right)}.\delta\psi\right\}dV=Cte$$

Traduces Noether's theorem: any invariance of physics by a continuous transformation leads to the conservation of a physical quantity.

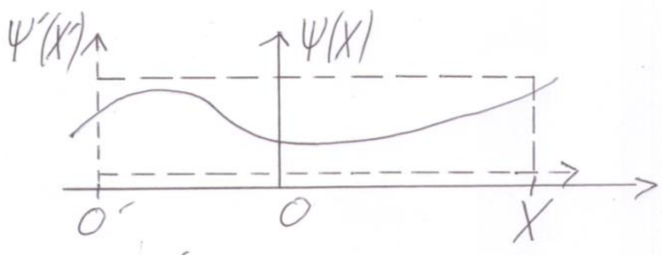
Remark: generalization of Noether's theorem in quantum domain by including non continuous transformations, discrete symmetries (e.g. inversions).

Postulate of 4-space homogeneity -> invariance under spatio-temporal translation:

$$\mathbf{X}\rightarrow\mathbf{X}'=\mathbf{X}+\mathbf{a}\Rightarrow\delta X_j=a_j \longrightarrow \forall\psi,\ \ \psi'\left(\mathbf{X}'\right)=\psi\left(\mathbf{X}\right)\Rightarrow\delta\psi=0$$

$\longrightarrow$ $P_j:=\int_{V_3} T_{4j}dV=Cte$ conservation of 4-momentum

$\longrightarrow$ $$\sum_i\partial_i f_i=0\rightarrow\sum_j\partial_j T_{4j}=0$$



Noether's theorem & conservation laws in field theory (5)

Def.: 3-momentum $$P_\mu := \int_{V_\infty = V_3} T_{4\mu} dV = -\int_{V_\infty = V_3} \frac{\partial L}{\partial(\partial_4 \psi)} \partial_\mu \psi dV \rightarrow \vec{P} = -\int_{V_\infty = V_3} \frac{\partial L}{\partial(\partial_4 \psi)} \vec{\nabla} \psi dV$$

involving field function $$\Pi(\mathbf{X}) := \frac{\partial L}{\partial(\partial_4 \psi)}$$ derivative w.r. time (similar to analytic mech.)

Role similar to momentum in analytical mechanics.

Def.: momentum of a field 3-vector built from the volumetric density

$$\vec{p}(\mathbf{X}) := -\Pi(\mathbf{X}) \vec{\nabla} \psi(\mathbf{X}) \rightarrow \vec{P} = -\int_{V_\infty = V_3} \Pi(\mathbf{X}) \vec{\nabla} \psi(\mathbf{X}) dV$$

Def.: energy of the field is remaining time-like component

$$P_4 = W := \int_{V_\infty = V_3} T_{44} dV = \int_{V_\infty = V_3} \left\{ -L + \Pi(\mathbf{X}) \partial_4 \psi \right\} dV$$

-> Energy density $$H := \Pi(\mathbf{X}) \partial_4 \psi - L$$

Role similar to Hamiltonian in classical mechanics.

Noether's theorem & conservation laws in field theory (6)

Kinetic moment of a field, vector **L** and *spin* of a field: last vector does not depend on the choice of an origin of space -> intrinsic property. To be defined later on.

Contrary to this: *total moment of the field*, vector **J**, conserved quantity depending upon selected origin of space, vanishing for a central field.

For a scalar field, spin is nil -> only one component of the field (# components = 2N+1).

Complex field -> Lagrangian invariant under a gauge transformation

$$\psi \to \psi' = e^{i\alpha}\psi$$

Approximate this finite transformation by $\psi' \cong (1+i\alpha)\psi \Rightarrow \delta\psi = i\alpha\psi, \quad \delta\psi^* = -i\alpha\psi^*$

Noether's th. leads to force $f_i = \frac{\partial L}{\partial(\partial_i\psi)}.\delta\psi + \frac{\partial L}{\partial(\partial_i\psi^*)}.\delta\psi^*$

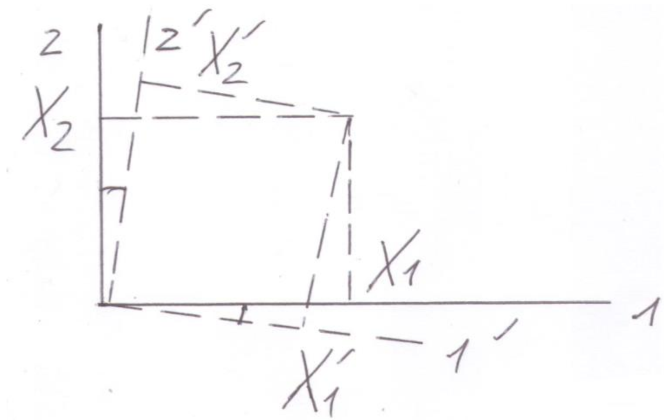
Associated current density: $j_i = \frac{\partial L}{\partial(\partial_i\psi)}.\psi - \psi^*.\frac{\partial L}{\partial(\partial_i\psi^*)}$

-> Conservation of field charge: $Q := \int_{V_3} j_0(\mathbf{X})dV = \int_{V_3} \left\{\Pi(\mathbf{X}).\psi(\mathbf{X}) - \psi^*(\mathbf{X}).\Pi^*(\mathbf{X})\right\} dV$

Noether's theorem & conservation laws in field theory (7)

Conservation of total moment of the field (sum of angular moment and spin)

-> transformation by infinitesimal rotation around axis x_3



$$\begin{cases} X'_1 = X_1\cos\varepsilon - X_2\sin\varepsilon \cong X_1 - \varepsilon X_2 \Rightarrow \delta X_1 = -\varepsilon X_2 \\ X'_2 = X_1\sin\varepsilon + X_2\cos\varepsilon \cong X_2 + \varepsilon X_1 \Rightarrow \delta X_2 = \varepsilon X_1 \\ X'_3 = X_3 \Rightarrow \delta X_3 = 0 \end{cases}$$

Thus $\delta\psi = I_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ infinitesimal rotation generator

* Isotropic space, get conserved quantity:

$$\int_{V_3} \left\{ \varepsilon\Pi(\mathbf{X}) I_3 \psi(\mathbf{X}) + T_{41}\delta X_1 + T_{41}\delta X_1 + T_{42}\delta X_2 \right\} dV = Cte$$

$$\longrightarrow \int_{V_3} \Pi(\mathbf{X}) I_3 \psi(\mathbf{X}) dV - \int_{V_3} \left\{ \Pi(\mathbf{X}).\left(X_1\partial_2\psi - X_2\partial_1\psi\right) \right\} dV = Cte$$

$\left(X_1\partial_2\psi - X_2\partial_1\psi\right)$ third component of vector product $\vec{r} \times \overrightarrow{\nabla\psi}$

$\longrightarrow$ Conserved quantity $J_3 := \int_{V_3} l_3 dV + \int_{V_3} \Pi\left(\mathbf{X}\right)\vec{\mathbf{I}}.\psi\left(\mathbf{X}\right)dV$ $\quad \vec{\mathbf{I}} := \left(\vec{I}_1 \quad \vec{I}_2 \quad \vec{I}_3\right)$

Def.: $L := \int_{V_3} \vec{r} \times \vec{p} dV$ angular moment of the field, with $\vec{p} := -\Pi.\overrightarrow{\nabla\psi}$

General case: invariance by rotation leads to conservation of total moment, sum of angular moment and spin $\vec{S} := \int_{V_3} \Pi(\mathbf{X})\vec{\mathbf{I}}\psi(\mathbf{X})dV$

Hamiltonian structure in dynamical elasticity

Lagrangian in hyperelasticity: $\mathrm{L} := \mathrm{K}[\dot{\mathbf{u}}] - \mathrm{E}[\mathbf{u}]$

$\mathbf{u}$ displacement, $\dot{\mathbf{u}} := \dfrac{\partial \mathbf{u}}{\partial t}$ velocity

$\mathrm{K}[\dot{\mathbf{u}}] := \int_V \frac{1}{2}\rho \dot{\mathbf{u}}^2 dV$ kinetic energy, $\mathrm{E}[\mathbf{u}] := \int_V W(\nabla \mathbf{u}) dV$ internal energy

Stationnarity condition: $\delta L := \int_V \rho \dot{\mathbf{u}}.\delta\dot{\mathbf{u}} dV + \int_V \mathrm{Div}\left(\dfrac{\partial W(\nabla \mathbf{u})}{\partial \nabla \mathbf{u}} = \mathbf{T}\right).\delta \mathbf{u} dV \equiv 0$

of the form $\partial_{\dot{u}} L.\delta\dot{\mathbf{u}} + \partial_u L.\delta \mathbf{u} = 0$

$\partial_{\dot{u}} L, \partial_u L$ co-vectors: $\partial_{\dot{u}} L : \delta\dot{\mathbf{u}} \mapsto \int_V \partial_{\dot{u}} L.\delta\dot{\mathbf{u}} dV$, $\partial_u L$: $\delta \mathbf{u} \to \int_V \partial_u L.\delta \mathbf{u} dV \equiv \int_V \mathrm{Div}\mathbf{T}.\delta \mathbf{u} dV$

On current trajectories, $\delta\dot{\mathbf{u}} = \dfrac{d\delta \mathbf{u}}{dt} \to \int_{t_1}^{t_2} dt \int_V \left(\dfrac{d\mathbf{u}}{dt} - \dot{\mathbf{u}}\right).\delta\dot{\mathbf{u}} dV = 0, \quad \forall t_1, t_2, \delta\dot{\mathbf{u}}$

$\Rightarrow \int_{t_1}^{t_2} dt \int_V \left(-\dfrac{d\dot{\mathbf{u}}}{dt}.\delta \mathbf{u} - \dot{\mathbf{u}}.\delta\dot{\mathbf{u}}\right).dV = 0$

Since $\rho\dot{\mathbf{u}} \equiv \partial_{\dot{u}} L \to -\dfrac{d(\partial_{\dot{u}} L)}{dt}.\delta \mathbf{u} - \partial_{\dot{u}} L.\delta\dot{\mathbf{u}} = 0$ (a)

Use next $\partial_{\dot{u}} L : \delta\dot{\mathbf{u}} = \partial_u L : \delta \mathbf{u} \to$ (a): $-\dfrac{d}{dt}(\partial_{\dot{u}} L) - \partial_u L = 0$

Variational form of equ. of motion: $\int_V (-\rho\ddot{\mathbf{u}} + \mathrm{Div}\mathbf{T}).\delta \mathbf{u} dV = 0, \quad \forall \delta \mathbf{u}$ (b)

Hamiltonian structure in dynamical elasticity (2)

Define Hamiltonian $H[\mathbf{u},\mathbf{p}:=\rho\dot{\mathbf{u}}]\equiv L[\mathbf{u},\dot{\mathbf{u}}]+\int_V \rho\dot{\mathbf{u}}.\dot{\mathbf{u}}dV$

$\rightarrow$ Rewrite (b) by setting $\mathbf{z}=\begin{pmatrix}\mathbf{u}\\ \mathbf{p}:=\rho\dot{\mathbf{u}}\end{pmatrix}$; $\mathrm{Grad}:=\begin{pmatrix}\partial/\partial\mathbf{u}\\ \partial/\partial\dot{\mathbf{u}}\end{pmatrix}$

$\rightarrow$ $(b)\int_V(-\rho\ddot{\mathbf{u}}+\mathrm{Div}\mathbf{T}).\delta\mathbf{u}dV=0,\ \ \forall\delta\mathbf{u}$ rewrites $\frac{d\mathbf{z}}{dt}=\boldsymbol{\omega}.\mathrm{GradH}(\mathbf{z}),\quad \boldsymbol{\omega}:=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$

Means that $\frac{d\mathbf{z}}{dt}$ is tangent to the iso-Hamitonian surfaces $H=Cte$

$\rightarrow$ Get Hamilton dynamical equations: $\frac{d\mathbf{u}}{dt}=\frac{\partial H}{\partial\mathbf{p}},\ \frac{d\mathbf{p}}{dt}=-\frac{\partial H}{\partial\mathbf{q}}$

Corollary: time evolution of any function $f=f(\mathbf{z})$:

$$\frac{df}{dt}=\mathrm{Gradf}_{|z}.\frac{d\mathbf{z}}{dt}=\mathrm{Gradf}.\boldsymbol{\omega}.\mathrm{GradH}(\mathbf{z})=\{f,H\}$$

Poissons bracket

Comparison with Hamiltonian formulation in classical mechanics

Hamiltonian formulation in new set of coordinates $\{q_\alpha, p_\alpha\}$

$$p_\alpha := \frac{\partial L}{\partial \dot{q}_\alpha} \rightarrow H[q_\alpha, p_\alpha, t] := p_\alpha \dot{q}_\alpha - L[q_\alpha, \dot{q}_\alpha, t]$$

-> Jacobi action: $S[q_\alpha] := \int_{t_1}^{t_2} \{p_\alpha \dot{q}_\alpha - L\} dt$

-> Euler-Lagrange equations: $\forall \alpha \in \{1, 2, ..., N\}, \quad \frac{dp_\alpha}{dt} = \frac{\partial L}{\partial q_\alpha}$

Dynamical equations in terms of the Hamiltonian: $\forall \alpha \in \{1, 2, ..., N\}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}; \quad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$

Interest of Hamilton formalism: get first integrals of motion using Poisson's bracket:

$$[f, g] := \frac{\partial f}{\partial p_\alpha} . \frac{\partial g}{\partial q_\alpha} - \frac{\partial f}{\partial q_\alpha} . \frac{\partial g}{\partial p_\alpha}$$

-> time derivative of a function $\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f]$

If f does not explicitly depend on time, get first integral of motion: $\boxed{[H, f] = 0}$

Specific cases: $[p_\alpha, q_\beta] = \delta_{\alpha\beta}; \quad [q_\alpha, q_\beta] = 0 = [p_\alpha, p_\beta]$

Historical vignette in field theory

Expression of the Lagrangian incorporating symmetries reflects laws of physics.

Two categories of symmetries:

* **External symmetries** acting on space-time coordinates of the scene of events;
* **Internal symmetries** (= gauge symmetries) acting on internal parameters (potentials, charges, wave function).

Both external and internal symmetries leave invariant laws of physics.

Concept common to special & general relativity: absence of absolute referential.

In special relativity: class of equivalent referentials defined by Poincaré group of transformations -> global symmetries.

In RG: postulated equivalence between gravitation field and inertial frame valid only locally (the orientation of the gravitation field varies from point to point) -> RG is a local theory.

-> Key idea of Weyl's gauge theory (1919): first historical attempt to extend idea of gravitation field described by connection giving relative orientation of frames in space-time.

Invariance of equ. (or action integral if any) by an internal symmetry = *gauge invariance.*

Extension: Noether's theorem in classical and quantum physics

Non observable	Symmetry	Conservation law
Absolute spatial position	Space translation	Linear momentum
Absolute time	Time translation	Energy
Absolute spatial direction	Rotation	Angular momentum
Absolute velocity	Lorentz Transformation	Generators of Lorentz group
Difference between identical particles	Permutation of identical particles	Fermi-Dirac or Bose-Einstein statistics
Absolute right or left	Inversion $\mathbf{X} \to -\mathbf{X}$	Parity
Absolute sign of the charge	Particles transformed into their antiparticles	Charge conjugation
Absolute phase of a charge matter field	Change of phase	Electrical charge, generators in $\mathrm{U}(1)$
Difference between coherent mixtures of colored quarks	Change of color	Color generator, belong to group $\mathrm{SU}(3)$
Difference between coherent mixtures of charged leptons and neutrinos	Transformation of a lepton in its neutrino	Weak isospin generators, belong to group $\mathrm{SU}(2)$

Noether's theorem in quantum physics: case of QED

Adopt system of units in which c=1.

Dirac equation satisfied by a fermion (spin is ½): linearized relativistic energy

$$E = \mathbf{p}.\mathbf{v} - L \qquad L = -m\left(1 - v^2\right) \longrightarrow E = p.v + (1 - v^2)m$$

Admits existence of a Hamiltonian of the same form: $H = \boldsymbol{\alpha}.\mathbf{p} + \beta m$

$(\boldsymbol{\alpha}, \beta)$ matrices

Eigenvalue problem for linearized Hamiltonian: $H\psi = E\psi \rightarrow (\boldsymbol{\alpha}.\mathbf{p} + \beta m)\psi = E\psi$

Correspondence principle: $E \rightarrow i\hbar \partial/\partial t, \quad p_i \rightarrow -i\hbar \partial/\partial x^i$

$i\partial_t \psi = (\boldsymbol{\alpha}.\mathbf{p} + \beta m)\psi$ Dirac equ., becomes: $i\left(\partial_0 + \alpha_k \partial_k\right)\psi - m\beta\psi = 0$

$i\partial_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} = 0$ Adjoint Dirac equ.

$$\overline{\psi} = \psi^+\gamma^0 = \begin{pmatrix}\psi_1^* & \psi_2^* & \psi_3^* & \psi_4^*\end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix}$$

$$\left\{\gamma^\mu, \gamma^\nu\right\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I$$

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

Noether's theorem in quantum physics: case of QED (2)

Lagrangian constructed based on both the Dirac equation and its adjoint

$$L = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\bar{\psi} - m\bar{\psi}\psi = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi = -\bar{\psi}\left(i\overrightarrow{\not{\partial}} - m\right)\psi = -\bar{\psi}\left(i\overleftarrow{\not{\partial}} + m\right)\psi$$

$$\not{\partial} := \gamma^{\mu}\partial_{\mu}$$

Get (easily) conservation law of electric current: $\partial_{\mu}\left(ej^{\mu}\left(\mathbf{x}\right)\right) = 0$

$j_k(\mathbf{x}) = \bar{\psi}(\mathbf{x})\gamma^k\psi(\mathbf{x})$ density of charges

Can be deduced from a <u>global gauge invariance of the Lagrangian</u>:

$$\psi_i(\mathbf{x}) \to \psi'_i(\mathbf{x}) = \exp\left(-i\Lambda T_{ij}\right)\psi_j(\mathbf{x})$$

Make 1st order Taylor expansion: $\psi'_i(\mathbf{x}) - \psi_i(\mathbf{x}) = \delta\psi_i(\mathbf{x}) = -i\Lambda T_{ij}\psi_j(\mathbf{x})$

-> variation of Lagrangian:

$$\delta L = \frac{\partial L}{\partial \psi_i}\delta\psi_i + \frac{\partial L}{\partial\left(\partial_{\mu}\psi_i\right)}\delta\left(\partial_{\mu}\psi_i\right) = \partial_{\mu}\left(\frac{\partial L}{\partial\left(\partial_{\mu}\psi_i\right)}\delta\psi_i\right) - \left(\partial_{\mu}\left(\frac{\partial L}{\partial\left(\partial_{\mu}\psi_i\right)}\right) - \frac{\partial L}{\partial\left(\partial_{\mu}\psi_i\right)}\right)\delta\psi_i$$

-> Conservation law of electric 4-current: $\delta L = 0 \Rightarrow \partial_{\mu}\left(-i\frac{\partial L}{\partial\left(\partial_{\mu}\psi_i\right)}T_{ij}\psi_j\right) = 0 \Rightarrow \partial_{\mu}j^{\mu} = 0$

Noether's theorem in quantum physics: case of QED (3)

Stronger condition of <u>local gauge invariance</u>: let group parameter depend on coordinates

$$\psi_i(\mathbf{x}) \to \psi'_i(\mathbf{x}) = \exp\left(-iq\Lambda(\mathbf{x})T_{ij}\right)\psi_j(\mathbf{x})$$

Modifies the Lagrangian to $L' = L + q\bar{\psi}\gamma^\mu\psi\partial_\mu\Lambda = L + qj^\mu\partial_\mu\Lambda$

Introduce covariant derivative: $D_\mu = \partial_\mu + iqA_\mu$

Field A_μ called *compensating field* or a *gauge field*

<u>Gauge field responsible for interactions between fermions and electromagnetic field</u>

Lagrangian invariant under previous local Lie group transformation when replacing partial derivatives by covariant derivatives:

$$L_F = L - qj^\mu A_\mu \equiv \bar{\psi}\left(i\not{D} - m\right)\psi \to L_F' = L_F$$

$$\not{D} := \gamma^\mu D_\mu$$

Lagrangian of QEM writes $L_{QED} = L_F + L_e = L_F - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

No kinetic term, thus vehicule of electromagnetic interactions (Photon) is masless!

Symmetry methods

Interest

- Many solution techniques for exact solution of ODE's & PDE's directly connected to symmetry properties: superposition principles, integral transforms, separated solutions, reduction of order, Green's function, travelling wave solutions.

- Invariance properties of governing equations important: conservation laws.

- Lie point symmetry framework provide systematic ways to study invariance properties of DEs w.r. continuous & discrete symmetry groups.

- Ex.: travelling wave solution validated by invariance under space-time translations.

$$x^i(X^1, X^2, t) = w^i(z, X^2), \quad z = X^1 - st, \quad i = 1, 2$$

- For variational PDE systems: equivalence of local conservation laws & variational symmetries via Noether's theorem.

Symmetries of differential equations (ODE, PDE)

Consider a general DE system

$$R^\sigma[\mathbf{u}] = \mathbf{R}^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N$$

with variables $\mathbf{x} = (x^1, ..., x^n)$, $\mathbf{u} = (u^1, ..., u^m)$.

Definition

A one-parameter Lie group of point transformations

$$x^* = f(x, u; a) = x + a\xi(x, u) + O(a^2),$$
$$u^* = g(x, u; a) = u + a\eta(x, u) + O(a^2)$$

(with the parameter a) is a *point symmetry* of $R^\sigma[u]$ if **the equation is the same** in new variables x^*, u^*.

Example 2: scaling

The scaling:

$$x^* = \alpha x, \quad t^* = \alpha^3 t, \quad u^* = \alpha u \quad (\alpha \in \mathbb{R})$$

also leaves the KdV equation invariant:

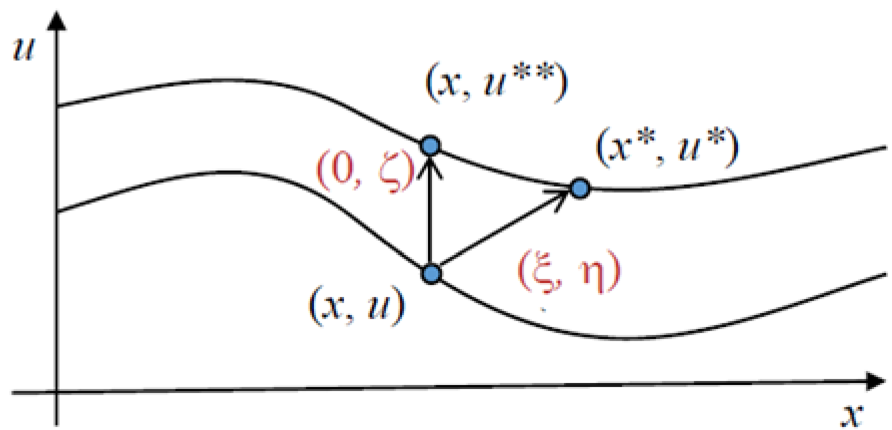
$$u_t + uu_x + u_{xxx} = 0 = u^*_{t^*} + u^* u^*_{x^*} + u^*_{x^* x^* x^*}.$$

Symmetries of differential equations (ODEs, PDEs)

A symmetry (in 1D case)

$$x^* = f(x, u; a) = x + a\xi(x, u) + O(a^2),$$
$$u^* = g(x, u; a) = u + a\eta(x, u) + O(a^2).$$

maps a solution $u(x)$ into $u^*(x^*)$, changing both x and u.



In the evolutionary form, the same curve mapping does not change x:

$$x^{**} = x, \quad u^{**} = u + a\,\zeta[u] + O(a^2),$$

$$\boxed{\zeta[u] = \eta(x, u) - \frac{\partial u}{\partial x}\xi(x, u).}$$

Application of symmetry methods to differential equations

Nonlinear DEs

- Numerical solutions: resource/time consuming; lack generality.
- Solution methods for linear DEs do not work.
- Symmetry analysis: a general systematic framework leading to useful results.

Symmetries for ODEs

- Reduction of order / complete integration.
- All known methods of solution of specific classes of ODEs follow from symmetries!

Symmetries for PDEs

- Exact symmetry-invariant (e.g., self-similar) solutions.
- Transformations: solutions $\Rightarrow$ new solutions.
- Mappings relating classes of equations; linearizations.
- Symmetry-preserving numerical methods.

Computation of Symmetries

- Lie point symmetries and other types are computed systematically for any DE.
- Literature widely available.
- Symbolic software packages available.
- A popular approach to analyze complicated DEs arising in applied science:
  - fluid and solid mechanics,
  - rocket science,
  - meteorology,
  - biological applications, ...

Conservation laws: general aspects

Definitions

Variables:

- Independent: $\mathbf{x} = (x^1, x^2, ..., x^n)$ or $(t, x^1, x^2, ...)$.
- Dependent: $\mathbf{u} = (u^1(\mathbf{x}), u^2(\mathbf{x}), ..., u^m(\mathbf{x}))$ or $(u(\mathbf{x}), v(\mathbf{x}), ...)$.

Partial derivatives:

- Notation: $\dfrac{\partial u^k}{\partial x^m} = u^k_{x^m} = u^k_m$.
- All first-order partial derivatives: $\partial \mathbf{u}$.
- All p^{th}-order partial derivatives: $\partial^p \mathbf{u}$.

Differential functions:

- A differential equation is an *algebraic equation* on components of $\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots$.
- A differential function is an expression that may involve independent and dependent variables, and derivatives of dependent variables to some order.

$$F[u] = F(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^p\mathbf{u}).$$

Conservation laws: general aspects (2)

The total derivative of a differential function:

- A basic chain rule.
- Example: $u = u(x, y)$, $g[u] = g(x, y, u, u_x)$, then

$$\begin{aligned} \mathrm{D}_x g[u] &\equiv \frac{\partial}{\partial x} g(x, y, u, u_x) \\ &= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} u_x + \frac{\partial g}{\partial u_x} u_{xx}. \end{aligned}$$

Conservation laws

- A local conservation law: a divergence expression equal to zero,

$$\mathrm{D}_i \Psi^i[\mathbf{u}] \equiv \operatorname{div} \mathbf{\Psi^i}[\mathbf{u}] = 0.$$

- For equations involving time evolution:

$$\mathrm{D}_t\, \Theta[\mathbf{u}] + \operatorname{div}_{\mathbf{x}}\ \mathbf{\Psi}[\mathbf{u}] = 0.$$

- $\Theta[\mathbf{u}]$: conserved density.
- $\mathbf{\Psi}[\mathbf{u}]$: flux vector.

Global conserved quantity (integral of motion)

$$\mathrm{D}_t \int_V \Theta\, dV = 0, \quad \text{if} \quad \oint_{\partial V} \mathbf{\Psi}[\mathbf{u}] \cdot d\mathbf{S} = 0.$$

Global conservation laws

- Given: a local CL for a time-dependent system,

$$\mathrm{D}_t\, \Theta[\mathbf{u}] + \mathrm{div}_{\mathbf{x}}\ \mathbf{\Psi}[\mathbf{u}] = 0.$$

- Integrate in the spatial domain:

$$\int_V \mathrm{D}_t\,\Theta\ dV + \int_V (\mathrm{div}_{\mathbf{x}}\ \mathbf{\Psi})\ dV \ = \ \int_V \mathrm{D}_t\,\Theta\ dV + \oint_{\partial V} \mathbf{\Psi}\cdot d\mathbf{S} \ = \ 0.$$

- When the total flux vanishes,

$$\oint_{\partial V} \mathbf{\Psi}[\mathbf{u}]\cdot d\mathbf{S} = 0,$$

one has

$$\frac{d}{dt}\int_V \Theta[\mathbf{u}]\ dV = 0,$$

i.e., a global conserved quantity (an integral of motion):

$$\boxed{Q = \int_V \Theta\ dV = \mathrm{const.}}$$

Global conservation laws: example with PDE

Example:

- Small oscillations of a string (transverse) or a rod (longitudinal) ⇔ 1D wave equation:

$$u_{tt} = c^2 u_{xx}. \qquad c^2 = T/\rho$$

- Independent variables: x, t; dependent: $u(x, t)$.

Conservation of momentum:

- Local conservation law: $\mathrm{D}_t(\rho u_t) - \mathrm{D}_x(T u_x) = 0$;
- Global conserved quantity: total momentum

$$M = \int_0^L \rho u_t \, dx = \mathrm{const}, \qquad \frac{d}{dt} M = 0$$

for Neumann homogeneous problems with $u_x(0, t) = u_x(L, t) = 0$.

Conservation of energy:

- Local conservation law:

$$\mathrm{D}_t \left(\frac{\rho u_t^2}{2} + \frac{T u_x^2}{2} \right) - \mathrm{D}_x(T u_t u_x) = 0;$$

- Global conserved quantity: total energy

$$E = \int_0^L \left(\frac{\rho u_t^2}{2} + \frac{T u_x^2}{2} \right) dx = \mathrm{const},$$

for both Neumann and Dirichlet homogeneous problems.

Extremum principles and conservation laws in hyperelasticity

Hyperelastic means local energy exists -> Strain energy function dictates constitutive law

$$W = W(\mathbf{X},\mathbf{F}) \Rightarrow T := \frac{\partial W(\mathbf{X},\mathbf{F})}{\partial \mathbf{F}}$$ nominal stress

$\mathbf{X}$ lagrangian coordinate, $\mathbf{F}$ transformation gradient: $\mathbf{F} := \nabla_X \mathbf{x}(\mathbf{X}, t)$

$\rightarrow$ action integral $S[u] = \int_\Omega L\left(\mathbf{X}, \mathbf{u}^{(n)}\right) d\Omega$

$\mathbf{X} = \{X_i, i = 1 \ldots 4\} = \{X_1 = t, X_2 = X, X_3 = Y, X_4 = Z\}$

$\Omega = V x I$ (I time interval) $\rightarrow d\Omega = dVdt = dX_1 dX_2 dX_3 dX_4$

Structure of Lagrangian:

$$L = K - \hat{W}(\mathbf{F}, \mathbf{X})$$

$$K := \frac{1}{2}\rho_0(\mathbf{X})\dot{\mathbf{x}}^2, \quad \hat{W}(\mathbf{F},\mathbf{X}) = W(\mathbf{F},\mathbf{X}) + \Phi(\mathbf{X})$$

K kinetic energy

$W(\mathbf{F},\mathbf{X})$ strain energy density

$\Phi(\mathbf{X})$ load potential / $f_0(\mathbf{X}) = -\nabla_X \Phi(\mathbf{X})$ body force vector

Extremum principles and conservation laws in hyperelasticity (2)

Evaluate variation of the action under a Lie group of transformations:

$$G: \begin{cases} \bar{x}_j = \bar{x}_j(\mathbf{x}, \mathbf{u}, \mu), \quad j = 1...4 & \text{independent variables (ex. : parameterization of material points)} \\ \bar{u}_j = \bar{u}_j(x, \mathbf{u}, \mu), \quad k = 1...q & \text{dependent variables (= fields)} \end{cases}$$

$$\bar{x}_j = x_j + \mu \frac{\partial \bar{x}_j}{\partial \mu}_{|\mu=0} + o(\mu) \equiv x_j + \mu \xi_j + o(\mu), \; \bar{u}_k = u_k + \mu \frac{\partial \bar{u}_k}{\partial \mu}_{|\mu=0} + o(\mu) \equiv u_k + \mu \eta_k + o(\mu)$$

(ξ_j, η_k) horizontal & vertical components of infinitesimal generator of the group

$$\longrightarrow \boxed{\delta S = \mu \int_\Omega \left(\frac{\partial L}{\partial u_k} - D_i \frac{\partial L}{\partial u_{k,i}} \right) (\phi_k - \xi_j u_{k,j}) d\Omega + \mu \int_{\partial\Omega} \left(L\xi_i + (\phi_k - \xi_j u_{k,j}) \frac{\partial L}{\partial u_{k,i}} \right) n_i d(\partial\Omega)}$$

$\left(\frac{\partial L}{\partial u_k} - D_i \frac{\partial L}{\partial u_{k,i}} \right)$ Euler operator, $(\phi_k - \xi_j u_{k,j})$ characteristic

$(\xi, \phi) = \mu(\delta \mathbf{x}, \delta \mathbf{u}) \rightarrow \delta S = \int_\Omega E_k(L) \delta u_k d\Omega + \mu \int_{\partial\Omega} \frac{\partial L}{\partial u_{k,i}} n_i \delta u_k d(\partial\Omega)$ for a purely vertical variation

Corollary: stationnarity condition of the action implies

$$\begin{cases} E_k(L) = 0 & \text{Euler equation of S} \\ \frac{\partial L}{\partial u_{k,i}} n_i = 0 & \text{for non fixed boundary conditions } (\delta u_{|\partial\Omega} \neq 0) \end{cases}$$

Extremum principles and conservation laws in hyperelasticity (3)

Compact writing:

$$L_X\omega = i_X d\omega + d\left(i_X\omega\right)$$ 'magic' Cartan formula

$\rightarrow$ Noether's theorem:

under condition $L_X\omega = 0$ (variational symmetry = invariance of S under G)

and $i_X d\omega = 0$ (Euler equ. are satisfied):

Conservation law: $d\left(i_X\omega\right) = 0 \Leftrightarrow \mathrm{Div}\left(L\xi_i + \left(\phi_k - \xi_j u_{k,j}\right)\dfrac{\partial L}{\partial u_{k,i}}\right) = 0$

<u>Case of a purely horizontal variation</u> (fields are fixed):

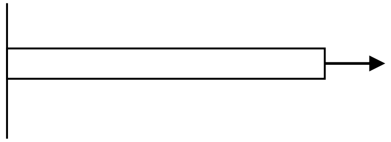
$$\delta S_{surf} = \int_{\partial\Omega} \mathbf{\Sigma}.\mathbf{n}.\delta\mathbf{X} d\left(\partial\Omega\right)$$

$\Sigma_{ij} := L\delta_{ij} - u_{k,j}\dfrac{\partial L}{\partial u_{k,i}} \leftrightarrow \mathbf{\Sigma} = L\mathbf{I} - \mathbf{F}^T.\dfrac{\partial L}{\partial \mathbf{F}}$ Eshelby stress / energy-momentum tensor

Important role in field theories

$\mathbf{\Sigma}.\mathbf{n}$ driving force for domain variation

Interpretation of Eshelby stress: 1D bar example



Bar length L submited to tensile force F at x=L

Linear elastic material modulus E_0

$\longrightarrow$ $u(x) = Fx/E_0$ displacement solution

Potential energy: $V[u] = \int_0^L W(x, u(x))dx - F.u(L)$

Strain energy density $W(x, u(x)) = \frac{1}{2}E_0 u'(x)^2 \equiv \frac{1}{2}E_0\left(\frac{F}{E_0}\right)^2 = \frac{1}{2}\frac{F^2}{E_0}$

Viewed as a function of L $V(L) = \frac{1}{2}\frac{F^2}{E_0}L - F.\left(\frac{FL}{E_0}\right) \equiv -\frac{1}{2}\frac{F^2 L}{E_0}$

Eshelby stress $\Sigma = W - \sigma.\varepsilon \equiv -\frac{1}{2}E_0 u'(x)^2 \equiv \frac{\delta V}{\delta L}$ expressing domain variation

$\longrightarrow$ Viewpoint of structural optimization

Further symmetries & associated conservation laws

1. Translation invariance of the Lagrangian: $\mathbf{u} \mapsto \mathbf{u} + \mathbf{c} \Rightarrow D_i\left(\frac{\partial L}{\partial u_{k,i}}\right) = 0,$ Euler equ.

More generally, L invariant under Euclidean group (material frame-indifference : observer in rigid body motion) $\mathbf{u} \mapsto R.\mathbf{u} + \mathbf{c}, \;\; R \in SO(3), \;\; \mathbf{c} \in R^3$

2. Rotation invariance of L -> conservation of angular momentum (in actual configuration):

$$L(\mathbf{X}, R.\nabla\mathbf{u}) = L(\mathbf{X}, \nabla\mathbf{u}), \;\; \forall R \to D_i\left(u_p \frac{\partial L}{\partial u_{q,i}} - u_q \frac{\partial L}{\partial u_{p,i}}\right) = 0, \;\; p, q = 1 \ldots 3$$

3. Isotropic materials: rotation w.r. reference configuration!

$$L(\mathbf{X}, \nabla\mathbf{u}.Q) = L(\mathbf{X}, \nabla\mathbf{u}), \;\; \forall Q \to D_i\left(\sum_{\alpha=1}^{q}\left(X^j u_k^\alpha - X^k u_j^\alpha\right)\right) = 0, \;\; u_k^\alpha := \frac{\partial u^\alpha}{\partial X^k}$$

associated to infinitesimal generators $X^k \frac{\partial}{\partial X^j} - X^j \frac{\partial}{\partial X^k}$

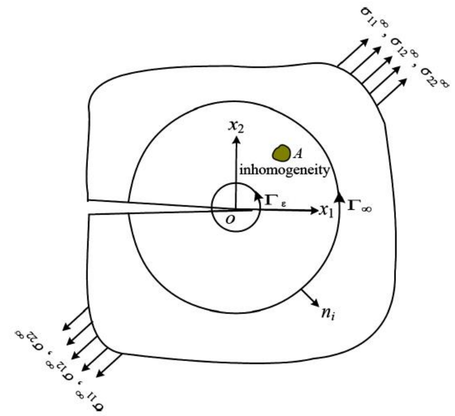
J_k,M,L integrals associated to material configurational forces

1. J_k integral: gradient of the Lagrangian function

$L = -W$ in statics, no body forces

$$\nabla L = -D_i W = \frac{dW}{dX^i} = -\left(\frac{\partial W}{\partial X^i}\right)_{expl} - T_{jk} u_{k,ji},$$

$$\mathbf{T} := \frac{\partial W(\nabla \mathbf{u})}{\partial \nabla \mathbf{u}}$$



[Li et al., Engng Fracture Mech., (2017), 171, 76-84]

Let $\mathbf{\Sigma} := W\mathbf{I} - \mathbf{F}^T.\mathbf{T}$, satisfy $\mathrm{Div}\mathbf{\Sigma} + \mathbf{R} = \mathbf{0}$, $\mathbf{R} := -\left(\frac{\partial W}{\partial X^i}\right)_{expl}$ material configurational force

Integrate configurational stress around closed contour enclosing the crack tip, thus

$J = J_1 = \oint_\Gamma \mathbf{e}_1.\mathbf{\Sigma}.\mathbf{n} dS$ includes material inhomogenity s.t. $W = W(\mathbf{F}, x_2, x_3)$

$$J_2 = \oint_\Gamma \mathbf{e}_2.\mathbf{\Sigma}.\mathbf{n} dS$$

$L_k = \int_\Gamma \varepsilon_{klm}(x_l b_{mj} + u_l \sigma_{mj}) n_j dS$, $M = \int_\Gamma \left[b_{ij} x_i + \frac{1}{2}\sigma_{ij}(2-\alpha)u_i \right] n_j dS$, $\alpha = 2$ (2D) or $\alpha = 3$ (3D)

derived from Noether's theorem from variational principle of elastostatics using rotations and similarity invariance; extend to large strains

J_k,M,L integrals associated to material configurational forces

2) M-integral: build Lagrangian momentum $L\mathbf{X}$

$$\rightarrow \mathrm{Div}\left(L\mathbf{X}\right) = -\left(WX_i\right)_{,i} = -mW - \left(\frac{\partial W}{\partial \mathbf{X}}\right)_{expl}.\mathbf{X} - \nabla\mathbf{F}.\mathbf{X}.\mathbf{T},\ m = \mathrm{Div}\mathbf{X} = 2$$ or 3 in dimension 2, 3 resp.

Configurational stress $\mathbf{M} := W\mathbf{X}.\mathbf{I} - \mathbf{T}.\nabla\mathbf{u}.\mathbf{X}$

$R := \left(\frac{\partial W}{\partial \mathbf{X}}\right)_{expl}.\mathbf{X}$ configurational force

$$\rightarrow \mathrm{Div}\mathbf{M} + \mathbf{R} = 0 \rightarrow M := \oint_\Gamma \mathbf{M}.\mathbf{n}dS$$

3) L-integral: identity $-\nabla_X\left(W\mathbf{X}\right)_m = -e_{mij}\left(WX_j\right)_{,i} = -e_{mij}\left\{\left(\frac{\partial W}{\partial X^i}\right)_{expl} X^j + T_{kl}u_{k,li}X_j\right\}$

$\mathbf{e}$ Levi-Civita permutation tensor

Configurational stress (second order tensor): $\mathbf{L} := \mathbf{e}:\left(W\mathbf{X}\otimes\mathbf{I} + \mathbf{T}\otimes\mathbf{u} - \mathbf{F}^T.\mathbf{T}\otimes\mathbf{X}\right)$

Configurational force (vector): $\mathbf{R} := -\mathbf{e}:\left(\frac{\partial W}{\partial \mathbf{X}}\right)_{expl}\otimes\mathbf{X}$

$$\rightarrow \mathrm{Div}\mathbf{L} + \mathbf{R} = \mathbf{0}$$

L_m-integral $L_3 := \oint_\Gamma L_{3l}n_l dS$

Applications of conservation laws

ODEs

- Constants of motion.
- Integration.

Applications to PDEs

- Rates of change of physical variables; constants of motion.
- Differential constraints (divergence-free or irrotational fields, etc.).
- Analysis: existence, uniqueness, stability.
- An infinite number of conservation laws may indicate integrability / linearization.
- Finite element/finite volume numerical methods may require conserved forms.
- Weak form of DEs for finite element numerical methods.
- Special numerical methods, conservation law-preserving methods (symplectic integrators, etc.).
- Numerical method testing.

Trivial conservation laws

Definition

A *trivial* local conservation law: a zero divergence expression that "does not carry a physical meaning".

A trivial CL, Type 1:

- Density and all fluxes vanish on all solutions of the given PDE system.
- Example: consider a wave equation on $u(x,t)$: $u_{tt} = u_{xx}$. The conservation law

$$\mathrm{D}_t(u(u_{tt} - u_{xx})) + \mathrm{D}_x(2x(u_{xtt} - u_{xxx})) = 0$$

is a trivial conservation law of the first type.

A trivial CL, Type 2:

- The conservation law vanishes as a differential identity.
- Example: for the wave equation on $u(x,t)$: $u_{tt} = u_{xx}$,

$$\mathrm{D}_t(u_{xx}) - \mathrm{D}_x(u_{xt}) \equiv 0$$

is a trivial conservation law of the second type.

Conservation laws equivalence

Definition

Two conservation laws $\mathrm{D}_i \Phi^i[\mathbf{u}] = 0$ and $\mathrm{D}_i \Psi^i[\mathbf{u}] = 0$ are *equivalent* if $\mathrm{D}_i(\Phi^i[\mathbf{u}] - \Psi^i[\mathbf{u}]) = 0$ is a trivial conservation law. An *equivalence class* of conservation laws consists of all conservation laws equivalent to some given nontrivial conservation law.

Definition

A set of ℓ conservation laws $\{\mathrm{D}_i \Phi^i_{(j)}[\mathbf{u}] = 0\}_{j=1}^{\ell}$ is *linearly dependent* if there exists a set of constants $\{a^{(j)}\}_{j=1}^{\ell}$, not all zero, such that the linear combination

$$\mathrm{D}_i(a^{(j)} \Phi^i_{(j)}[\mathbf{u}]) = 0$$

is a trivial conservation law. In this case, up to equivalence, one of the conservation laws in the set can be expressed as a linear combination of the others.

- In practice, one is interested in finding linearly independent sets of (nontrivial) conservation laws of a given PDE system.

Flux computation: different methods

Flux Computation Problem

Suppose for a given PDE system, a set of CL multipliers has been found, and one has

$$\Lambda_\sigma[\mathbf{u}]R^\sigma[\mathbf{u}] \equiv \mathrm{D}_i\Phi^i[\mathbf{u}] = 0.$$

- How does one compute $\{\Phi^i[\mathbf{u}]\}$?

Some methods [cf. *Wolf (2002), Cheviakov (2010)*]:

- Direct
- Homotopy 1 [*Bluman & Anco (2002)*]
- Homotopy 2 [*Hereman et al (2005)*]
- Scaling (when a specific scaling symmetry is present) [*Anco (2003)*]

Conservation laws from variational principles

Action integral

$$J[\mathbf{U}] = \int_\Omega \mathcal{L}(\mathbf{x}, \mathbf{U}, \partial\mathbf{U}, \ldots, \partial^k\mathbf{U})\, dx.$$

Principle of extremal action

Variation of $\mathbf{U}$: $\mathbf{U}(\mathbf{x}) \to \mathbf{U}(\mathbf{x}) + \delta\mathbf{U}(\mathbf{x})$; $\quad \delta\mathbf{U}(\mathbf{x}) = \varepsilon\mathbf{v}(\mathbf{x})$; $\quad \delta\mathbf{U}(\mathbf{x})\big|_{\partial\Omega} = 0.$

Variation of action: $\delta J \equiv J[\mathbf{U} + \varepsilon\mathbf{v}] - J[\mathbf{U}] = \int_\Omega (\delta\mathcal{L})\, dx = o(\varepsilon).$

Variation of the Lagrangian

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}(\mathbf{x}, \mathbf{U} + \varepsilon\mathbf{v}, \partial\mathbf{U} + \varepsilon\partial\mathbf{v}, \ldots, \partial^k\mathbf{U} + \varepsilon\partial^k\mathbf{v}) - \mathcal{L}(\mathbf{x}, \mathbf{U}, \partial\mathbf{U}, \ldots, \partial^k\mathbf{U}) \\
&= \varepsilon\left(\frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma} v^\sigma + \frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma_j} v^\sigma_j + \cdots + \frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma_{j_1\cdots j_k}} v^\sigma_{j_1\cdots j_k}\right) + O(\varepsilon^2) \\
&\overset{\text{by parts}}{=} \varepsilon(v^\sigma \mathrm{E}_{U^\sigma}(\mathcal{L}[\mathbf{U}])) + \mathrm{div}(\ldots) + O(\varepsilon^2)
\end{aligned}$$

Euler-Lagrange equations, Euler operators:

$$\begin{aligned}
\mathrm{E}_{U^\sigma}(\mathcal{L}[\mathbf{U}]) &= \frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma} + \cdots + (-1)^k \mathrm{D}_{j_1}\cdots\mathrm{D}_{j_k} \frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma_{j_1\cdots j_k}} = 0, \\
\sigma &= 1, \ldots, m.
\end{aligned}$$

Conservation laws from variational principles (2)

Example 1: Harmonic oscillator, $U = x = x(t)$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \qquad \mathrm{E}_x\mathcal{L} = -m(\ddot{x} + \omega^2 x) = 0, \qquad \omega^2 = k/m.$$

Example 2: Wave equation for $U = u(x,t)$

$$\mathcal{L} = \frac{1}{2}\rho {u_t}^2 - \frac{1}{2}T{u_x}^2, \qquad \mathrm{E}_u\mathcal{L} = -\rho(u_{tt} - c^2 u_{xx}) = 0, \qquad c^2 = T/\rho.$$

- A number of physical non-dissipative systems have a variational formulation.
- The vast majority of PDE systems do not have a variational formulation.
- A PDE system follows from a variational principle (as it stands) $\Leftrightarrow$ the linearization operator is *self-adjoint* (*symmetric*).
  - # equations = # unknowns.
  - For a single PDE, only even-order derivatives.
  - The system has to be written in a "right" way!
- No systematic way to tell if a given system has a variational formulation.

Conservation laws from variational principles (3)

Definition

A DE system $\mathbf{R}[\mathbf{u}] = 0$ is variational if its equations are Euler-Lagrange equations for some variational principle:

$$R^\sigma[\mathrm{U}] \;=\; \mathrm{E}_{U^\sigma}(\mathcal{L}[\mathrm{U}]), \qquad \sigma = 1, \ldots, m.$$

- Example

Wave equation for $U = u(x, t)$

$$\mathcal{L} = K - P = \frac{1}{2}\rho {u_t}^2 - \frac{1}{2} T {u_x}^2$$

$$\mathrm{E}_u = \frac{d}{du} - \mathrm{D}_t \frac{d}{du_t} - \mathrm{D}_x \frac{d}{du_x}$$

$$\mathrm{E}_u \mathcal{L} = -\rho(u_{tt} - c^2 u_{xx}) = 0, \qquad c^2 = T/\rho$$

Construction of the Lagrangian

PDE linearization

- Given PDE or system: $\mathbf{R}[\mathbf{u}] = 0$.
- Linearized system (Fréchet derivative): $\mathbf{L}[\mathbf{u}]\mathbf{v}(\mathbf{x}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathbf{R}[\mathbf{u} + \epsilon\mathbf{v}] = 0$.
- Adjoint Linearized system:

$$\mathbf{w}(\mathbf{x}) \cdot (\mathbf{L}[\mathbf{u}]\ \mathbf{v}(\mathbf{x})) \overset{\text{by parts}}{=} (\mathbf{L}^*[\mathbf{u}]\ \mathbf{w}(\mathbf{x})) \cdot \mathbf{v}(\mathbf{x}) + (\text{divergence}).$$

Self-adjointness

- Given system $\mathbf{R}[\mathbf{u}] = 0$ is self-adjoint if

$$\mathbf{L}[\mathbf{u}]\,\mathbf{v}(\mathbf{x}) = \mathbf{L}^*[\mathbf{u}]\,\mathbf{v}(\mathbf{x}).$$

Homotopy Formula for a Lagrangian:

$$\boxed{\mathcal{L} = \int_0^1 \mathbf{u} \cdot \mathbf{R}[\lambda\mathbf{u}]\ d\lambda.}$$

Construction of the Lagrangian (2)

Example 1: Wave equation for $u(x,t)$

$$R[u] = u_{tt} - c^2 u_{xx} = 0;$$

Linearization (already linear!)

$$L[u]\, v(x,t) = v_{tt} - c^2 v_{xx} = 0;$$

Adjoint linearization operator:

$$w(x,t)\, L[u]\, v(x,t) = w(v_{tt} - c^2 v_{xx}) = (w_{tt} - c^2 w_{xx}) v(x,t) + (v_t w - v w_t)_t - c^2 (v_x w - v w_x)_x;$$

Result:

$$L^*[u]\, v(x,t) = L[u]\, v(x,t),$$

so $R[u]$ is self-adjoint.

Lagrangian:

$$\mathcal{L} = \frac{1}{2} {u_t}^2 - \frac{1}{2} c^2 {u_x}^2.$$

Construction of the Lagrangian (3)

Example 2:

- Heat equation for $u(x,t)$: $R[u] = u_t - u_{xx} = 0$.
- Linearization: $L[u]\,v(x,t) = v_t - v_{xx} = 0$.
- Adjoint linearization operator: $L^*[u]\,w(x,t) = -w_t - w_{xx} = 0$,
- NOT self-adjoint!

Append the adjoint:

- $\mathbf{R}[u^1, u^2] = \{u^1_t - u^1_{xx} = 0,\ -u^2_t - u^2_{xx} = 0\}$.
- Self-adjoint!
- Lagrangian: $\mathcal{L} = \frac{1}{2}\left(-u^1(u^2_t - u^2_{xx}) + u^2(u^1_t - u^1_{xx})\right)$.
- Euler-Lagrange equations:

$$\mathrm{E}_{u^1}(\mathcal{L}) = -u^2_t - u^2_{xx} = R^2, \qquad \mathrm{E}_{u^2}(\mathcal{L}) = u^1_t - u^1_{xx} = R^1.$$

- This technique can be used to make *any* PDE system self-adjoint.
- Non-physical Lagrangian (pseudo-Lagrangian).

Construction of the Lagrangian (4)

Example 4:

- KdV for $u(x,t)$ $R[u] = u_t + uu_x + u_{xxx} = 0$.
- Odd-order, clearly NOT self-adjoint.

... a differential substitution:

- $u = q_x$, $\widehat{R}[q] = q_{xt} + q_x q_{xx} + q_{xxxx} = 0$;
- Self-adjoint!
- Lagrangian for $\widehat{R}[q]$: $\mathcal{L} = \frac{1}{2}q_{xx}^2 - \frac{1}{6}q_x^3 - \frac{1}{2}q_x q_t$.

Result:

- For a given PDE/system, it is not simple to conclude whether it follows from a variational principle.
  - Much depends on the "right" writing.
  - Tricks can make equations variational...
- A feasible tool: comparison of local variational symmetries and local conservation laws.
  - This is based on the **first Noether's theorem**.

Variational symmetries

Consider a general DE system

$$R^\sigma[\mathbf{u}] = \mathbf{R}^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N$$

that follows from a variational principle with $J[\mathbf{u}] = \int_\Omega \mathcal{L}[\mathbf{u}]\, dx$.

Definition

A symmetry of $R^\sigma[u]$ given by

$$\mathbf{x}^* = f(\mathbf{x}, \mathbf{u}; a) = x + a\,\xi(\mathbf{x}, \mathbf{u}) + O(a^2),$$
$$\mathbf{u}^* = g(\mathbf{x}, \mathbf{u}; a) = \mathbf{u} + a\,\eta(\mathbf{x}, \mathbf{u}) + O(a^2)$$

is a **variational symmetry** of $R^\sigma[\mathbf{u}]$ if it preserves the action $J[\mathbf{u}]$.

Example 2: scaling for the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad \mathcal{L} = \frac{1}{2}{u_t}^2 - \frac{c^2}{2}{u_x}^2.$$

The scaling $x^* = x, \quad t^* = t, \quad u^* = u/\alpha$ is not a variational symmetry: $J^* = \alpha^2 J$.

Noether's theorem

Theorem

Given:

1. *a PDE system* $R^\sigma[\mathbf{u}] = 0$, $\sigma = 1, \dots, N$, *following from a variational principle;*
2. *a variational symmetry*

$$(x^i)^* = f^i(\mathbf{x}, \mathbf{u}; a) = x^i + a\xi^i(\mathbf{x}, \mathbf{u}) + O(a^2),$$
$$(u^\sigma)^* = g^\sigma(\mathbf{x}, \mathbf{u}; a) = u^\sigma + a\eta^\sigma(\mathbf{x}, \mathbf{u}) + O(a^2).$$

Then *the system* $R^\sigma[\mathbf{u}]$ *has a* **conservation law** $\mathrm{D}_i\Phi^i[\mathbf{u}] = 0$.
In particular,

$$\mathrm{D}_i\Phi^i[\mathbf{u}] \equiv \Lambda_\sigma[\mathbf{u}]R^\sigma[\mathbf{u}] = 0,$$

where the multipliers are given by

$$\Lambda_\sigma \equiv \zeta^\sigma[\mathbf{u}] = \eta^\sigma(\mathbf{x}, \mathbf{u}) - \frac{\partial u^\sigma}{\partial x_i}\xi^i(\mathbf{x}, \mathbf{u}).$$

Noether's theorem: example

Example 2

- **Equation:** Wave equation $u_{tt} = c^2 u_{xx}, \quad u = u(x,t)$.
- **Time Translation Symmetry:**

$$\begin{array}{ll} t^* = t + a, & \xi^t = 1; \\ x^* = x, & \xi^x = 0, \\ u^* = u, & \eta = 0, \end{array}$$

- **Multiplier**: $\Lambda = \zeta = \eta - 0 \cdot u_x - 1 \cdot u_t = -u_t$;
- **Conservation law (Energy):**

$$\Lambda R = -u_t(u_{tt} - c^2 u_{xx}) = -\left[D_t \left(\frac{u_t^2}{2} + c^2 \frac{u_x^2}{2} \right) - D_x \left(c^2 u_t u_x \right) \right] = 0.$$

Linearization operator & variational formulations

Recollect:

- Given PDE or system: $\mathbf{R}[\mathbf{u}] = 0$.
- Linearized system (Fréchet derivative): $\mathbf{L}[\mathbf{u}]\mathbf{v}(\mathbf{x}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathbf{R}[\mathbf{u} + \epsilon \mathbf{v}] = 0$.
- Adjoint Linearized system:

$$\mathbf{w}(\mathbf{x}) \cdot (\mathbf{L}[\mathbf{u}]\ \mathbf{v}(\mathbf{x})) \overset{\text{by parts}}{=} (\mathbf{L}^*[\mathbf{u}]\ \mathbf{w}(\mathbf{x})) \cdot \mathbf{v}(\mathbf{x}) + (\text{divergence}).$$

Facts:

- Symmetry components $\zeta^\sigma[\mathbf{u}]$ are solutions of the linearized system.
- Conservation law multipliers $\Lambda_\sigma[\mathbf{u}]$ are solutions of the adjoint linearized system.

A self-adjointness test:

- Check # equations = # unknowns.
- In some writing, CL multipliers and symmetries are “similar”?
- The test is not systematic... the “correct” writing of the system is not prescribed!

More general method:
Direct construction method of conservation laws

Definition

The *Euler operator* with respect to U^j:

$$\mathrm{E}_{U^j} = \frac{\partial}{\partial U^j} - \mathrm{D}_i \frac{\partial}{\partial U^j_i} + \cdots + (-1)^s \mathrm{D}_{i_1} \ldots \mathrm{D}_{i_s} \frac{\partial}{\partial U^j_{i_1 \ldots i_s}} + \cdots, \quad j = 1, \ldots, m.$$

Theorem

Let $\mathbf{U}(\mathbf{x}) = (U^1, \ldots, U^m)$. *The equations* $\mathrm{E}_{U^j} F[\mathbf{U}] \equiv 0$, $j = 1, \ldots, m$, *hold for arbitrary* $\mathbf{U}(\mathbf{x})$ *if and only if*

$$F[\mathbf{U}] \equiv \mathrm{D}_i \Psi^i[\mathbf{U}]$$

for some functions $\Psi^i[\mathbf{U}]$.

Idea:

- Seek conservation laws in the characteristic form $\boxed{\mathrm{D}_i \Phi^i = \Lambda_\sigma R^\sigma = 0}$

  (based on Hadamard's lemma for systems of maximal rank).

Hadamard lemma for differentiable functions

Given:

- A *totally nondegenerate* PDE system $R^\sigma[\mathbf{u}] = 0$, $\sigma = 1, \ldots, N$ [cf. *Olver (1993)*].
- A nontrivial local CL: $\mathrm{D}_i \Phi^i[\mathbf{u}] = 0$.
- Denote $G[\mathbf{U}] = \mathrm{D}_i \Phi^i[\mathbf{U}]$.

Hadamard lemma for differential functions:

A differential function $G[\mathbf{U}]$ vanishes on solutions of a PDE system $\boldsymbol{\mathcal{R}}$ if and only if it has the form

$$G[\mathbf{U}] = P^\alpha_\sigma[\mathbf{U}] \, \mathrm{D}_\alpha \, R^\sigma[\mathbf{U}].$$

Characteristic form of a CL:

Using the product rule, one has

$$G[\mathbf{U}] = \mathrm{D}_i \Phi^i[\mathbf{U}] = \Lambda_\sigma[\mathbf{U}] \, R^\sigma[\mathbf{U}] + \mathrm{div} \; \mathbf{H}[\mathbf{U}],$$

where $\mathbf{H}[\mathbf{U}]$ is linear in R^σ; $\mathrm{div} \; \mathbf{H}[\mathbf{u}] = 0$ is a trivial CL.

Hence every CL $\mathrm{D}_i \Phi^i[\mathbf{u}] = 0$ has an equivalent characteristic form

$$\mathrm{D}_i \tilde{\Phi}^i[\mathbf{u}] = \Lambda_\sigma[\mathbf{u}] \, R^\sigma[\mathbf{u}] = 0, \qquad \tilde{\Phi}^i = \Phi^i - H^i.$$

- CL multipliers (characteristics): $\{\Lambda_\sigma[\mathbf{u}]\}_{\sigma=1}^N$.

Direct construction method: general idea

Consider a general system $\mathbf{R}[\mathbf{u}] = 0$ of N PDEs.

Direct Construction Method

- Specify dependence of multipliers: $\Lambda_\sigma = \Lambda_\sigma(\mathbf{x}, \mathbf{U}, ...)$, $\quad \sigma = 1, ..., N$.

- Solve the set of determining equations

$$\mathrm{E}_{U^j}(\Lambda_\sigma R^\sigma) \equiv 0, \quad j = 1, \ldots, m,$$

  for arbitrary $\mathbf{U}(\mathbf{x})$ (off of solution set!) to find all such sets of multipliers.

- Find the corresponding fluxes $\Phi^i[\mathbf{U}]$ satisfying the identity

$$\Lambda_\sigma R^\sigma \equiv \mathrm{D}_i \Phi^i.$$

- Each set of fluxes, multipliers yields a local conservation law

$$\mathrm{D}_i \Phi^i[\mathbf{u}] = 0,$$

  holding for all solutions $\mathbf{u}(\mathbf{x})$ of the given PDE system.

Completeness of the direct construction method

Extended Kovalevskaya form

A PDE system $\mathbf{R}[\mathbf{u}] = 0$ is in *extended Kovalevskaya form* with respect to an independent variable x^j, if the system is solved for the highest derivative of each dependent variable with respect to x^j, i.e.,

$$\frac{\partial^{s_\sigma}}{\partial (x^j)^{s_\sigma}} u^\sigma = G^\sigma(x, u, \partial u, \dots, \partial^k u), \quad 1 \le s_\sigma \le k, \quad \sigma = 1, \dots, m, \tag{1}$$

where all derivatives with respect to x^j appearing in the right-hand side of each PDE in (1) are of lower order than those appearing on the left-hand side.

Theorem [*R. Popovych, A. C.*]

Let $\mathbf{R}[\mathbf{u}] = 0$ be a PDE system in the extended Kovalevskaya form (1). Then every its local conservation law has an equivalent conservation law in the characteristic form,

$$\Lambda_\sigma R^\sigma \equiv \mathrm{D}_i \Phi^i = 0,$$

such that neither Λ_σ nor Φ^i involve the leading derivatives or their differential consequences.

2D incompressible hyperelasticity for fiber reinforced materials

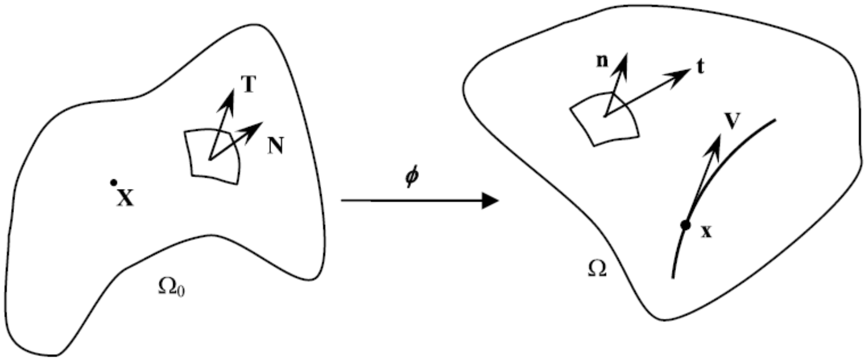


Fig. 1. Material and Eulerian coordinates.

Material picture

- A solid body occupies the reference (Lagrangian) volume $\Omega_0 \subset \mathbb{R}^3$.
- Actual (Eulerian) configuration: $\Omega \subset \mathbb{R}^3$.
- Material points are labeled by $\mathbf{X} \in \Omega_0$.
- The actual position of a material point: $\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \in \Omega$.
- Jacobian matrix (deformation gradient): $J = \det \mathbf{F} > 0$.

Material picture

- Boundary force (per unit area) in Eulerian configuration: $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$.
- Boundary force (per unit area) in Lagrangian configuration: $\mathbf{T} = \mathbf{P}\mathbf{N}$.
- $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ is the Cauchy stress tensor.
- $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$ is the first Piola-Kirchhoff tensor.
- Density in reference & actual configuration: $\rho_0 = \rho_0(\mathbf{X})$, $\rho = \rho(\mathbf{X}, t) = \rho_0/J$.

Equations of motion

Dynamical BVP:

$$\begin{cases} J = \det \mathbf{F} = 1, & \rho(\mathbf{X}, t) = \rho_0(\mathbf{X})/J = \rho_0(\mathbf{X}) \\ \rho_0 \mathbf{x}_{tt} = \mathrm{div}_{(X)} \mathbf{P} + \rho_0 \mathbf{R}, & (\mathrm{div}_{(X)} \mathbf{P})^i = \frac{\partial P^{ij}}{\partial X^j}. \quad \text{Lagrangian divergence} \\ \mathbf{P} = -p\ \mathbf{F}^{-T} + \rho_0 \frac{\partial W}{\partial \mathbf{F}}, & \\ \mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T & \text{conservation of linear momentum} \end{cases}$$

Additive split of strain energy density: $W^h = W_{iso} + W_{aniso}$

Mooney-Rivlin constitutive model: $W_{iso} = a(I_1 - 3) + b(I_2 - 3)$

Left & right Cauchy-Green tensors: $\mathbf{B} = \mathbf{F}\mathbf{F}^T, \quad B^{ij} = F^i_k F^j_k, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad C_{ij} = F^k_i F^k_j.$

Mapping of fiber material vector: $\mathbf{a} = \mathbf{a}(\mathbf{X}, t) = \mathbf{F}\mathbf{A}/|\mathbf{F}\mathbf{A}| = \mathbf{F}\mathbf{A}/\lambda,$

Anisotropic contribution: $W_{aniso} = c(I_5 - (I_4)^2)$

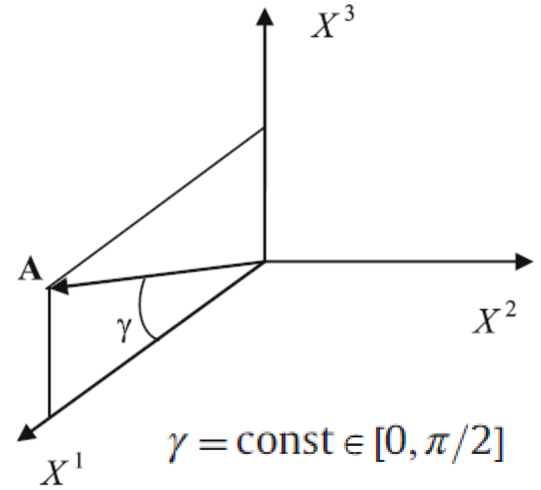
$$I_4 = \mathbf{A}^T\mathbf{C}\mathbf{A} \qquad I_5 = \mathbf{A}^T\mathbf{C}^2\mathbf{A}.$$

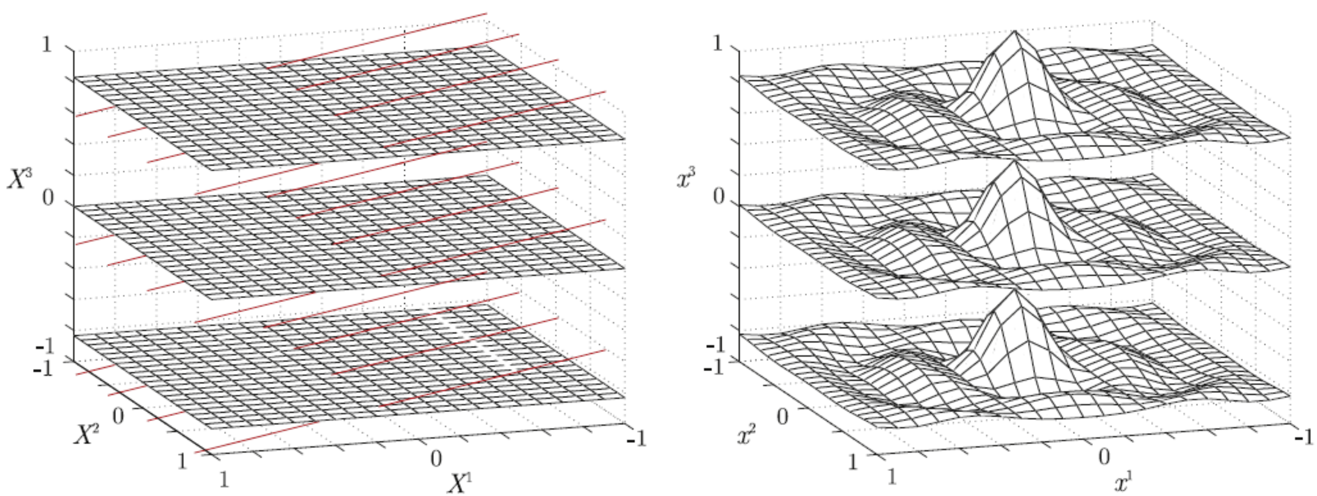
Motion transverse to a plane

Coined polarized motion or anti-plane shear

$$\mathbf{x} = \mathbf{X} + \mathbf{G}(\mathbf{X}, t) \longrightarrow \mathbf{x} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 + G\left(X^1, X^2, t\right) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}$$ Fiber family in reference configuration



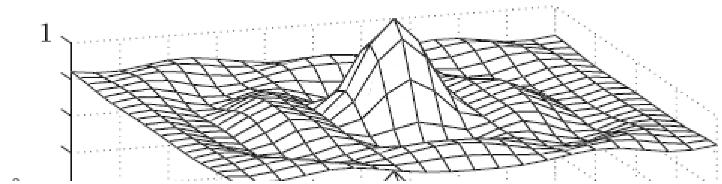


Fiber direction. The reference (Lagrangian) mesh with fibers. A sample deformed mesh (Eulerian configuration)

-> Strongly nonlinear boundary value problem (BVP) difficult to solve in general.

Symmetry classification of nonlinear elastodynamics BVP: motion transverse to a plane

$$\mathbf{x} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 + G\left(X^1, X^2, t\right) \end{bmatrix}$$



BVP for one fiber reinforced material governed by two PDE's:

$$\frac{\partial^2 G}{\partial t^2} = \alpha\left(\frac{\partial^2 G}{\partial\left(X^1\right)^2} + \frac{\partial^2 G}{\partial\left(X^2\right)^2}\right) + \beta\cos^2\gamma\left(3\cos^2\gamma\left(\frac{\partial G}{\partial X^1}\right)^2 + 6\cos\gamma\sin\gamma\frac{\partial G}{\partial X^1} + 2\sin^2\gamma\right)\frac{\partial^2 G}{\partial\left(X^1\right)^2}$$

$$b\frac{\partial}{\partial X^1}\left[\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial X^1 \partial X^2} - \frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial\left(X^1\right)^2}\right] =$$

$$= \frac{\partial}{\partial X^2}\left[\beta\cos^3\gamma\frac{\partial^2 G}{\partial\left(X^1\right)^2}\left(\cos\gamma\frac{\partial G}{\partial X^1} + \sin\gamma\right) + b\left(\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial X^1 \partial X^2} - \frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial\left(X^2\right)^2}\right)\right]$$

Search Lie groups of point transformations satisfied by these two PDE's:

$$G^* = g(X^1, X^2, t, G; \varepsilon) = G + \varepsilon\eta(X^1, X^2, t, G) + O(\varepsilon^2),$$

$$(X^*)^i = f^i(X^1, X^2, t, G; \varepsilon) = z^i + \varepsilon\xi^i(X^1, X^2, t, G) + O(\varepsilon^2), \quad i = 1, 2$$

$$t^* = h(X^1, X^2, t, G; \varepsilon) = u^\mu + \varepsilon\tau(X^1, X^2, t, G) + O(\varepsilon^2),$$

Symmetry classification of the BVP

-> search infinitesimal generators of Lie algebra: $\mathrm{Y} = \xi^i(X^1, X^2, t, G)\frac{\partial}{\partial X^i} + \tau(X^1, X^2, t, G)\frac{\partial}{\partial t} + \eta(X^1, X^2, t, G)\frac{\partial}{\partial G}$

Parameters	Symmetries
Arbitrary	$\mathrm{Y}^1 = \frac{\partial}{\partial t}, \mathrm{Y}^2 = \frac{\partial}{\partial X^1},\ \mathrm{Y}^3 = \frac{\partial}{\partial X^2},\ \mathrm{Y}^4 = \frac{\partial}{\partial G},\ \mathrm{Y}^5 = t\frac{\partial}{\partial G},\ \mathrm{Y}^6 = X^1\frac{\partial}{\partial X^1} + X^2\frac{\partial}{\partial X^2} + t\frac{\partial}{\partial t} + G\frac{\partial}{\partial G}$
$\gamma = \pi/2$ or $\beta = 4q = 0$	$\mathrm{Y}^1,\ \mathrm{Y}^2,\ \mathrm{Y}^3,\ \mathrm{Y}^4,\ \mathrm{Y}^5,\ \mathrm{Y}^6,\ \mathrm{Y}^7 = -X^2\frac{\partial}{\partial X^1} + X^1\frac{\partial}{\partial X^2},\ \mathrm{Y}^8 = G\frac{\partial}{\partial G}$

Y^1, Y^4: space & time translations.

Y^5: Galilean group in the direction of displacement.

Y^6: homogeneous space-time coupling.

* Specific case q cos γ =0 (in Table): fiber bundle orthogonal to (X^1, X^2) plane.

Y^7: rotation.

Y^8: scaling of G.

* 1D wave propagation independent of fiber direction: displacement only function of X^2

$$\mathbf{x} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 + Q(X^2, t) \end{bmatrix} \longrightarrow \boxed{\frac{\partial^2 Q}{\partial t^2} = \alpha\frac{\partial^2 Q}{\partial (X^2)^2},\quad \frac{\partial P}{\partial X^1} = \frac{\partial P}{\partial X^2} = 0}$$

Linear equ.

Simplified 1D motion (dependent on fiber direction)

$$\mathbf{x} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 + G(X^1, t) \end{bmatrix}$$ motion

$$G_{tt} = \underbrace{\left(\alpha + \beta\cos^2\gamma\left[(3\cos^2\gamma)(G_x)^2 + (6\sin\gamma\cos\gamma)G_x + 2\sin^2\gamma\right]\right)}_{= C^2} G_{xx}$$ dynamical hyperbolic equ.

$$0 = p_x - 2\beta\rho_0\cos^3\gamma(\cos\gamma G_x + \sin\gamma)G_{xx}, \longrightarrow p = \beta\rho_0\cos^3\gamma(\cos\gamma G_x + 2\sin\gamma)G_x + f(t)$$ explicit

Look for Lie point symmetries:

$$Z = \xi(x,t,G)\frac{\partial}{\partial x} + \tau(x,t,G)\frac{\partial}{\partial t} + \eta(x,t,G)\frac{\partial}{\partial G}$$

γ angle between fiber & wave propagation direction

Parameters	Symmetries $\gamma \neq 0, \pi/2$
Arbitrary	$Z^1 = \frac{\partial}{\partial x}$, $Z^2 = \frac{\partial}{\partial t}$, $Z^3 = \frac{\partial}{\partial G}$, $Z^4 = t\frac{\partial}{\partial G}$, $Z^5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + G\frac{\partial}{\partial G}$
$4\alpha \leq \beta$, $\sin^2 2\gamma = \frac{4\alpha}{\beta}$	Z^1, Z^2, Z^3, Z^4, Z^5, $Z^6 = 2t\cos\gamma\frac{\partial}{\partial t} + x\cos\gamma\frac{\partial}{\partial x} - x\sin\gamma\frac{\partial}{\partial G}$

Pr.: 1D PDE is hyperbolic for any fiber orientation γ iff $4\alpha > \beta$.

Necessary condition for loss of hyperbolicity = vanishing coeff. of G_{xx}, is $4\alpha < \beta$ & $\sin^2(2\gamma) \geq \frac{4\alpha}{\beta}$

Ex.: model of rabbit artery (Holzapfel, 2000):
a=1.5 kPa, q=1.18 kPa (media) / a=0.15 kPa, q=0.28 kPa (adventitia).
-> $4\alpha > \beta$, model remains hyperbolic.

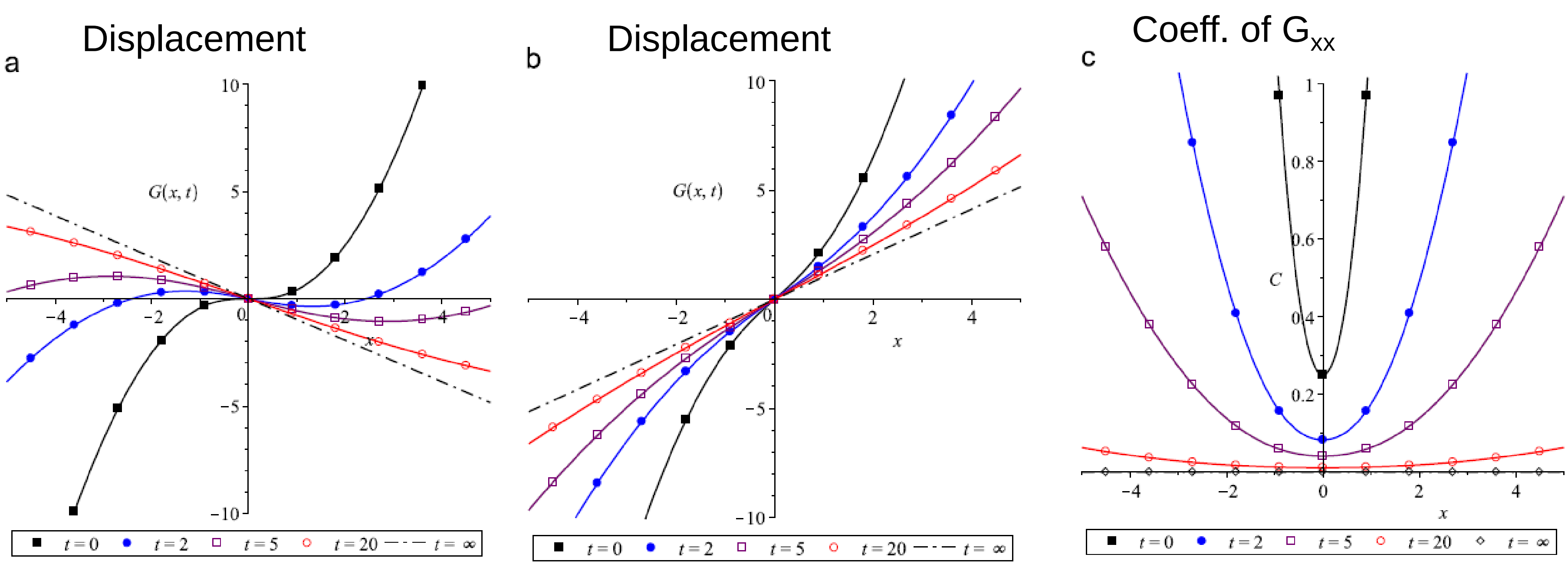
1D motion dependent on fiber direction: exact invariant solution

Case of fiber angle with direction of wave propagation s.t. $\sin^2 2\gamma = \frac{4\alpha}{\beta}$

→ Necesary but not sufficient condition (!) for loss of hyperbolicity satisfied.

Search for solutions invariant under Z^6: $y = \frac{x}{\sqrt{t}}, \quad M(y) = \sqrt{12\beta}\cos^2\gamma\,(G(x,t) + x\tan\gamma).$

→ Reduced 2nd order ODE: $M''(y) = \frac{3M'(y)}{(M'(y))^2 - y^2}$ IC: $M(0) = 0, \quad M'(0) = m_0.$



$\alpha = 1/12,\ \beta = 1/3,\ m_0 = 1,\ \gamma = \pi/4,$

$\gamma = -\pi/4.$

-> BVP remains hyperbolic

Motion transverse to an axis

$$\mathbf{X}=\begin{bmatrix} X^1 \\ X^2+H\left(X^1,t\right) \\ X^3+G\left(X^1,t\right) \end{bmatrix}.$$

Motion orthogonal to X^1 axis; wave propagation in X^1 direction

-> incompressibility condition automatically satisfied.

BVP:

$$0=p_x-2\beta\rho_0\cos^3\gamma[(\cos\gamma G_x+\sin\gamma)G_{xx}+\cos\gamma H_x H_{xx}], \longrightarrow p=\beta\rho_0\cos^3\gamma\left[\cos\gamma\,(G_x^2+H_x^2)+2\sin\gamma\,G_x\right]+f(t).$$

$$H_{tt}=\alpha H_{xx}+\beta\cos^3\gamma\left[\cos\gamma([G_x^2+H_x^2]H_{xx}+2G_xH_xG_{xx})+2\sin\gamma\frac{\partial}{\partial x}(G_xH_x)\right]$$

$$G_{tt}=\alpha G_{xx}+\beta\cos^2\gamma\left[2\sin^2\gamma\,G_{xx}+\cos^2\gamma\left(2G_xH_xH_{xx}+\left(H_x^2+3G_x^2\right)G_{xx}\right)\right.$$
$$\left.+\sin 2\gamma(3G_xG_{xx}+H_xH_{xx})\right],$$

$$\alpha=2(a+b)>0, \qquad \beta=4a>0$$

$$\longrightarrow \begin{cases} H_{tt}=\dfrac{\partial}{\partial x}\left(\left[\alpha+\beta\cos^3\gamma\left\{(G_x^2+H_x^2)\cos\gamma+2G_x\sin\gamma\right\}\right]H_x\right), \\ G_{tt}=\dfrac{\partial}{\partial x}\left(\alpha G_x+\beta\cos^2\gamma\left[2\sin^2\gamma\,G_x+\cos^2\gamma(G_x^2+H_x^2)G_x\right.\right. \\ \qquad\left.\left.+\sin\gamma\cos\gamma(3G_x^2+H_x^2)\right]\right). \end{cases}$$

Reduced form of PDE's
in conserved form

Motion transverse to an axis (2)

Lie point symmetries in fully non-linear situation:

$$W = \xi(x,t,H,G)\frac{\partial}{\partial x} + \tau(x,t,H,G)\frac{\partial}{\partial t} + \eta(x,t,H,G)\frac{\partial}{\partial H} + \zeta(x,t,H,G)\frac{\partial}{\partial G}$$

Parameters	Symmetries
Arbitrary	$W^1 = \frac{\partial}{\partial x},\ W^2 = \frac{\partial}{\partial t},\ W^3 = \frac{\partial}{\partial H},\ W^4 = \frac{\partial}{\partial G},\ W^5 = t\frac{\partial}{\partial H},$ $W^6 = t\frac{\partial}{\partial W},\ W^7 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + H\frac{\partial}{\partial H} + G\frac{\partial}{\partial G},$ $W^8 = \cos\gamma\left(H\frac{\partial}{\partial G} - G\frac{\partial}{\partial H}\right) - x\sin\gamma\frac{\partial}{\partial H}$
$4\alpha \le \beta,$ $\sin^2 2\gamma = \frac{4\alpha}{\beta}$	$W^1,\ W^2,\ W^3,\ W^4,\ W^5,\ W^6,\ W^7,\ W^8,$ $W^9 = 2t\cos\gamma\frac{\partial}{\partial t} + x\cos\gamma\frac{\partial}{\partial x} - x\sin\gamma\frac{\partial}{\partial G}$

- Special symmetry W^9 when $\sin^2 2\gamma = \frac{4\alpha}{\beta}$ necesary condition for loss of hyperbolicity
- Galilean group W^5 in x^2 direction
- Fiber-dependent (via γ) rotation group W^8:

$$t^* = t, \quad x^* = x,$$
$$H^* = H\cos\phi + G\sin\phi + x\tan\gamma\sin\phi,$$
$$G^* = -H\sin\phi + G\cos\phi - x\tan\gamma(1-\cos\phi),$$

Motion transverse to an axis: exact travelling wave solutions

Symmetry generator: $W_{tw} = c\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t}$ $\longrightarrow$ Invariants: $r = x - ct, \quad H(x,t) = h(r), \quad G(x,t) = g(r).$

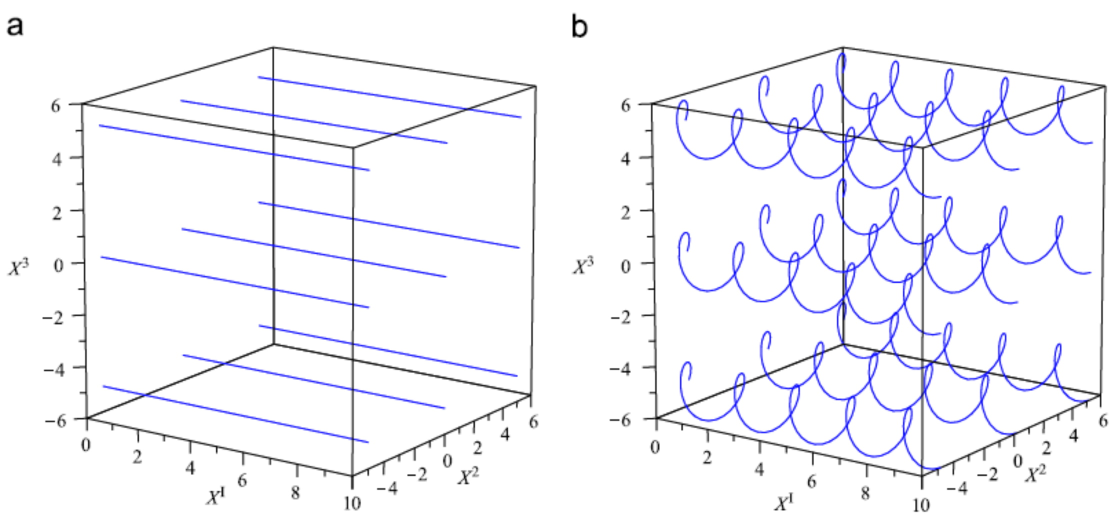
$\longrightarrow$ Balance of momentum:
(2 equ. for g(r), h(r))

$$\left[\alpha - c^2 + \beta\cos^4\gamma(3(h')^2 + (g')^2) + 2\beta\sin\gamma g'\right]h'' + 2\beta\cos^3\gamma[\cos\gamma\, g' + \sin\gamma]h'g'' = 0, \quad 2\beta\cos^3\gamma[\cos\gamma\, g' + \sin\gamma]h'h'' + \left[\alpha - c^2 + \beta\cos^2\gamma \times \left(\cos^2\gamma\,[(h')^2 + 3(g')^2] + 3\sin 2\gamma\, g' + 2\sin^2\gamma\right)\right]g'' = 0$$

<u>Example of solution</u>: harmonic functions $h(r) = A\cos(kr + \phi_0), \quad g(r) = A\sin(kr + \phi_0), \quad A = R/k.$

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t) = \begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix} + A\begin{bmatrix} 0 \\ \cos(k[X^1 - ct] + \phi_0) \\ \sin(k[X^1 - ct] + \phi_0) \end{bmatrix}$$

Time-periodic perturbation of stress-free state



Travelling helical shear waves

Few material lines for X^2=Const, X^3= const in reference config. (a). Same lines in actual configuration (b)

Equivalence transformations

Definition: PDE model with M constitutive parameters (K_1, …, K_M).

Equivalence transformations map independent variables, dependent variables & constitutive parameters into new ones s.t. form of PDE's is preserved.
Ex.: scalings, translations.

-> Reduces number of parameters & simplifies form of PDEs.

Consider PDE system

$$E^{\sigma}(z, u, \partial u, \ldots, \partial^{k} u) = 0, \quad \sigma = 1..N$$

$$n \text{ independent variables } z = \left(z^1, \ldots, z^n\right)$$

$$m \text{ dependent variables } u(z) = \left(u^1(z), \ldots, u^m(z)\right)$$

$$L \text{ constitutive functions / parameters } K = \left(K_1, \ldots, K_l\right)$$

One-parameter Lie group of equivalence transformations:

$$z^* = f(z, u; \varepsilon), \quad u^* = g(z, u; \varepsilon),$$

$$K^* = G_l(z, u, K; \varepsilon)$$

Equivalence transformations: example

2D BVP of nonlinear elasticity $\mathbf{F}=\begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\mathbf{C}_2=\begin{bmatrix} F^2_{\ 2} & -F^2_{\ 1} \\ -F^1_{\ 2} & F^1_{\ 1} \end{bmatrix}$

Equ. of motion:

$$\rho_0(x^1)_{tt}-\frac{\partial P^{11}}{\partial X^1}-\frac{\partial P^{12}}{\partial X^2}-\rho_0 R^1=0,$$
$$\rho_0(x^2)_{tt}-\frac{\partial P^{21}}{\partial X^1}-\frac{\partial P^{22}}{\partial X^2}-\rho_0 R^2=0,$$

$$\rho_0=\rho_0(X^1,X^2),\qquad P^{ij}=\rho_0(X^1,X^2)\frac{\partial W}{\partial F_{ij}},\quad i,j=1,2$$

Ciarlet-Mooney-Rivlin model: $W=aI_1+bI_2-cI_3-\frac{1}{2}d\ \ln I_3,\quad a>0,\ b,c,d\geq 0$

2D elasticity BVP admits following equivalence transformations:

$$\begin{aligned}
&\widetilde{t}=e^{\varepsilon_2}t+\varepsilon_1,\\
&\widetilde{X}^1=e^{\varepsilon_3}\left(X^1\cos\varepsilon_7-X^2\sin\varepsilon_7\right)+\varepsilon_4,\\
&\widetilde{X}^2=e^{\varepsilon_3}\left(X^1\sin\varepsilon_7+X^2\sin\varepsilon_7\right)+\varepsilon_5,\\
&\widetilde{x}^1=e^{2\varepsilon_2}x^1+f^1(t),\\
&\widetilde{x}^2=e^{2\varepsilon_2}x^2+f^2(t),\\
&\widetilde{\rho}_0=e^{\varepsilon_6}\rho_0,\\
&\widetilde{R}^1=R^1+\frac{d^2f^1(t)}{dt^2},\qquad \widetilde{R}^2=R^2+\frac{d^2f^2(t)}{dt^2},\\
&\widetilde{a}=-b+e^{2\varepsilon_3-2\varepsilon_2}(a+b),\qquad \widetilde{b}=b,\\
&\widetilde{c}=-b+e^{4\varepsilon_3-6\varepsilon_2}(b+c),\qquad \widetilde{d}=e^{2\varepsilon_2}d,
\end{aligned}$$

$$\begin{aligned}
&\widetilde{a}=a+\varepsilon_8 G(a,b,c,d)+O(\varepsilon^2),\\
&\widetilde{b}=b-\varepsilon_8 G(a,b,c,d)+O(\varepsilon^2),\\
&\widetilde{c}=c+\varepsilon_8 G(a,b,c,d)+O(\varepsilon^2),\\
&\widetilde{d}=d.
\end{aligned}$$

Th.: the dynamics of Ciarlet-Mooney-Rivlin models in 2D depends on only 3 parameters

$$\mathbf{P}_2=\rho_0\left[A\mathbf{F}_2+BJ\mathbf{C}_2-\frac{d}{J}\mathbf{C}_2\right]\qquad A=2(a+b)\geq 0,\ B=2(b+c)\geq 0$$

Symmetry , but also Symmetry breaking!

In biology: at least one asymmetric carbon atom in the molecules = ‘bricks of life’ (nucleotides, amino acids).
-> homochirality of biomolecules: amino acids left-handed (levorotatory compound) / nucleotides right-handed (dextrorotary compound).

Such chiral molecules non-superposable with their mirror image.

Two possible explanations for this asymmetry:

- Amplification of random fluctuations by some self-catalytic process.
- More fundamental dissymmetry of universe.

Ex.: violation of parity of weak interactions: electrons (matter) that dominate over positrons are left-handed / positrons are right-handed.

Irreversible processes have a dual role: destroy order close to equilibrium / generate order far from equilibrium.

Transition towards organized states in non-equilibrium situations due to generation of order by amplification of fluctuations & percolation phenomena.

-> Generates dissipative structures, like crystals [Prigogine], like in turbulence.

Summary- Outlook

- Symmetry classification & search for conservation laws implemented in symbolic package GEM (A. Sheviakov, Univ. Saskachewan).

- Divergence-type conservation laws useful in the analysis of BVP & numerics.

- Conservation laws obtained systematically through Direct Construction Method.

- For variational DE systems: conservation laws correspond to variational symmetries.

- Noether’s theorem not a preferred way to derive unknow conservation laws.

- Symmetry methods as reduction methods.

- Numerical schemes preserving symmetries & cons. laws for dissipative systems.