Géométrisation de la thermodynamique des milieux continus, classique et relativiste

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La Rochelle, 3-7 juin 2019

GDR Géométrie différentielle et Mécanique

LaMCUBE Laboratoire de mécanique, multiphysique, multiéchelle

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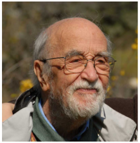
Idea debate

General Relativity

is not solely a theory of gravitation which would be reduced to predict tiny effects but –may be above all– it is a consistent framework for mechanics and physics of continua

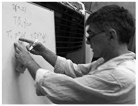
Inspiration sources :

Jean-Marie Souriau
Lect. Notes in Math. 676 (1976)



Claude Vallée, IJES (1981)

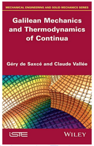
Patrick Iglesias, DGMP (1983)



Some key ideas :

- We take the Relativity as model, process termed "geometrization", but with Galileo symmetry group
- The **entropy** is generalized in the form of a 4-vector and the **temperature** in the form of a 5-vector
- We generalize the **energy-momentum** tensor by associating the "mass" with it
- We decompose the new object into reversible and dissipative parts
- We obtain a covariant and more compact writing of the 1^{st} and 2^{nd} principles

Book with **Claude Vallée :**
Galilean Mechanics and Thermodynamics of Continua (ISTE-Wiley, 2016)



Galilean transformations

- Event $\boldsymbol{X}$ occuring at position x and at time t

$$X = \begin{pmatrix} t \\ x \end{pmatrix} \text{ coordinates of } \boldsymbol{X} \in \textbf{ time-space } \mathcal{M}$$

- The **Galilean transformations** are space-time transformations preserving the **Uniform Straight Motion**, the **durations**, the **distances**, the **angles** and the **oriented volumes**, then affine of the form $X = P\,X' + C$ with :

$$P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

where $u \in \mathbb{R}^3$ is the **Galilean boost** and R is a rotation

- Their set is **Galileo's group**, a Lie group of dimension 10

Galilean vectors

- A Galilean vector $\vec{\boldsymbol{V}}$, represented by a column V, has a transformation law $V = P\,V'$ where P is a Galilean linear transformation
- The 4-velocity $\vec{\boldsymbol{U}} \in T_{\boldsymbol{X}}\mathcal{M}$ represented by the column

$$U = \frac{dX}{dt} = \begin{pmatrix} 1 \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix}$$

- Its transformation law $U = P\,U'$ provides the velocity addition formula

$$v = u + R\,v'$$

The fifth dimension...

FIGURE – 5D simulator (La Foux, Saint-Tropez)

Bargmannian transformations

- The space-time $\mathcal{M}$ is embedded into a space $\hat{\mathcal{M}}$ of dimension 5 : $\mathcal{M} \to \hat{\mathcal{M}} \,:\, \boldsymbol{X} \mapsto \hat{\boldsymbol{X}} = \hat{f}(\boldsymbol{X})$
- We built a group of affine transformations $\hat{X}' \mapsto \hat{X} = \hat{P}\,\hat{X}' + \hat{C}$ of $\mathbb{R}^5$ which are Galilean when acting onto the space-time hence of the form :

$$\hat{P} = \begin{pmatrix} P & 0 \\ \Phi & \alpha \end{pmatrix} ,$$

where P is Galilean, Φ and α must have a physical meaning linked to the energy

- Thus we know that, under the action of a boost u and a rotation R, the kinetic energy is transformed according to :

$$e = \frac{1}{2}\,m \parallel u + R\,v' \parallel^2 = \frac{1}{2}\,m \parallel u \parallel^2 + m\,u \cdot (R\,v') + \frac{1}{2}\,m \parallel v' \parallel^2 \ .$$

Bargmannian transformations

- We claim that the fifth dimension is linked to the energy by :

$$dz = \frac{e}{m}\, dt = \frac{1}{2} \parallel u \parallel^2 dt' + u^T R\, dx' + dz'$$

that leads to consider the **Bargmannian transformations** of $\mathbb{R}^5$ of which the linear part is :

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 \\ u & R & 0 \\ \frac{1}{2} \parallel u \parallel^2 & u^T R & 1 \end{pmatrix}$$

Their set is the **Bargmann's group**,
a Lie group of dimension 11,
introduced in quantum mechanics for cohomologic reasons
but which turns out very useful in Thermodynamics !

Geometrization process

- Reciprocal temperature $\beta = \dfrac{1}{\theta} = \dfrac{1}{k_B T}$
- Isothermal reversible transformation Clausius (1865)
  Entropy $\mathcal{S} = \dfrac{\mathcal{Q}_R}{\theta} = \mathcal{Q}_R \cdot \beta$
- Geometrization process

$$\begin{array}{ccccc} \mathcal{S} & = & \mathcal{Q}_R & \cdot & \beta \\ \downarrow & & \downarrow & & \downarrow \\ S & = & \hat{T}_R & \cdot & \hat{W} \end{array}$$

where

- S is the entropy 4-flux
- $\hat{T}_R$ is the momentum tensor
- $\hat{W}$ is the temperature 5-vector

Temperature 5-vector

- The reciprocal temperature β is generalized as a Bargmannian 5-vector :

$$\hat{W} = \begin{pmatrix} W \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix} ,$$

- The transformation law $\hat{W}' = \hat{P}^{-1}\hat{W}$ leads to :

$$\beta' = \beta \ , \quad w' = R^T(w - \beta u), \quad \zeta' = \zeta - w \cdot u + \frac{\beta}{2} \parallel u \parallel^2$$

- Picking up $u = w \, / \, \beta$, we obtain the **reduced form**

$$\hat{W}' = \begin{pmatrix} \beta \\ 0 \\ \zeta_{int} \end{pmatrix}$$

interpreted as the temperature vector of a volume element **at rest**

Temperature 5-vector

Boost method :
Starting from the reduced form, we apply the Galilean transformation of boost v, that gives :

$$\hat{W} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix} , = \begin{pmatrix} \beta \\ \beta\, v \\ \zeta_{int} + \frac{\beta}{2} \parallel v \parallel^2 \end{pmatrix} .$$

where ζ is **Planck's potential**

Friction tensor

Friction tensor

The **friction tensor** is a mixed 1-covariant and 1-contravariant tensor :

$$\boldsymbol{f} = \nabla\, \vec{\boldsymbol{W}}$$

represented by the 4×4 matrix $f = \nabla\, W$

- This object introduced by Souriau merges the temperature gradient and the strain velocity
- In dimension 5, we can also introduce

$$\hat{\boldsymbol{f}} = \nabla\, \hat{\vec{\boldsymbol{W}}}$$

represented by a 5×4 matrix

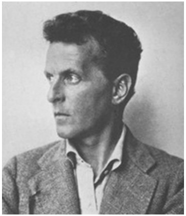
$$\hat{f} = \nabla\, \hat{W} = \begin{pmatrix} f \\ \nabla\, \zeta \end{pmatrix}$$

Momentum tensor

Method

Taking care **to walk up and down the rough ground of the reality** (Wittgenstein),

we want to work, in dimension 4 ou 5, with tensors of which the transformation law respects the physics



The meaning of the components is not given *a priori* but results, through the transformation law, from the choice of the symmetry group

Momentum tensor

Momentum tensor

Linear map from the tangent space to $\hat{\mathcal{M}}$ at $\hat{\boldsymbol{X}} = \hat{f}(\boldsymbol{X})$ into the tangent space to $\mathcal{M}$ at $\boldsymbol{X}$, hence a **mixed tensor** $\hat{\boldsymbol{T}}$ of rank 2

- **Galilean momentum tensors :** represented by a 4×5 matrix of the form :

$$\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ k & \sigma_\star & p \end{pmatrix}$$

  where $\sigma_\star$ is a 3×3 symmetric matrix

- In matrix form, the transformation law is :

$$\hat{T}' = P\ \hat{T}\ \hat{P}^{-1}$$

To reveal the physical meaning of the components ...

Momentum tensor

... we let the symmetry group act !

- The transformation law provides :

$$\rho' = \rho \,, \quad p' = R^T\,(p - \rho u), \quad \sigma'_\star = R^T\,(\sigma_\star + u\,p^T + p\,u^T - \rho\,u\,u^T)\,R$$

$$\mathcal{H}' = \mathcal{H} - u \cdot p + \frac{\rho}{2} \parallel u \parallel^2, \quad k' = R^T\,(k - \mathcal{H}' u + \sigma_* u + \frac{1}{2} \parallel u \parallel^2\, p)$$

- which leads to the **reduced form** :

$$\hat{T}' = \begin{pmatrix} \rho\, e_{int} & 0 & \rho \\ h' & \sigma' & 0 \end{pmatrix} ,$$

interpreted as the momentum of a volume element **at rest**

Momentum tensor

Boost method : starting from the reduced form, we apply a Galilean transformation law of boost v and rotation R and we interpret :

- ρ as the **density**
- $p = \rho\, v$ as the **linear momentum**
- $\sigma_\star = \sigma - \rho v v^T$ as the **dynamical stresses**
- $\mathcal{H} = \rho\,(e_{int} + \frac{1}{2} \parallel v \parallel^2)$ as the **total energy**
- $k = h + \mathcal{H} v - \sigma v$ as the **energy flux** by conduction convection stress

with :

- the **heat flux** $h = R\, h'$
- the **statical stresses** $\sigma = R\, \sigma'\, R^T$

First principle

Momentum divergence

5-row $div\ \hat{T}$ such that, for all smooth 5-vector field $\hat{W}$:

$$Div\ (\hat{T}\ \hat{W}) = (Div\ \hat{T})\ \hat{W} + Tr\ \left(\hat{T}\ \nabla\ \hat{W}\right)$$

Covariant form of the 1^{st} principle

$$Div\ \hat{T} = 0$$

First principle

In absence of gravity, we recover the balance equations of :

- mass : $\dfrac{\partial \rho}{\partial t} + div\,(\rho\, v) = 0$
- linear momentum : $\rho \left[\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x}\, v \right] = (div\ \sigma)^T$,
- energy : $\dfrac{\partial \mathcal{H}}{\partial t} + div\ \ (h + \mathcal{H} v - \sigma\, v) = 0$

First principle

Reversible medium

if ζ is a function of

- the Lagrangian strain measure $E = \frac{1}{2}(F^T F - I)$
- the temperature vector W
- and the Lagrangean coordinates s'

then the 4×4 matrix $T_R = U\, \Pi_R + \begin{pmatrix} 0 & 0 \\ -\sigma_R v & \sigma_R \end{pmatrix}$

with $\Pi_R = -\rho\, \frac{\partial \zeta}{\partial W}$ $\quad \sigma_R = -\frac{2\rho}{\beta}\, F\, \frac{\partial \zeta}{\partial C}\, F^T$ is such that :

♢ $Tr\left(\hat{T}_R \nabla \hat{W}\right) = 0$

♡ $\hat{T}_R = \begin{pmatrix} T_R & N \end{pmatrix}$ with $N = \rho\, U$ represents a momentum tensor $\hat{\boldsymbol{T}}_R$

♣ $\hat{T}_R\, \hat{W} = \left(\zeta - \frac{\partial \zeta}{\partial W}\, W\right)\, N$

First principle

ζ is the prototype of **thermodynamic potentials** :

- the **internal energy** $e_{int} = -\dfrac{\partial \zeta_{int}}{\partial \beta}$
- the **specific entropy** $s = \zeta_{int} - \beta \, \dfrac{\partial \zeta_{int}}{\partial \beta}$ of which the Galilean 4-vector $\vec{\boldsymbol{S}} = \hat{\mathbf{T}}_R \, \hat{\vec{\boldsymbol{W}}}$ is the 4-flux

$$\vec{\boldsymbol{S}} = s \, \vec{\boldsymbol{N}}$$

- the **free energy** $\psi = -\dfrac{1}{\beta} \, \zeta_{int} = -\theta \, \zeta_{int}$ allows to recover

$$-e_{int} = \theta \, \frac{\partial \psi}{\partial \theta} - \psi, \qquad -s = \frac{\partial \psi}{\partial \theta}$$

The interest of Planck's potential ζ is that it generates all the other ones

Second principle

Additive decomposition of the momentum tensor

$\hat{\mathbf{T}} = \hat{\mathbf{T}}_R + \hat{\mathbf{T}}_I$ with

- the reversible part $\hat{\mathbf{T}}_R$ represented by :

$$\hat{T}_R = \begin{pmatrix} \mathcal{H}_R & -p^T & \rho \\ \mathcal{H}_R v - \sigma_R v & \sigma_R - v p^T & \rho v \end{pmatrix}$$

- the irreversible one $\hat{\mathbf{T}}_I$ represented by :

$$\hat{T}_I = \begin{pmatrix} \mathcal{H}_I & 0 & 0 \\ h + \mathcal{H}_I v - \sigma_I v & \sigma_I & 0 \end{pmatrix}$$

where σ_I are the **dissipative stresses** and $\mathcal{H}_I = -\rho q_I$ is the dissipative part of the energy due to the **irreversible heat sources** q_I

- **Thermodynamics of irreversible processes (TIP)** : Planck's potential depends on extra internal variables (for instance the plastic strain E^p)

Second principle

Time arrow

Linear form $\boldsymbol{e}^0 = dt$ represented by an invariant row under Galilean transformation :

$$e^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

Covariant form of the second principle

The **local production of entropy** of a medium caracterized by a temperature vector $\hat{\vec{\boldsymbol{W}}}$ and a momentum tensor $\hat{\mathbf{T}}$ is non negative :

$$\Phi = \boldsymbol{Div}\ \left(\hat{\boldsymbol{T}}\ \hat{\vec{\boldsymbol{W}}}\right) - \left(\boldsymbol{e}^0(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\ \left(\boldsymbol{e}^0(\boldsymbol{T}_l(\vec{\boldsymbol{U}}))\right) \geq 0$$

and vanishes if and only if the process is reversible
[de Saxcé & Vallée IJES 2012]

Second principle

- The local production of entropy

$$\Phi = \boldsymbol{Div}\ \left(\hat{\boldsymbol{T}}\ \hat{\vec{\boldsymbol{W}}}\right) - \left(\boldsymbol{e}^0(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\ \left(\boldsymbol{e}^0(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right)$$

is a **Galilean invariant !**

- After some manipulations, it can be putted in the classical form of **Clausius-Duhem inequality**

$$\Phi = \rho\ \frac{ds}{dt} - \frac{\rho}{\theta}\ \frac{dq_I}{dt} + div\ \left(\frac{h}{\theta}\right) \geq 0$$

Galilean charts

Theorem

A necessary and sufficient condition for the Jacobian matrix $P = \frac{\partial X'}{\partial X}$ of a coordinate change $X \mapsto X'$ being a linear Galilean transformation is that this change is compound of a rigid motion and a clock change :

$$x' = (R(t))^T (x - x_0(t)), \qquad t' = t + \tau_0$$

- The local charts that are deduced one from each other by such changes are called **Galilean charts**.
- G being the group of linear Galilean transformations, this theorem shows that the G-**structure** [Kobayashi 1963] is **integrable**

Galilean gravitation

Theorem

The **Galilean connexions**, that is the symmetric connections of which the matrix Γ belongs to the Lie algebra of Galileo's group, are such that :

$$\Gamma(dX) = \begin{pmatrix} 0 & 0 \\ \Omega \times dx - g\, dt & j(\Omega)\, dt \end{pmatrix},$$

where $j(u)$ is the unique skew-symmetric matrix such that $j(u)v = u \times v$

- g is the classical **gravity**
- Ω is a new object called **spinning**

Equation of motion of a particle

- $T = m\,U$ being the linear 4-momentum, the covariant equation of motion reads :

$$\nabla T = dT + \Gamma(dX)\,T = 0$$

or in tensor notations

$$\nabla T^{\alpha} = dT^{\alpha} + \Gamma^{\alpha}_{\mu\beta} dX^{\mu} T^{\beta} = 0$$

- In the Galilean charts, its general form is

$$\dot m = 0, \quad \dot p = m\,(g - 2\,\Omega \times v)$$

[Souriau, Structure des systèmes dynamiques, 1969]

- It allows to explain simply the motion of Foucault's pendulum without neglecting the centripetal force as in the classical textbook

The hidden agenda ...

- Galileo's group does not preserve space-time metrics
- Bargmann's group preserves the metrics $\hat{ds}^2 = \| dx \|^2 - 2\, dz\, dt$, then the space $\hat{\mathcal{M}}$ is a riemanniennian manifold and, in this case, the G-structure is not in general integrable, the obstruction being the curvature.
- In other words, we are going to work, up to now, in linear frames which are not associated to local coordinates (moving frames)
- Hence we have to find frames associated to coordinate systems (natural frames)

Thermodynamics and Galilean gravitation

- With the **potentials of the Galilean gravitation** ϕ, A such that

$$g = -grad\,\phi - \frac{\partial A}{\partial t}, \quad \Omega = \frac{1}{2}\,curl\,A$$

the Lagrangian is $\mathcal{L}(t, x, v) = \frac{1}{2}\,m \parallel v \parallel^2 - m\,\phi + m\,A \cdot v$

- that suggests to introduce a coordinate change

$$dz' = \frac{\mathcal{L}}{m}\,dt = dz - \phi\,dt + A \cdot dx\ , \quad dt' = dt, \quad dx' = dx$$

- In the new chart, the **Bargmannian connection** is

$$\hat{\Gamma}(d\hat{X}) = \begin{pmatrix} 0 & 0 & 0 \\ j(\Omega)\,dx - g\,dt & j(\Omega)\,dt & 0 \\ \left(\frac{\partial \phi}{\partial t} - A \cdot g\right)\,dt & [(grad\,\phi - \Omega \times A)\,dt & 0 \\ +\,(grad\,\phi - \Omega \times A) \cdot dx & -grad_s A\,dx^T & \end{pmatrix}$$

Thermodynamics and Galilean gravitation

The developments are similar to the ones in absence of gravitation but with some exceptions :

- Planck's potential becomes $\zeta = \zeta_{int} + \frac{\beta}{2} \parallel v \parallel^2 - \beta\,\phi + A \cdot w$
- the Hamiltonian becomes $\mathcal{H} = \rho \left(e_{int} + \frac{1}{2} \parallel v \parallel^2 + \phi - q_I \right)$,
- the linear momentum becomes $p = \rho\,(v + A)$.

In presence of gravitation, the first principle restitutes the balance equations of the mass and of

- the linear momentum : $\rho\,\dfrac{dv}{dt} = (div\ \sigma)^T + \rho\,(g - 2\,\Omega \times v)$
- the energy : $\dfrac{\partial \mathcal{H}}{\partial t} + div\ \ (h + \mathcal{H} v - \sigma v) = \rho \left(\dfrac{\partial \phi}{\partial t} - \dfrac{\partial A}{\partial t} \cdot v \right)$

A smidgen of relativistic Thermodynamics

- We come back to the relativistic model with **Lorentz-Poincaré** symmetry group

- In this approach, the temperature is transformed according to

$$\theta' = \frac{\theta}{\gamma} = \theta \sqrt{1 - \frac{\| v \|^2}{c^2}}$$

  This the **temperature contraction** !

- thanks to the space-time Minkowski's metrics $ds^2 = c^2 dt^2 - \| dx \|^2$, we can associate to the 4-velocity $\vec{\boldsymbol{U}}$ one and only one linear form $\boldsymbol{U}^*$ represented by

$$U^T G = \left(\gamma, \gamma\, v^T\right) \begin{pmatrix} c^2 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{pmatrix} = c^2 \left(\gamma, -\frac{1}{c^2} \gamma\, v^T\right) ,$$

  which approaches $c^2 \boldsymbol{e}^0$ when c approaches $+\infty$

A smidgen of relativistic Thermodynamics

On this ground, we replace $\boldsymbol{e}^0$ by $\boldsymbol{U}^*/\ c^2$ in the Galilean expression of the 2^{nd} principle, that lead to

Relativistic form of the 2^{nd} principle

The **local production of entropy** of a medium characterized by a temperature vector $\vec{\mathbf{W}}$, a momentum tensor $\hat{\mathbf{T}}$, a potential ζ and a 4-flux of mass $\vec{\boldsymbol{N}}$ is non negative :

$$\Phi = \boldsymbol{Div}\ \left(\boldsymbol{T}\,\vec{\boldsymbol{W}} + \zeta\,\vec{\boldsymbol{N}}\right) - \frac{1}{c^2}\left(\boldsymbol{U}^*(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\frac{1}{c^2}\left(\boldsymbol{U}^*(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right) \geq 0\ ,$$

and vanishes if and only if the process is reversible

Thank you !

A fragance of symplectic geometry

- In Physics, a powerful tool is the **symmetry group**, a Lie group G acting on a symplectic manifold $(\mathcal{N}, \omega)$ by $a \mapsto a \cdot \eta$

  We denote $\mathfrak{g}^*$ the dual of its Lie algebra $\mathfrak{g}$

- $\eta \mapsto \mu = J(\eta) \in \mathfrak{g}^*$ is a **momentum map** (Souriau) if

$$\forall Z \in \mathfrak{g}, \qquad \omega(Z_{\mathcal{N}}, \bullet) = -d(\langle J(\eta), Z \rangle)$$

  It allows to recover **integrals of the motion** (modern version of Noether's theorem)

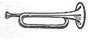
- G naturally acts on $\mathfrak{g}$ by the adjoint representation $Ad(a)\, Z = a\, Z\, a^{-1}$ and on $\mathfrak{g}^*$ by the induced action Ad^* (**coadjoint representation**)

A fragance of symplectic geometry

- **Theorem** Souriau

  There exists $\sigma : G \mapsto \mathfrak{g}^*$ called a **symplectic cocycle** such that $J(a \cdot \eta) - Ad^*(a)\, J(\eta) = \sigma(a)$

- modulo a coboundary, it defines a **class of symplectic cohomology** $[\sigma] \in H^1(G; \mathfrak{g}^*)$, generally null.

-  A noticeable exception is **Galileo's group**, the symmetry group of the classical mechanics

- **Bargmann's group** $\hat{G} = G \times \mathbb{R}$, equipped with the operation :

$$(a, b)\,(a', b') = \left(aa', b + b' + f\,(a, a')\right) \ ,$$

is a **central extension** of G by $\mathbb{R}$ where f is constructed from σ in such way that **its class of symplectic cohomology is null**