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SUR LES LOIS DE COMPORTEMENT

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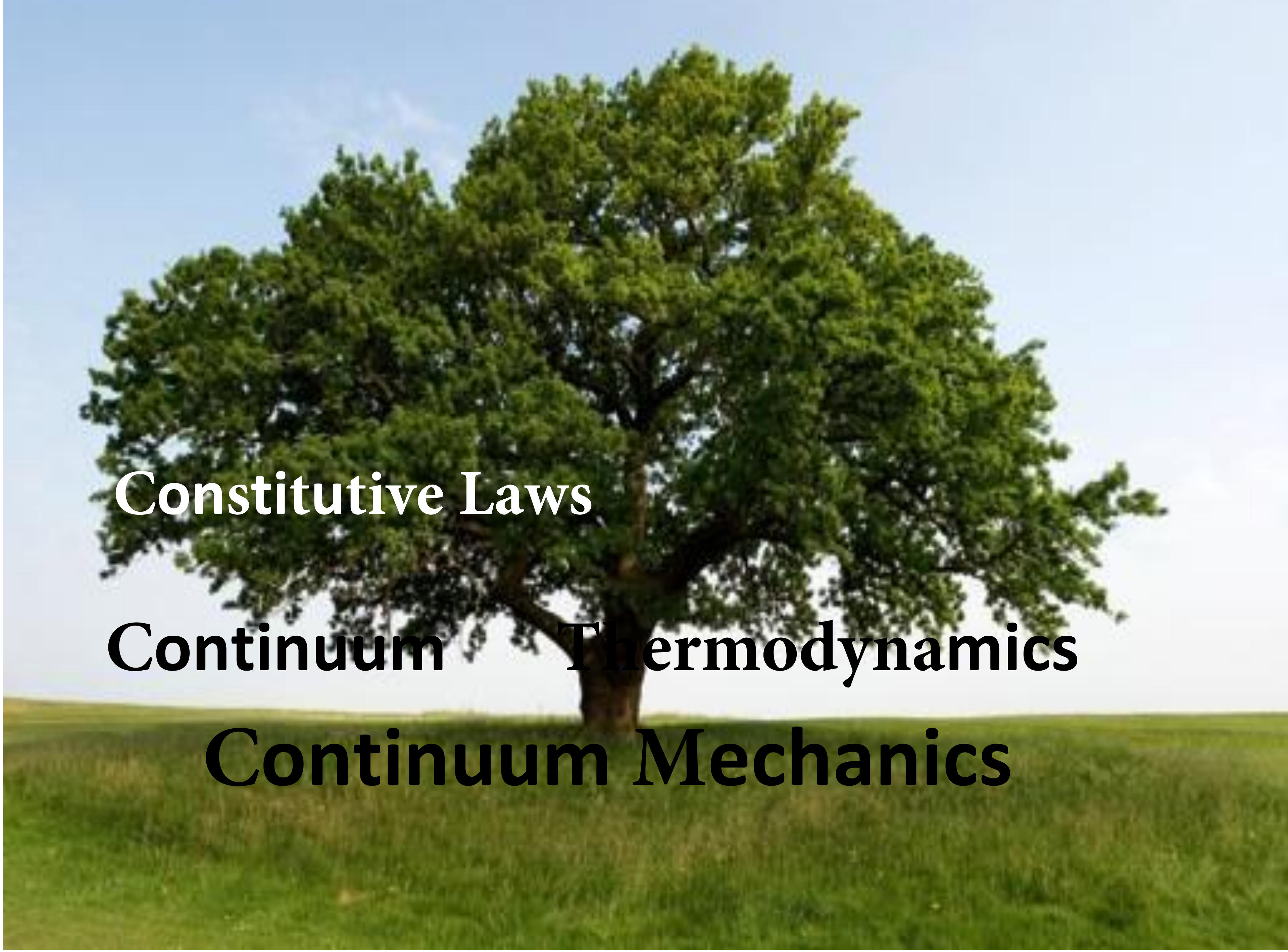
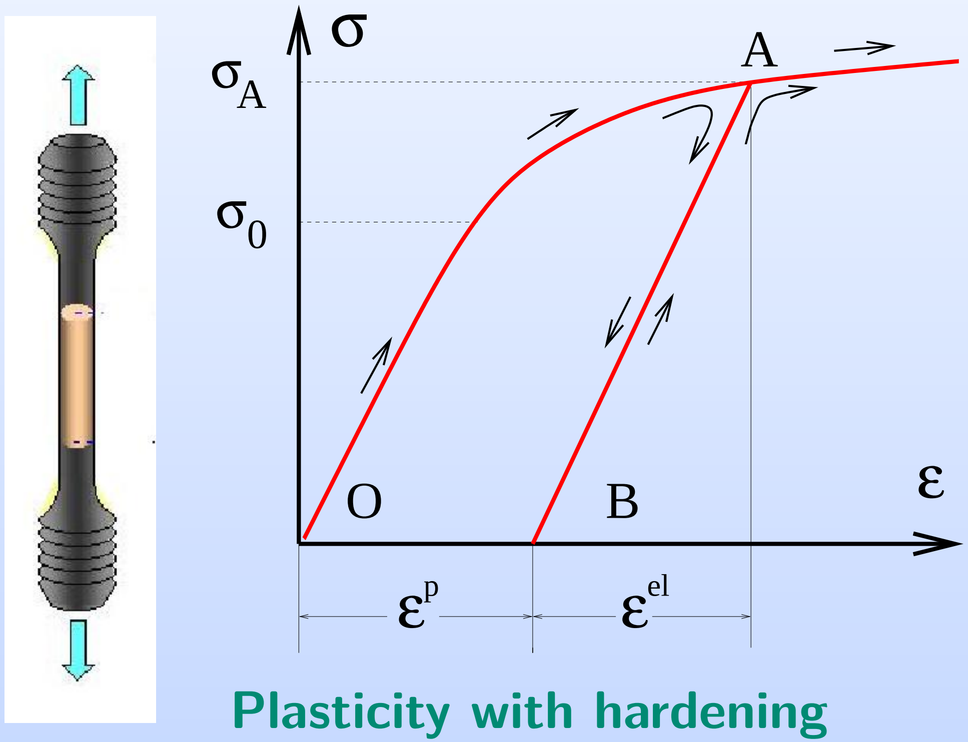
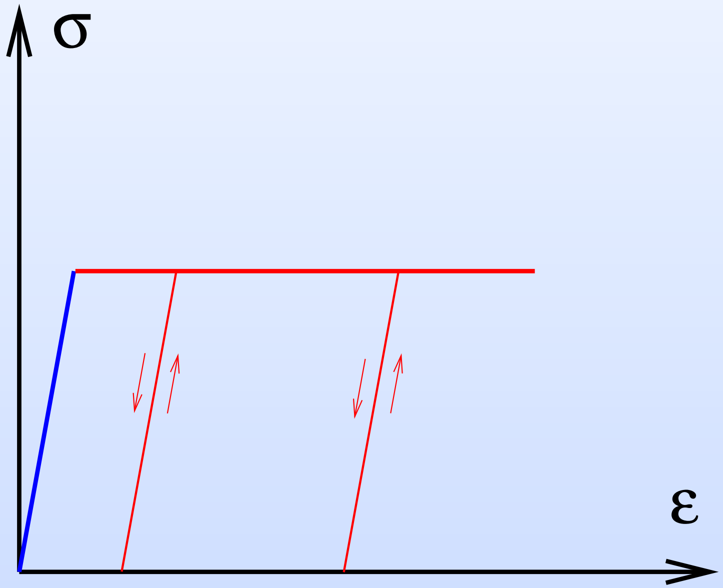
Thanks to Habibou Maitournam, Imsia, Ensta


Constitutive Laws
Continuum
Thermodynamics
Continuum Mechanics

a macroscopic view of plasticity

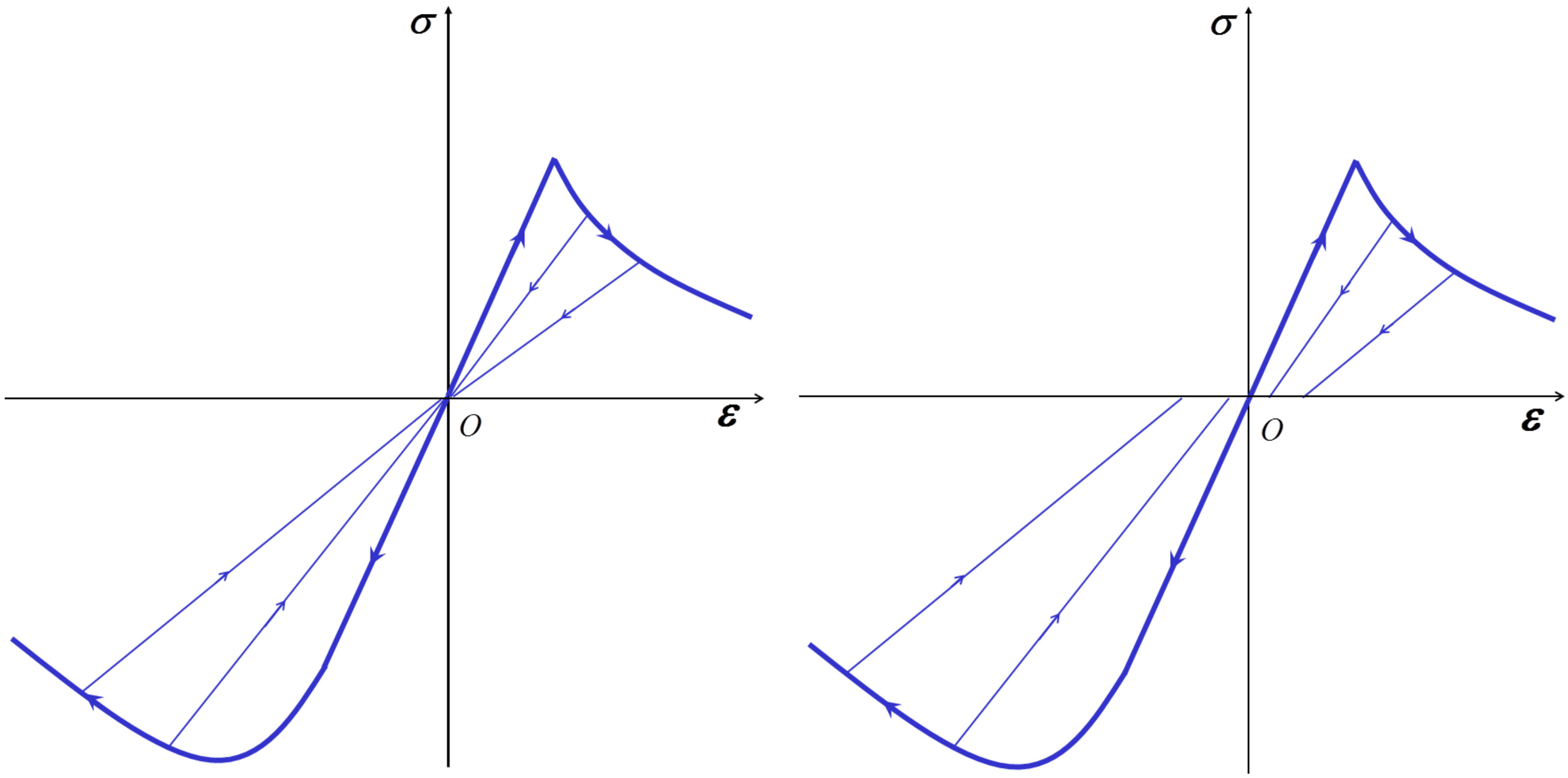


Plasticity with hardening



Perfect (ideal) Plasticity

Damage manifestation in quasi brittle materials

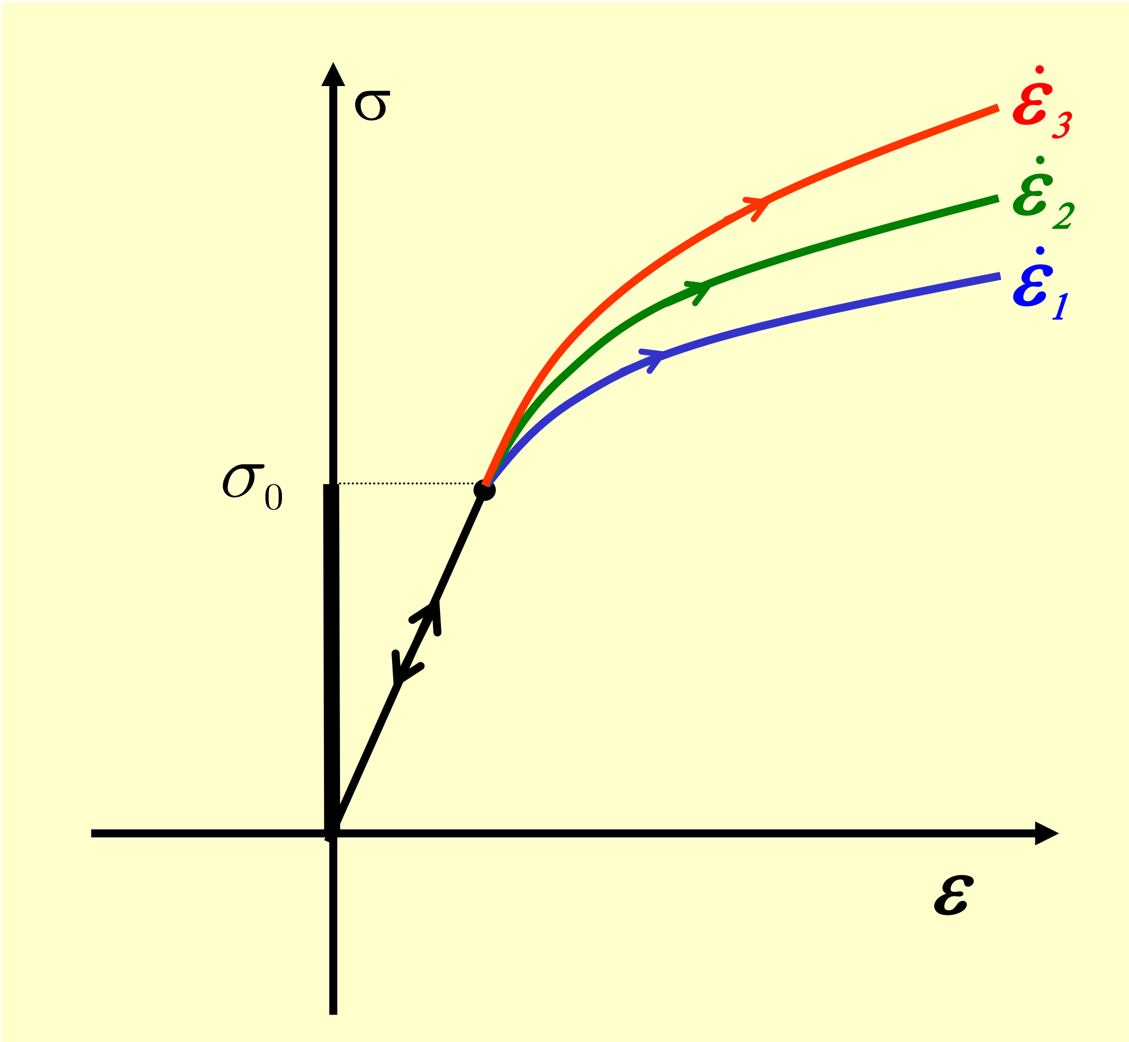


Mechanical response under tension-compression

- Degradation of materials properties during loading
- Occurence of a softening regime
- Dissymetry between tension & compression

Courbes de traction à différentes vitesses de déformation

Chaque essai est effectué avec une vitesse de déformation constante.



$$\dot{\varepsilon}_3 > \dot{\varepsilon}_2 > \dot{\varepsilon}_1$$

Feuille de route!

Objectif : Introduire un cadre général de formulation des lois de comportements thermomécaniques et montrer quelques illustrations

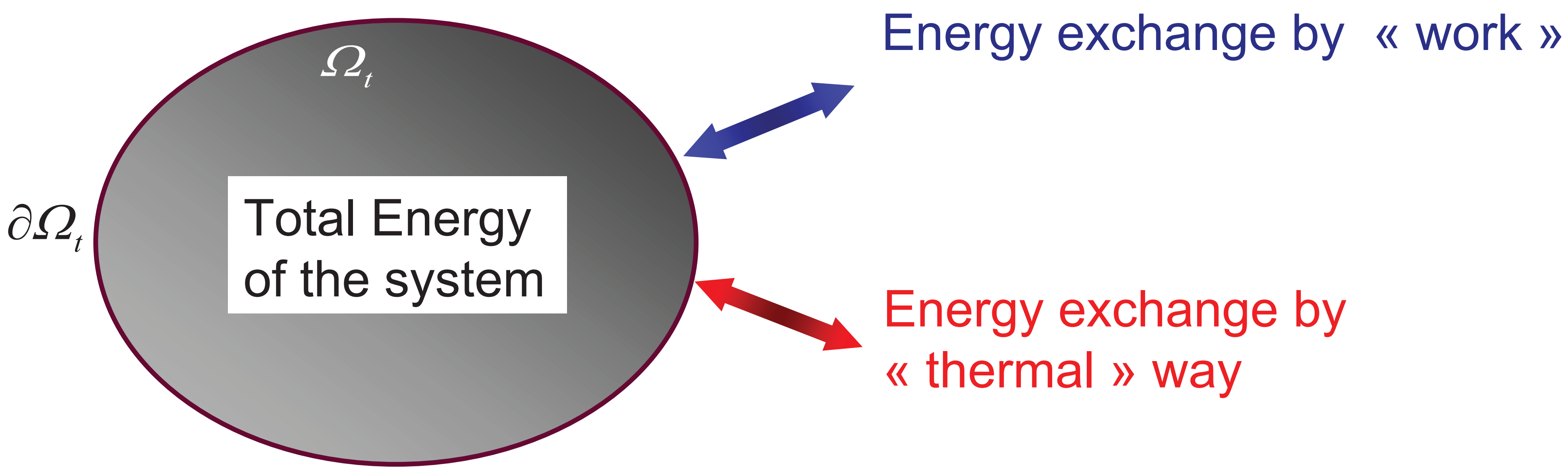
Cadre thermodynamique des lois de comportements dissipatifs

Application à l'élasto(visco)plasticité

Couplage élasticité - endommagement

First principle of thermodynamics

There exists a state function, called the internal energy, S, such that:



Variation of internal energy + Variation of kinetics energy = Quantity of the internal power + Heat supply per unit time

Variation of total energy

Note that the kinetics energy is not an objective quantity

- **1st principe :** $\dot{E} + \dot{C} = P_{(e)} + \overset{o}{Q}$
- **Th. of K.E. :** $P_{(i)} + P_{(e)} = \dot{C}$

$\Rightarrow$ $\dot{E} = -P_{(i)} + \overset{o}{Q}$

global form

with (unity= Joule):

$$E = \int_{\Omega_t} \rho(\underline{x},t)\, e(\underline{x},t)\, \mathrm{d}\Omega_t$$

$$P_{(i)} = \int_{\Omega_t} -\underline{\underline{\sigma}}(\underline{x},t) : \dot{\underline{\underline{\varepsilon}}}(\underline{x},t)\, \mathrm{d}\Omega_t$$

and (unity= watt)

$$\overset{0}{Q} = \int_{\partial\Omega_t} -\underline{q}(\underline{x},t).\underline{n}\, \mathrm{d}a + \int_{\Omega_t} r(\underline{x},t)\, \mathrm{d}\Omega_t$$

r=heat generated within the volume by external agencies (inductive heating for instance)

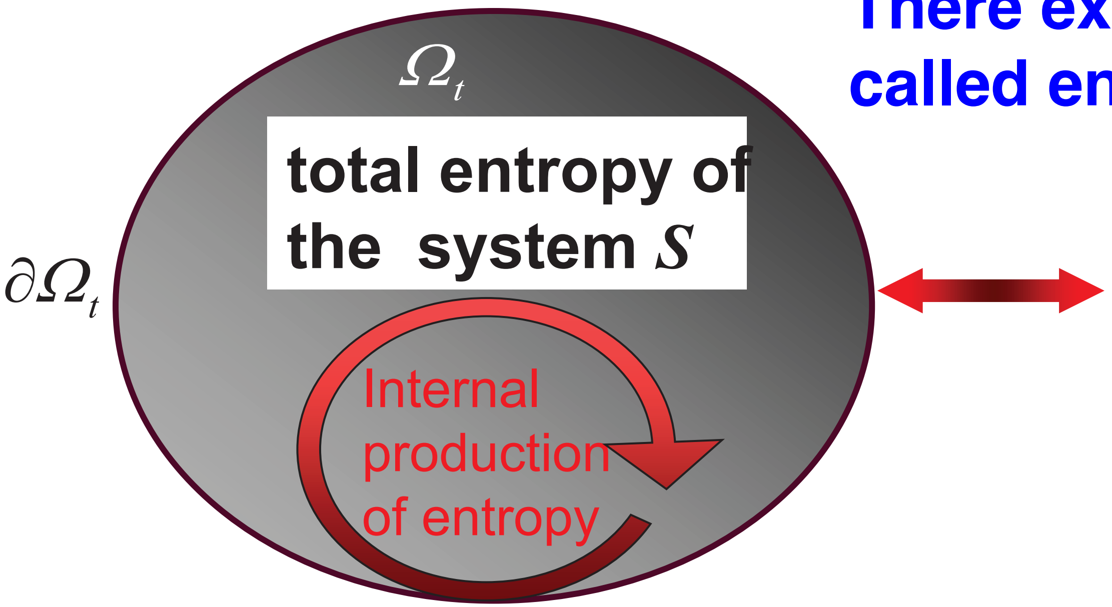
q= heat received by conduction through the boundary.

==> local form :

$$\rho\dot{e} = \underbrace{\underline{\underline{\sigma}} : \dot{\underline{\underline{\varepsilon}}}}_{\text{"Internal power"}} + \underbrace{r - \operatorname{div} \underline{q}}_{\text{heat}}$$

Second principle of thermodynamics

There exists another state function, S, called entropy, such that:



Entropy Exchange by « thermal » way

$$\underbrace{\dot{S} - \int_{\Omega_t} \frac{r(\underline{x},t)}{T(\underline{x},t)}\, \mathrm{d}\Omega_t + \int_{\partial\Omega_t} \frac{\underline{q}(\underline{x},t)\,.\,\underline{n}(\underline{x})}{T(\underline{x},t)}\, \mathrm{d}a}_{\textit{Entropy Production}\ \geq\ 0} \geq 0$$

For a reversible process, the Entropy production is equal to 0

Second principle : $\dot{S} - \int_{\Omega_t} \frac{r}{T}\,d\Omega_t + \int_{\partial\Omega_t} \frac{\underline{q}.\underline{n}}{T}\,da \geq 0$

$$S = \int_{\Omega_t} \rho(\underline{x},t)s(\underline{x},t)\,d\Omega_t$$

$$\int_{\partial\Omega_t} \frac{\underline{q}.\underline{n}}{T}\,da = \int_{\Omega_t} \mathrm{div}\left(\frac{\underline{q}}{T}\right) d\Omega$$

Local form : $\rho\dot{s} + \mathrm{div}\left(\frac{\underline{q}}{T}\right) - \frac{r}{T} \geq 0$ ***Multipliying by T one gets :***

$$\rho T\dot{s} + \left(\mathrm{div}\,\underline{q} - r\right) - \frac{\underline{q}.\nabla T}{T} \geq 0$$

D **: total dissipation**

Clausius-Duhem inequality

Total dissipation :

$$D = \rho \dot{T} s + \left(\operatorname{div} \underline{q} - r\right) - \frac{\underline{q}.\nabla T}{T} \geq 0$$

$$\rho \dot{e} = \underline{\underline{\sigma}} : \underline{\underline{\dot{\varepsilon}}} + r - \operatorname{div} \underline{q}$$

Clausius-Duhem Inequality:

$$D = \rho T \dot{s} - \rho \dot{e} + \underline{\underline{\sigma}} : \underline{\underline{\dot{\varepsilon}}} - \frac{\underline{q}.\nabla T}{T}$$

D_1 INTRINSIC dissipation

D_2 THERMAL dissipation

Assumption of Decoupled Dissipations

$D = D_1 + D_2 \geq 0$ replaced by $\begin{cases} D_1 \geq 0 \\ \text{and} \\ D_2 \geq 0 \end{cases}$

Again, for a thermodynamically reversible transformation, the dissipation is null

Method of local thermodynamics state

- **Thermodynamics state** at a time *t* : defined at any point point *M* by a finite number of **state variables**

$$= \chi_i$$

i.e. all thermodynamics quantities characterizing this state are defined by means of these variables

Example of state variables in continuum mechanics of solids:

- *the température :* $\chi_1 = T$ *or entropy* $\chi_1 = s$,
- *the deformation :* $\chi_2 = \underline{\underline{\varepsilon}}$,
- *anelastic deformation :* $\chi_3 = \underline{\underline{\varepsilon}}^{an}$,
- *damage :* $\chi_4 = \beta$, *COUPLED PHENOMENA, etc.*

- The deformation and T are called Normal State Variables
-
- The other (and independent) state variables are called internal variables

- It is generally preferable to consider ***T*** as state variable at the place of ***s.***

This motivates the consideration of an Helmholtz energy potential, $w(T, \underline{\underline{\varepsilon}}, \alpha)$, defined by means of the following transformation:

$$w(T, \underline{\underline{\varepsilon}}, \alpha) = e(s, \underline{\underline{\varepsilon}}, \alpha) - Ts$$

- With the free energy, the state equations then read:

State laws :

$$\begin{cases} s = -\dfrac{\partial w}{\partial T} \\ \underline{\underline{\sigma}}^{rev} = \rho \dfrac{\partial w}{\partial \underline{\underline{\varepsilon}}} \\ A^{rev} = \rho \dfrac{\partial w}{\partial \alpha} \end{cases}$$

Returning to materials with dissipative mechanisms

The intrinsic dissipation inequality can now be put in the following form:

$$D = \rho T\dot{s} - \rho\dot{e} \cdot \underline{\underline{\sigma}}:\underline{\underline{\dot{\varepsilon}}} - \frac{\underline{q}.\underline{\nabla T}}{T} \geq 0$$

$$e = w + Ts$$

$$\dot{e} = \dot{w} + T\dot{s} + s\dot{T}$$

$$D = \underline{\underline{\sigma}}:\underline{\underline{\dot{\varepsilon}}} - \rho\left(\dot{w} + s\dot{T}\right) - \frac{\underline{q}.\underline{\nabla T}}{T} \geq 0$$

Clausius-Duhem Inequality

$$\dot{w} = \frac{\partial w}{\partial \underline{\underline{\varepsilon}}}:\underline{\underline{\dot{\varepsilon}}} \quad \frac{\partial w}{\partial \alpha}.\dot{\alpha} + \frac{\partial w}{\partial T}\dot{T}$$

$$D = \left(\underline{\underline{\sigma}} - \rho\frac{\partial w}{\partial \underline{\underline{\varepsilon}}}\right):\underline{\underline{\dot{\varepsilon}}} - \rho\frac{\partial w}{\partial \alpha}.\dot{\alpha} - \rho\left(s + \frac{\partial w}{\partial T}\right)\dot{T} - \frac{\underline{q}}{T}.\underline{\nabla T} \geq 0$$

$$D = \underline{\underline{\sigma}}^{irr}:\underline{\underline{\dot{\varepsilon}}} + A^{irr}.\dot{\alpha} - \frac{\underline{q}}{T}.\underline{\nabla T} \geq 0$$

$$\begin{cases} \underline{\underline{\sigma}}^{irr} = \underline{\underline{\sigma}} - \underline{\underline{\sigma}}^{rev} \\ A^{irr} = -A^{rev} \end{cases}$$

Notation: $A = A^{irr}$

EVOLUTION LAWS: Normal Dissipativity (J-J. Moreau (1970)

By denoting $\dot{\chi} = (\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}, \underline{\nabla T})$ et $\boldsymbol{F} = (\underline{\underline{\sigma}}^{irr}, A, -\frac{\underline{q}}{T})$

An efficient way, conforted by experiments in numerous cases, to formulate these complementary laws is **to assume a normal dissipativit** by deducing

the irreversible thermodynamics forces $\boldsymbol{F}$ from a **a dissipation potential**,

convex function of fluxes $\dot{\chi}$ and the present state, non negative & null at $\dot{\chi}$ =***0***.

In the differentiable case, $\boldsymbol{F} = \frac{\partial \varphi(\dot{\chi})}{\partial \dot{\chi}}$

EX : THERMAL DISSIPATION : Fourier Law

$$\underline{q} = -K\,\underline{\nabla T}$$

which derives from a quadratic dissipation potential

Interest of the normal dissipativity assumption

It automatically ensures the positivity of the dissipation,

that is : $$\boxed{\mathcal{F} \cdot \dot{\chi} \geq 0}$$

Indeed, as $\varphi(\dot{\chi})$ is convex , one has :

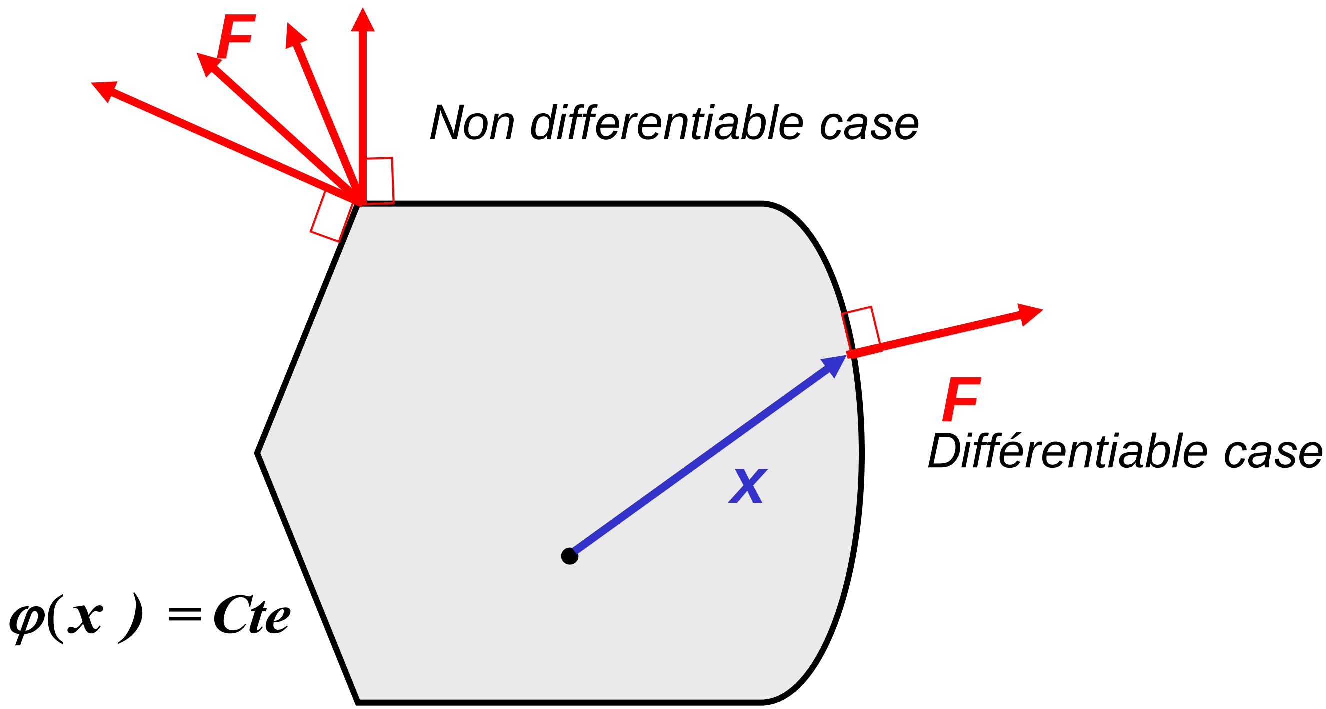
$$\forall \dot{Z} \qquad \varphi(\dot{\chi}) - \varphi(\dot{Z}) \leq \partial\varphi(\dot{\chi}) \cdot (\dot{\chi} - \dot{Z})$$

$$\text{pour } \dot{Z} = 0 \qquad \varphi(\dot{\chi}) \leq \partial\varphi(\dot{\chi}) \cdot \dot{\chi}$$

$$\text{and then} \qquad 0 \leq \varphi(\dot{\chi}) \leq \underbrace{\partial\varphi(\dot{\chi}) \cdot \dot{\chi}}_{\mathcal{F} \cdot \dot{\chi}}$$

In the case of a non differentiable potential, the classical notion of gradient must be generalized to that of sub-gradient defined as :

$$\mathcal{F} \in \partial\varphi(x) \quad \Leftrightarrow \quad \forall y \quad \varphi(y) - \varphi(x) \geq \mathcal{F}.(y - x)$$



Normal Dissipativity : $\left(\underline{\underline{\sigma}}^{irr}, A\right) \in \partial\varphi_{\left(\dot{\underline{\underline{\varepsilon}}},\dot{\alpha}\right)}\left(\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}, \underline{\underline{\varepsilon}}, \alpha, T\right)$

Generalized Standard Materials (GSM)

Application of normal dissipativity assumption to $\left(\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}\right)$

Materials defined by two potentials (Haphen & Nguyen, 1975, Nguyen, 2000):

1. Thermodynamics Potential (ex : Helmholtz Energy ***w***)
providing the state laws

$$\underline{\underline{\sigma}}^{rev} = \rho \frac{\partial w}{\partial \underline{\underline{\varepsilon}}}, \quad A = -\rho \frac{\partial w}{\partial \alpha}, \quad s = - \frac{\partial w}{\partial T}$$

2. Dissipation Potential

- convex with respect of flux $\left(\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}\right)$
- minimum in $\left(\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}\right) = (0,0)$

$$\underline{\underline{\sigma}}^{irr} = \frac{\partial \varphi\left(\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}, \underline{\underline{\varepsilon}}, \alpha, T\right)}{\partial \dot{\underline{\underline{\varepsilon}}}}, \quad A = \frac{\partial \varphi\left(\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}, \underline{\underline{\varepsilon}}, \alpha, T\right)}{\partial \dot{\alpha}}$$

or

$$\left(\underline{\underline{\sigma}}^{irr}, A\right) \in \partial \varphi_{\left(\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}\right)}\left(\dot{\underline{\underline{\varepsilon}}}, \dot{\alpha}, \underline{\underline{\varepsilon}}, \alpha, T\right)$$

Dual Potential

Inversion of of the above evolution laws:

$\left(\dot{\varepsilon},\dot{\alpha}\right)$ as function of $\left(\underline{\underline{\sigma}}^{irr},A\right)$

The dual potential, is defined by means of a Legendre-Fenchel transform :

$$\varphi^*(X)=\sup_Y\left(X.Y-\varphi(Y)\right)$$

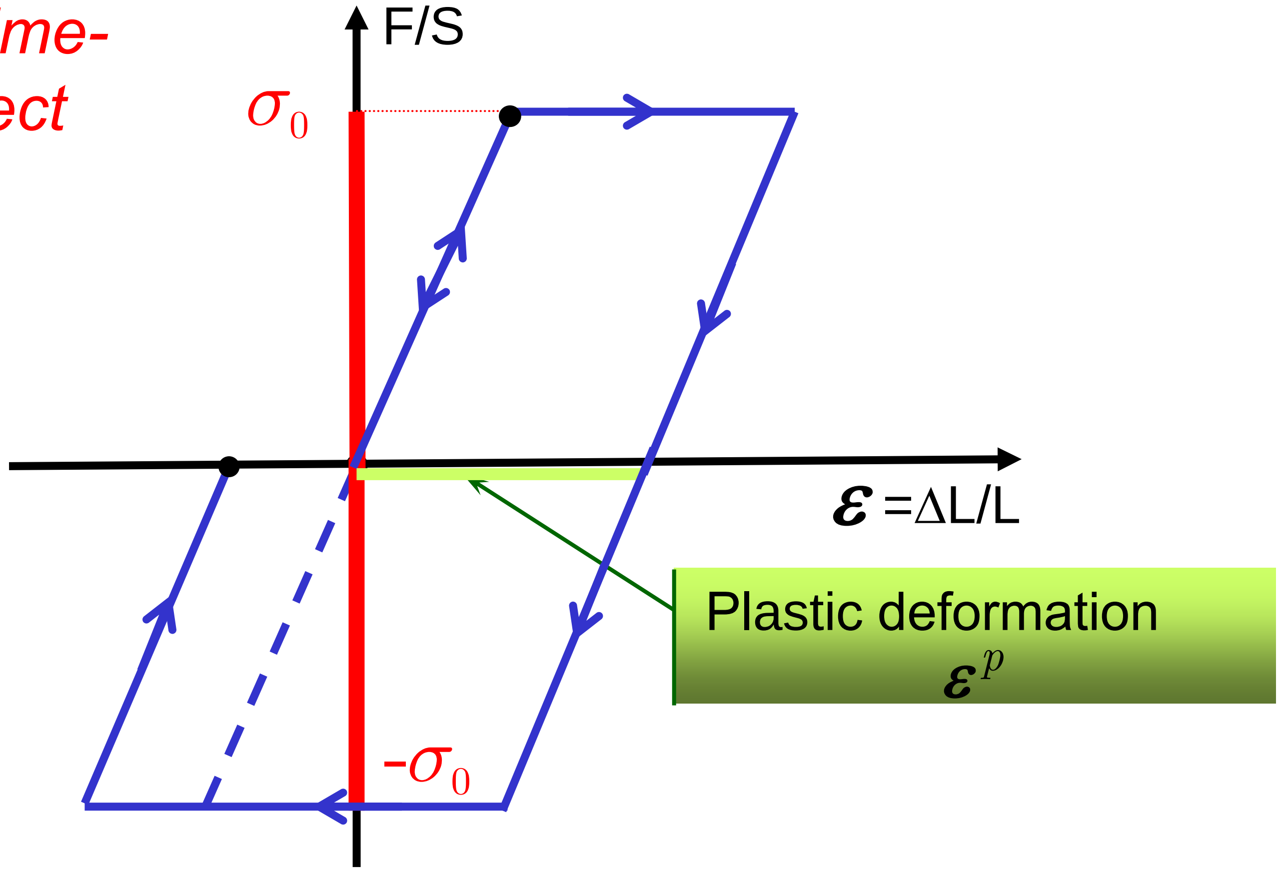
One has then : $\left(\underline{\underline{\dot{\varepsilon}}},\dot{\alpha}\right)\in\partial\varphi^*_{(\underline{\underline{\sigma}}^{irr},A)}\left(\underline{\underline{\sigma}}^{irr},A,\underline{\underline{\varepsilon}},\alpha,T\right)$

In the differentiable case:

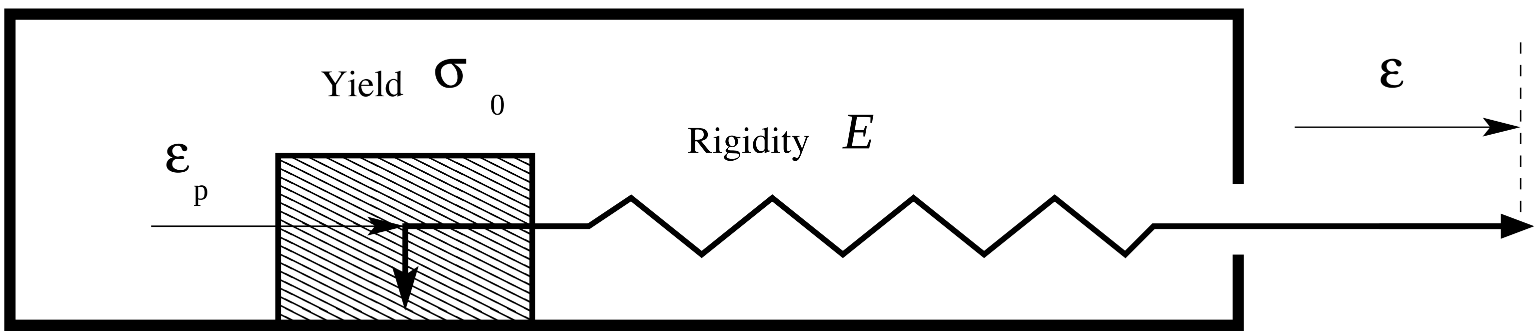
$$\begin{cases}\underline{\underline{\dot{\varepsilon}}}=\dfrac{\partial\varphi^*\left(\underline{\underline{\sigma}}^{irr},A,\underline{\underline{\varepsilon}},\alpha,T\right)}{\partial\underline{\underline{\sigma}}^{irr}}\\[2ex]\dot{\alpha}=\dfrac{\partial\varphi^*\left(\underline{\underline{\sigma}}^{irr},A,\underline{\underline{\varepsilon}},\alpha,T\right)}{\partial A}\end{cases}$$

ELASTO(VISCO)PLASTICITY

1D isothermal & time-independent Perfect Elastoplasticity



Corresponding Rheological model



- **State variables :** $(\varepsilon\,,\ \varepsilon^{p}\ )$

- **Thermodynamics potential :**

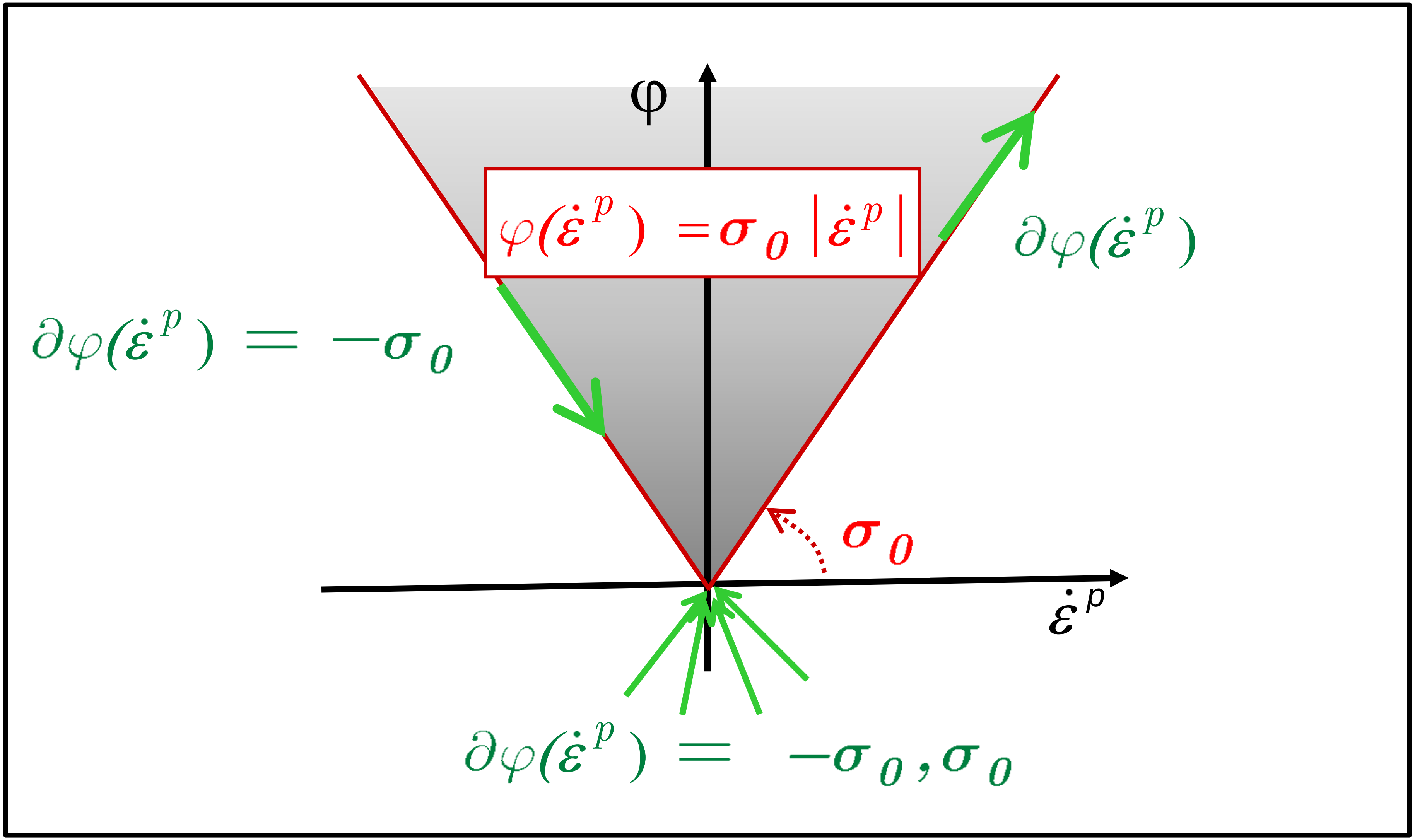
$$w\ (\varepsilon\,,\ \varepsilon^{p}) = \frac{1}{2}E(\varepsilon - \varepsilon^{p}\ )^{2} = \frac{1}{2}E\ \ {\varepsilon^{e}}^{\ 2}$$

$$\begin{cases} \sigma^{\text{rev}} = \dfrac{\partial w}{\partial \varepsilon} = E\ (\varepsilon - \varepsilon^{p}\ ) \\ A_{\varepsilon^{p}} = -\dfrac{\partial w}{\partial \varepsilon^{p}} = E\ (\varepsilon - \varepsilon^{p}\ ) = \sigma^{\text{rev}} \end{cases}$$

- **Identification of the dissipation potential :**

The dissipation potential : $\sigma_{0}\,|\dot{\varepsilon}^{p}|$

- **The sub-gradient of φ :**



• The dual potential

$$\varphi^*(\boldsymbol{\sigma}) = \sup_{\boldsymbol{x}}(\boldsymbol{\sigma x} - \varphi(\boldsymbol{x})) = \sup_{\boldsymbol{x}}(\boldsymbol{\sigma x} - \boldsymbol{\sigma_0}|\boldsymbol{x}|)$$

$$= \sup_{|\boldsymbol{x}|}(|\boldsymbol{\sigma}||\boldsymbol{x}| - \boldsymbol{\sigma_0}|\boldsymbol{x}|) = \sup_{|\boldsymbol{x}|}(|\boldsymbol{\sigma}| - \boldsymbol{\sigma_0})|\boldsymbol{x}|$$

$$\varphi^*(\boldsymbol{\sigma}) = \begin{cases} 0 & \text{if} \quad |\boldsymbol{\sigma}| - \boldsymbol{\sigma_0} \leq 0 \\ +\infty & \text{if} \quad |\boldsymbol{\sigma}| - \boldsymbol{\sigma_0} > 0 \end{cases}$$

then

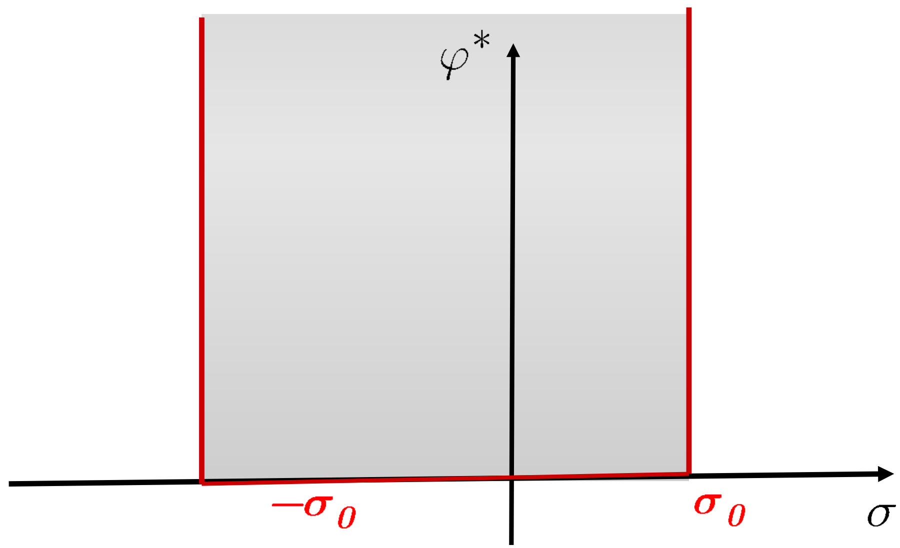
$$\varphi^*(\boldsymbol{\sigma}) = \begin{cases} 0 & \text{if} \quad \boldsymbol{\sigma} \in C \\ +\infty & \text{if} \quad \boldsymbol{\sigma} \notin C \end{cases}$$

avec $C = [-\boldsymbol{\sigma_0}, \boldsymbol{\sigma_0}]$

The dual potential is the indicator function of the convex $C = [-\boldsymbol{\sigma_0}, \boldsymbol{\sigma_0}]$

- **The dual potential**

$$\varphi^*(\sigma) = \sup_{\dot{\boldsymbol{\varepsilon}}^p} \left(\sigma \dot{\boldsymbol{\varepsilon}}^p - \varphi(\dot{\boldsymbol{\varepsilon}}^p)\right)$$



- **The sub-gradient of the dual potential:**

$$\dot{\boldsymbol{\varepsilon}}^p \in \partial\varphi^*(\sigma) = \begin{cases} 0 & \text{if} \quad \boldsymbol{\sigma} \in \overset{o}{C} =]-\boldsymbol{\sigma_0}, \boldsymbol{\sigma_0}[\\ \lambda \dfrac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} & \text{if} \quad |\boldsymbol{\sigma}| = \boldsymbol{\sigma_0} \quad \textit{with} \quad \lambda \geq 0, \; f(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}| - \boldsymbol{\sigma_0} \end{cases}$$

3D perfect Elastoplastic model

State Variables: $\underline{\underline{\boldsymbol{\varepsilon}}}\ ,\ \underline{\underline{\boldsymbol{\varepsilon}}}^p$

- **Helmholtz Free Energy**

$$\rho w\,(\underline{\underline{\boldsymbol{\varepsilon}}}\,,\ \underline{\underline{\boldsymbol{\varepsilon}}}^p) = \rho w\,(\underline{\underline{\boldsymbol{\varepsilon}}} - \underline{\underline{\boldsymbol{\varepsilon}}}^p) = \frac{1}{2}(\underline{\underline{\boldsymbol{\varepsilon}}} - \underline{\underline{\boldsymbol{\varepsilon}}}^p) : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : (\underline{\underline{\boldsymbol{\varepsilon}}} - \underline{\underline{\boldsymbol{\varepsilon}}}^p)$$

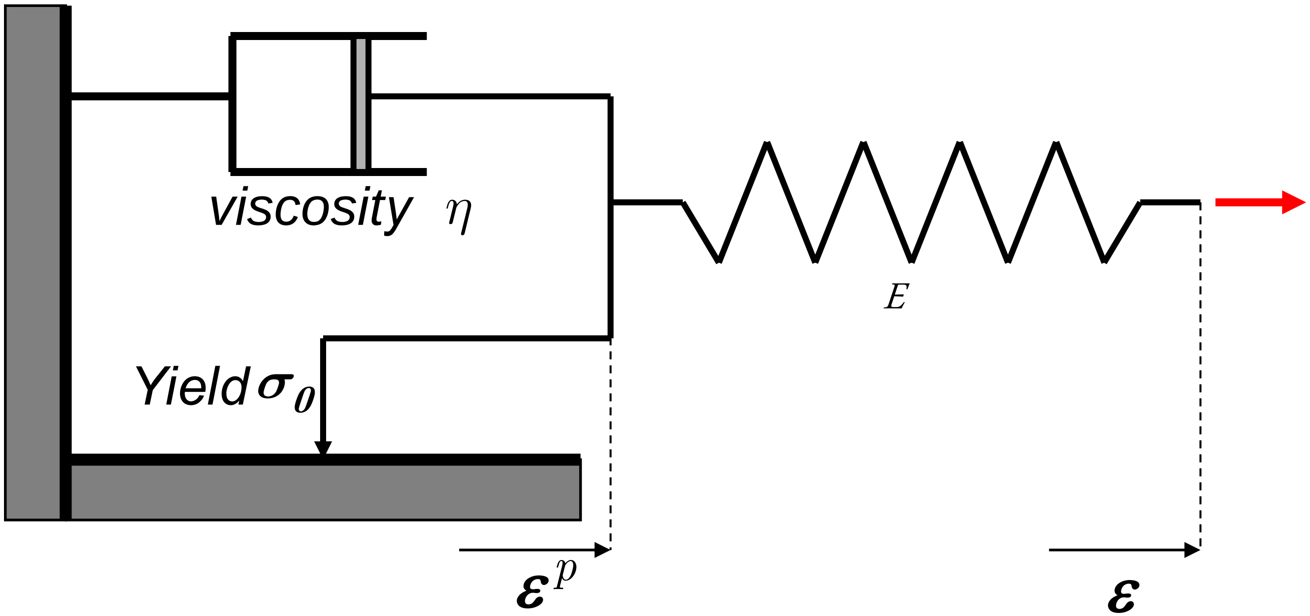
$$\underline{\underline{\boldsymbol{\sigma}}} = \rho\frac{\partial w}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}} = \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : (\underline{\underline{\boldsymbol{\varepsilon}}} - \underline{\underline{\boldsymbol{\varepsilon}}}^p) \qquad A_{\varepsilon^p} = -\rho\frac{\partial w}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}^p} = \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : (\underline{\underline{\boldsymbol{\varepsilon}}} - \underline{\underline{\boldsymbol{\varepsilon}}}^p)$$

- **Convex Yield Function *f***

The evolution laws are given by the normality rule

$$\dot{\underline{\underline{\boldsymbol{\varepsilon}}}}^p = \lambda\frac{\partial f(\underline{\underline{\boldsymbol{\sigma}}})}{\partial \underline{\underline{\boldsymbol{\sigma}}}} \quad \text{with} \quad \lambda f(\underline{\underline{\boldsymbol{\sigma}}})=0, \quad \lambda\geq 0, \quad f(\underline{\underline{\boldsymbol{\sigma}}})\leq 0$$

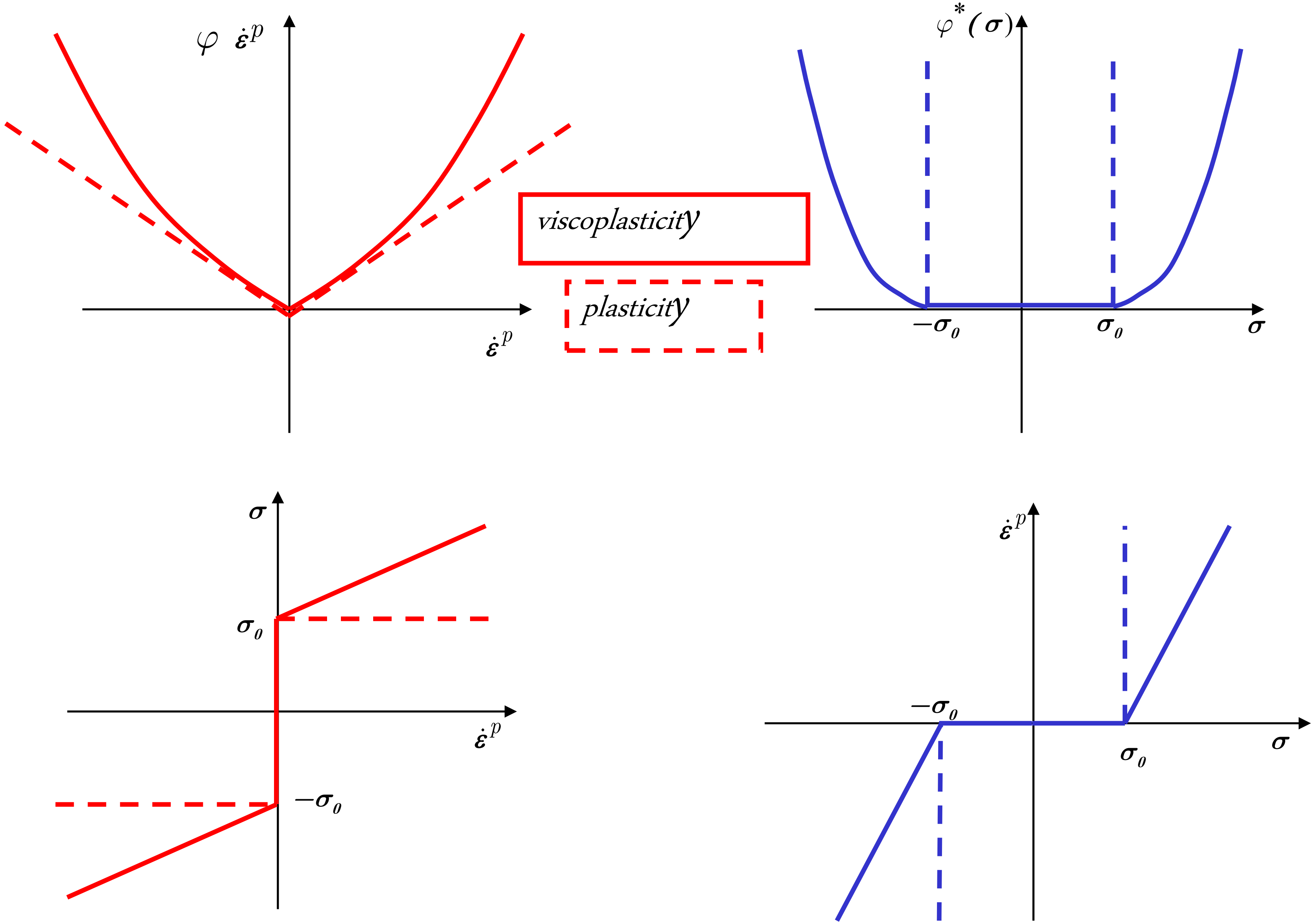
Viscoplastic regularization (Bingham Fluid model)



- **State variables :** $\boldsymbol{\varepsilon}$ **,** $\boldsymbol{\varepsilon}^{p}$
- **Thermodynamics potential :**
  - ➢ Same Free energy and state laws as before

- **Dissipation potential :**

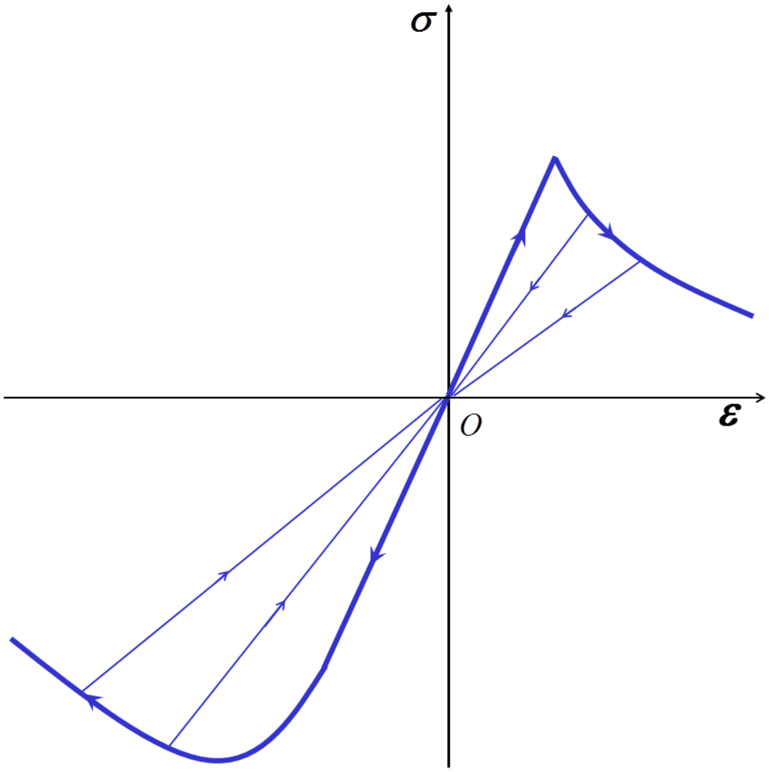
Dissipation potential : $\varphi(\dot{\boldsymbol{\varepsilon}}^{p}) = \boldsymbol{\sigma_0} \left| \dot{\boldsymbol{\varepsilon}}^{p} \right| + \frac{1}{2} \eta (\dot{\boldsymbol{\varepsilon}}^{p})^{2}$

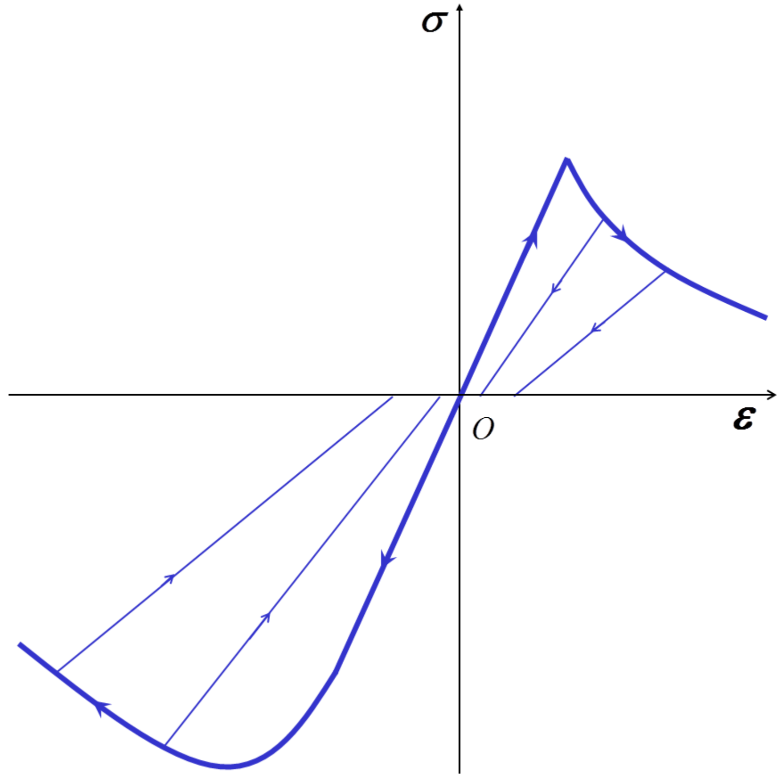


$\varphi \quad \dot{\varepsilon}^p$
$\dot{\varepsilon}^p$
viscoplasticity
plasticity
$\varphi^*(\sigma)$
$-\sigma_0$
σ_0
σ
σ
σ_0
$\dot{\varepsilon}^p$
$-\sigma_0$
$\dot{\varepsilon}^p$
$-\sigma_0$
σ_0
σ

Damage modeling

Damage modeling





Mechanical response under tension-compression

- **Degradation of materials properties during loading**

Un exemple de modèle d'endommagement régularisé

Cadre thermo. approprié, voir Nguyen Q. S. "Quasi-static responses and variational principles in gradient plasticity", JMPS, 97 (2016) 156–167

Choix des potentiels

$$\begin{cases} \rho w\big(\underline{\varepsilon}, d\big) = \frac{1}{2}\underline{\varepsilon} : \underline{\underline{\underline{C}}}(d) : \underline{\varepsilon} \\ \varphi(\dot{d}, \underline{\nabla}\dot{d}; d, \underline{\nabla} d) = Y_c(d)\dot{d} + l^2 w_1 \underline{\nabla} d \cdot \underline{\nabla}\dot{d} + I_+(\dot{d}), \quad Y_c(d) > 0 \end{cases}$$

Équations d'évolution

$$(\Sigma_2) \begin{cases} \dot{d} \geqslant 0 \\ l^2 w_1 \Delta d - \frac{1}{2}\underline{\varepsilon} : \underline{\underline{\underline{C}}}'(d) : \underline{\varepsilon} - Y_c(d) \leqslant 0 \text{ dans } \Omega \\ \underline{\nabla} d \cdot \underline{n} = 0 \text{ sur } \partial\Omega \end{cases}$$

Formulation variationnelle en vitesse

$$J(\dot{\underline{\varepsilon}}^*, \dot{d}^*) \;=\; \int_\Omega \Big(\rho\dot{w}^* \;+\; Y_c(d^*)\dot{d}^* \;+\; l^2 w_1 \underline{\nabla} d^* \cdot \underline{\nabla}\dot{d}^*\Big) d\Omega \;-\; \int_{\partial\Omega} \underline{T} \cdot \dot{\underline{u}}^* \; dS$$

Cas des systèmes à dissipation simple

L'exemple qui précède conduit à un système à dissipation simple (voir Ehrlacher, 1985 ; Fedelich et Ehrlacher, 1988).

Formulation variationnelle totale

Par intégration en temps, et en considérant que les données au bord sont en déplacement, la fonctionnelle à minimiser prend la forme

$$\Psi(\underline{u}, d) = \int_{\Omega} \Big(\underbrace{\frac{1}{2} \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{C}}}}(d) : \underline{\underline{\varepsilon}}}_{\text{énergie élastique}} + \underbrace{w(d) + \frac{1}{2} l^2 w_1 \underline{\nabla} d . \underline{\nabla} d}_{\text{énergie dissipée}} \Big) d\Omega$$

.

La fonctionnelle $\Psi(\underline{u}, d)$ est celle établie par Pham et al. (2013) par une autre voie, en lien avec les approches variationnelles de la rupture (voir Ambrosio-Torterelli, 1991 ; Francfort et Marigo, 1998 ; Bourdin et al., 2000)

.

On notera des formulations alternatives : la TLS (Thick Level Set) du groupe de N. Moes ou une propositio n récente de Fremond et Stolz (2017).