GDR Géométrie Différentielle et Mécanique

La Rochelle, 4-7 juin 2019

Sur l'anisotropie en mécanique plane

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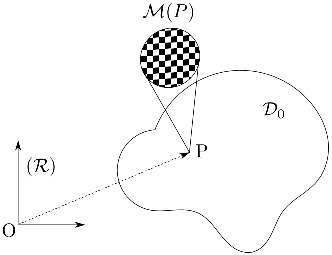
June 5, 2019

Physical motivations

The elasticity tensor is an image of the microstructure of the material.

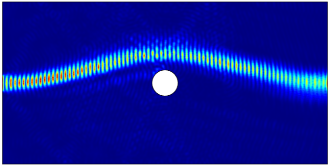
Optimal design

- Response to specifications;
- Achieving non standard properties;
- Controlling wave propagation;
- ...



Evolution of the matter

- Damaging of the matter;
- Growth of bio-material;
- Control of smart materials;
- ...



Optimal design using invariants

2D anisotropic Elasticity, small strains, small displacements

Structural problem

$\updownarrow$

Equivalent behavior
Invariant parametrization

$\updownarrow$

Micro structure level
Geometrical parametrization

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Structural optimization

Objective function : Mass, Buckling, Energy
Optimization parameters : Invariants

Optimal design using invariants

2D anisotropic Elasticity, small strains, small displacements

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Micro structure level
Geometrical parametrization

Structural optimization

Objective function : Mass, Buckling, Energy
Optimization parameters : Invariants

Optimal design of achitectured materials

Objective function : f(Invariants)
Optimization parameters : Geometry

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5. Designing anisotropy

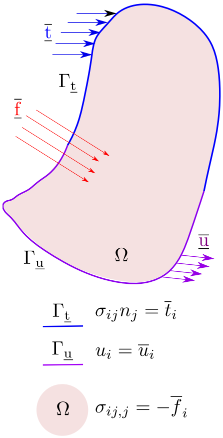
What is elasticity ?

Consider :

- a domain $\Omega \subset \mathbb{R}^d$;
- $\underline{u}$ the displacement field over Ω;
- $\underline{f}$ the bulk force field over Ω;
- ε the strain field over Ω;
- σ the stress field over Ω;

Ω is at rest provided:

$$\begin{cases} \underline{\text{div}}\sigma + \underline{f} = 0 \\ \sigma \cdot \underline{n} = \overline{\underline{t}} \end{cases}$$



Constitutive law

A constitutive model is needed to relate σ to ε

The elasticity tensor

Hooke's law

Linear relation between the stress tensor $\sigma \in S^2(\mathbb{R}^2)$ and the strain tensor $\varepsilon \in S^2(\mathbb{R}^2)$:

$$\sigma = \mathbb{C} : \varepsilon$$

Properties

$\mathbb{C}$ is an element of the vector space $\mathbb{E}\mathrm{la} := S^2(S^2(\mathbb{R}^2))$;
$\mathbb{C}$ is positive definite :

$$\forall \varepsilon \neq 0, \quad \varepsilon : \mathbb{C} : \varepsilon > 0$$

An elastic material: a O(2) -orbit

O(2)-action

O(2) acts on $\mathbb{E}\text{la}$ through standard $\star$ defined by :

$$\begin{aligned} \star : \mathrm{O}(2) \times \mathbb{E}\text{la} &\to \mathbb{E}\text{la} \\ (\mathrm{Q}, \mathbb{C}) &\mapsto \mathrm{Q} \star \mathbb{C} := Q_{ip} Q_{jq} Q_{kr} Q_{ls} C_{pqrs} \end{aligned}$$

Orbit

The set of tensors of $\mathbb{E}\text{la}$ O(2)-conjugate to $\mathbb{C}$ constitutes its O(2)-orbit :

$$\mathrm{Orb}(\mathbb{C}) := \left\{ \overline{\mathbb{C}} = \mathrm{Q} \star \mathbb{C} \mid \mathrm{Q} \in \mathrm{O}(2) \right\}.$$

The orbits space is the quotient space $\mathbb{E}\text{la}/\mathrm{O}(2)$.

Symmetry Class

Symmetry Group

Symmetry group of an elasticity tensor:

$$G_{\mathbb{C}} := \{Q \in O(2), \quad Q \star \mathbb{C} = \mathbb{C}\}.$$

Symmetry Class

The class of symmetry is the conjugacy class of a symmetry group:

$$[G_{\mathbb{C}}] := \left\{Q G_{\mathbb{C}} Q^{-1}, \quad Q \in O(2)\right\}.$$

$\mathbb{E}$la is divided into strata of different symmetry classes.

2D Symmetry classes

In 2D, the space of elasticity tensors is divided into 4 strata:

$$\mathbb{E}\text{la} = \Sigma_{[\mathrm{Z}_2]} \cup \Sigma_{[\mathrm{D}_2]} \cup \Sigma_{[\mathrm{D}_4]} \cup \Sigma_{[\mathrm{O}(2)]}$$

- Z_2: cyclic group generated by $\mathrm{R}(\pi)$, a rotation of angle π;
- D_k: dihedral group generated by $\mathrm{R}(2\pi/k)$ and $\mathrm{P}(\underline{e}_2)$ (mirror transformation through the x axis),
- $\mathrm{O}(2)$: orthogonal group.

	Biclinic	Orthotropic	Tetragonal	Isotropic
$[G_{\mathbb{C}}]$	$[\mathrm{Z}_2]$	$[\mathrm{D}_2]$	$[\mathrm{D}_4]$	$[\mathrm{O}(2)]$
$\#_{\text{indep}}(\mathbb{C})$	6 (5)	4	3	2

1. Elastic material and Symmetry class

2. Fundamentals of the polar formalism

3. Some theoretical results

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The polar components

Following the work of Verchery, we introduce non polynomial quantities, the polar components, to represent the Cartesian components.

Example of 2nd order symmetric tensors

$$T = \frac{L_{11} + L_{22}}{2}, \qquad L_{11} = T + R\cos 2\Phi$$

$$L_{22} = T - R\cos 2\Phi$$

$$Re^{2i\Phi} = \frac{L_{11} - L_{22}}{2} + iL_{12} \qquad L_{12} = R\sin 2\Phi$$

- T, R, and Φ are the so-called polar components
- T and R^2 are polynomial invariants
- Φ is an angle that determines the tensor orientation
- T represents the spherical part and $Re^{2i\Phi}$ the deviatoric part

Case of 4th order elasticity tensor [Verchery 1979]

$$
\begin{aligned}
T_0 &= \frac{1}{8}(\mathbb{E}_{1111} - 2\mathbb{E}_{1122} + 4\mathbb{E}_{1212} + \mathbb{E}_{2222})\\
T_1 &= \frac{1}{8}(\mathbb{E}_{1111} + 2\mathbb{E}_{1122} + \mathbb{E}_{2222})\\
R_0 e^{4i\Phi_0} &= \frac{1}{8}\left[\mathbb{E}_{1111} - 2\mathbb{E}_{1122} - 4\mathbb{E}_{1212} + \mathbb{E}_{2222} + 4i(\mathbb{E}_{1112} - \mathbb{E}_{12222})\right]\\
R_1 e^{2i\Phi_1} &= \frac{1}{8}\left[\mathbb{E}_{1111} - \mathbb{E}_{2222} + 2i(\mathbb{E}_{1112} + \mathbb{E}_{1222})\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_{1111} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 + 4R_1\cos 2\Phi_1\\
\mathbb{E}_{1112} &= R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1\\
\mathbb{E}_{1122} &= -T_0 + 2T_1 - R_0\cos 4\Phi_0\\
\mathbb{E}_{1212} &= T_0 - R_0\cos 4\Phi_0\\
\mathbb{E}_{1222} &= -R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1\\
\mathbb{E}_{2222} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 - 4R_1\cos 2\Phi_1
\end{aligned}
$$

Transformation of a 4th order elasticity tensor

$$\mathrm{R}(\theta) : \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \mathrm{P}(\underline{\mathrm{e}}_2) : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbb{E} = (T_0, T_1, R_0 e^{4i\Phi_0}, R_1 e^{2i\Phi_1})$$

$$\mathrm{R}(\theta) \star \mathbb{E} = (T_0, T_1, R_0 e^{4i(\Phi_0+\theta)}, R_1 e^{2i(\Phi_1+\theta)})$$

$$\mathrm{P}(\underline{\mathrm{e}}_1) \star \mathbb{E} = (T_0, T_1, R_0 e^{4i\Phi_0}, R_1 e^{-2i\Phi_1})$$

2D elasticity tensor invariants

O(2)-integrity basis of $\mathbb{E}\mathrm{la}$

The following quantities

$$I_1 = T_1 \qquad J_1 = T_0 \qquad I_2 = R_1^2 \qquad J_2 = R_0^2 \qquad I_3 = R_0 R_1^2 \cos 4(\Phi_0 - \Phi_1)$$

- Constitute an integrity basis for the O(2)-action;
- The algebra $\mathbb{R}[\mathbb{E}\mathrm{la}]^{\mathrm{O}(2)}$ is free.

2D Symmetry classes

- Isotropy

$$R_0 = 0 \text{ and } R_1 = 0$$

- Square symmetry (tetragonal)

$$R_1 = 0$$

- Orthotropy

$$\cos 4(\Phi_0 - \Phi_1) = \pm 1 \quad \Leftrightarrow \quad \Phi_0 - \Phi_1 = K\frac{\pi}{4} \quad K \in \{0, 1\}$$

- Anisotropy (biclinic)

Polar decomposition of the strain energy

- Putting the strain energy $V=\frac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}$ in the form $V=V_{sph}+V_{dev}$

$$V_{sph}:=\frac{1}{2}\boldsymbol{\varepsilon}_{sph}:\boldsymbol{\sigma}_{sph}=T\ t,$$
$$V_{dev}:=\frac{1}{2}\boldsymbol{\varepsilon}_{dev}:\boldsymbol{\sigma}_{dev}=R\ r\cos 2(\varPhi-\varphi),$$

we get

$$V_{sph}=4T_1t^2+4R_1r\ t\cos 2(\varPhi_1-\varphi),$$
$$V_{dev}=2r^2\left[T_0+R_0\cos 4(\varPhi_0-\varphi)\right]+4R_1r\ t\cos 2(\varPhi_1-\varphi).$$

- T_1 affects only V_{sph},
T_0 and R_0 only V_{dev},
while R_1 couples V_{sph} with V_{dev}.
- For materials with $R_1=0$, the two parts are uncoupled.

Bounds on the polar invariants

- The positive definiteness of $\mathbb{E}$ can be expressed in terms of bounds on its polar invariants
- It can be shown that the positive definiteness reduces to the following

$$T_0 - |R_0| > 0,$$
$$T_1(T_0^2 - R_0^2) - 2R_1^2\left[T_0 - R_0 \cos 4(\Phi_0 - \Phi_1)\right] > 0$$

- The above conditions $\Rightarrow T_0 > 0, T_1 > 0$.

Polar parameters of the inverse tensor

- The polar components of $\mathbb{S}=\mathbb{E}^{-1}$, denoted by the lower-case letters t_0, t_1, r_0, r_1 and $\varphi_0-\varphi_1$, are:

$$
\begin{aligned}
t_0 &= \frac{2}{\Delta}\left(T_0T_1-R_1^2\right),\\
t_1 &= \frac{1}{2\Delta}\left(T_0^2-R_0^2\right),\\
r_0e^{4i\varphi_0} &= \frac{2}{\Delta}\left[(R_1e^{2i\Phi_1})^2-T_1R_0e^{4i\Phi_0}\right],\\
r_1e^{2i\varphi_1} &= \frac{1}{\Delta}\left[R_0e^{4i\Phi_0}R_1e^{-2i\Phi_1}-T_0R_1e^{2i\Phi_1}\right].
\end{aligned}
$$

- Δ is an invariant positive quantity, defined by

$$\Delta = 8T_1\left(T_0^2-R_0^2\right)-16R_1^2\left[T_0-R_0\cos 4\left(\Phi_0-\Phi_1\right)\right]$$

- An important result for the symmetry analysis is the fact that

$$R_1=0 \Leftrightarrow r_1=0, \qquad R_0=0 \not\Leftrightarrow r_0=0.$$

ARE ALL THE SYMMETRIES MECHANICALLY EQUIVALENT?

Let distinguish between ordinary orthotropies and a special orthotropy:

- $\Phi_0 - \Phi_1 = K\frac{\pi}{4}$ **and** $R_0 \neq 0$ $\rightarrow$ ordinary orthotropies ($K \in \{0, 1\}$) (determined by a polynomial cubic invariant);
- $R_0 = 0 \rightarrow$ special orthotropy (determined by a polynomial quadratic invariant).

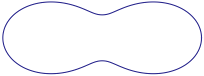
Ordinary orthotropies

- For the same set of invariants T_0, T_1, R_0 and R_1 two possible and distinct ordinary orthotropies exist: one with $K=0$ and the other one with $K=1$.
- For ordinary orthotropic materials

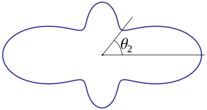
$$
\begin{aligned}
\mathbb{E}_{1111} &= T_0 + 2T_1 + (-1)^K R_0 \cos 4\Phi_1 + 4R_1 \cos 2\Phi_1\\
\mathbb{E}_{1112} &= R_0 \sin 4\Phi_1 + 2R_1 \sin 2\Phi_1\\
\mathbb{E}_{1122} &= -T_0 + 2T_1 - (-1)^K R_0 \cos 4\Phi_1\\
\mathbb{E}_{1212} &= T_0 - (-1)^K R_0 \cos 4\Phi_1\\
\mathbb{E}_{1222} &= -(-1)^K R_0 \sin 4\Phi_1 + 2R_1 \sin 2\Phi_1\\
\mathbb{E}_{2222} &= T_0 + 2T_1 + (-1)^K R_0 \cos 4\Phi_1 - 4R_1 \cos 2\Phi_1
\end{aligned}
$$

- $\mathbb{E}_{1111}$ can be of 3 types, depending upon K and $\rho = \dfrac{R_0}{R_1}$

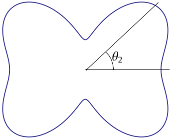
$\rho < 1, K = 0, 1$



$\rho > 1, K = 0$



$\rho > 1, K = 1$



- The type of ordinary orthotropy is not necessarily the same for stiffness and compliance:

$$\left.\begin{array}{l} K=0 \ \text{ and } \ R_1^2>T_1R_0 \\ \text{or} \\ K=1 \end{array}\right\} \ \Rightarrow \ k=0,$$

$$K=0 \ \text{ and } \ R_1^2<T_1R_0 \ \Rightarrow \ k=1.$$

 the combination $(K=1, k=1)$ cannot exist.

- The value of K strongly affects the solution of an optimal design with in anisotropic elasticity: switching the value of K transforms the best in the worst solution (or viceversa).

R_0 SPECIAL-ORTHOTROPY

- $R_0 = 0$ identifies the so-called $R_0 - orthotropy$ (J Elas, 2002).
- With $R_0 = 0$, we get

$$
\begin{aligned}
\mathbb{E}_{1111} &= T_0 + 2T_1 + 4R_1 \cos 2\Phi_1 \\
\mathbb{E}_{1112} &= +2R_1 \sin 2\Phi_1 \\
\mathbb{E}_{1122} &= -T_0 + 2T_1 \\
\mathbb{E}_{1212} &= T_0 \\
\mathbb{E}_{1222} &= +2R_1 \sin 2\Phi_1 \\
\mathbb{E}_{2222} &= T_0 + 2T_1 - 4R_1 \cos 2\Phi_1
\end{aligned}
$$

- Two components are isotropic,
  the others rotates like those of a 2nd order tensor.

- Because $R_0 = 0 \nLeftrightarrow r_0 = 0$, the dual case exists too: r_0-orthotropy.
- It concerns the compliance tensor $\mathbb{S}$. In such a case, it can be shown that

$$R_0 = \frac{R_1^2}{T_1}, \quad K = 0.$$

- It is interesting to notice that just the above relations distinguish this case from that of ordinary orthotropy of $\mathbb{E}$, otherwise indetectable.
- The invariance of S_{1212} implies that of G_{12}:

$$G_{12} = \frac{1}{4S_{1212}} = \frac{1}{4t_0}.$$

- This is the special orthotropy typical of paper (J Elas, 2010)

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Some theoretical results obtained using the polar formalism

- Special orthotropies of planar materials (J Elas, 2002, 2010)
- Invariants of the piezoelectric tensor (Int J Sol Str, 2007)
- Polar invariants and optimal solutions (Int J Mech Sc, 2009)
- Optimization of the orthotropy directions (Vincenti & Desmorat, J Elas, 2009)
- Anisotropy of special elastic materials (Int J Sol Str, 2010; Math Meth Appl Sc, 2016)
- Bounds on the elastic moduli of laminates (J Elas, 2012)
- Interaction between geometry and anisotropy (Math Meth Appl Sc, 2012)
- General theory of thermostable anisotropic laminates (J Elas, 2013)
- Invariants of strength criteria for an anisotropic ply (Math Meth Appl Sc, 2012)
- Theory of wrinkled anisotropic membranes (J Elas, 2013)
- Tensor form of the polar method: the polar projectors (Math Meth Appl Sc, 2014)
- Bounds on damage-induced anisotropy (Int J Sol Str, 2015)
- Extrema of Young's modulus for transverse isotropy (Appl Math & Comput, 2015)

Beyond classical elasticity

- What about the anisotropy of non-classical elastic materials?
- Complex materials (Int J Sol Str, 2010), e.g.

$$\sigma_{ij}\neq\sigma_{ji},\quad \varepsilon_{kl}\neq\varepsilon_{lk}\Rightarrow \mathbb{E}_{ijkl}\neq\mathbb{E}_{jikl}\neq\mathbb{E}_{ijlk}\neq\mathbb{E}_{jilk},\quad \mathbb{E}_{ijkl}=\mathbb{E}_{klij}$$

$$\left\{\begin{array}{llllll}
\mathbb{E}_{1111} = & T_0+T_1+T_2 & +R_0\cos 4\Phi_0 & +2R_1\cos 2\Phi_1 & +2R_2\cos 2\Phi_2\\
\mathbb{E}_{1112} = & -T_3 & +R_0\sin 4\Phi_0 & & +2R_2\sin 2\Phi_2\\
\mathbb{E}_{1121} = & T_3 & +R_0\sin 4\Phi_0 & +2R_1\sin 2\Phi_1 & \\
\mathbb{E}_{1122} = & -T_0+T_1+T_2 & -R_0\cos 4\Phi_0 & & \\
\mathbb{E}_{1212} = & T_0+T_1-T_2 & -R_0\cos 4\Phi_0 & +2R_1\cos 2\Phi_1 & -2R_2\cos 2\Phi_2\\
\mathbb{E}_{1221} = & T_0-T_1+T_2 & -R_0\cos 4\Phi_0 & & \\
\mathbb{E}_{1222} = & -T_3 & -R_0\sin 4\Phi_0 & +2R_1\sin 2\Phi_1 & \\
\mathbb{E}_{2121} = & T_0+T_1-T_2 & -R_0\cos 4\Phi_0 & -2R_1\cos 2\Phi_1 & +2R_2\cos 2\Phi_2\\
\mathbb{E}_{1112} = & T_3 & -R_0\sin 4\Phi_0 & & +2R_2\sin 2\Phi_2\\
\mathbb{E}_{2222} = & T_0+T_1+T_2 & +R_0\cos 4\Phi_0 & -2R_1\cos 2\Phi_1 & -2R_2\cos 2\Phi_2
\end{array}\right.$$

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$$\sigma_{ij} \neq \sigma_{ji}, \quad \varepsilon_{kl} \neq \varepsilon_{lk} \Rightarrow \mathbb{E}_{ijkl} \neq \mathbb{E}_{jikl} \neq \mathbb{E}_{ijlk} \neq \mathbb{E}_{jilk}, \quad \mathbb{E}_{ijkl} = \mathbb{E}_{klij}$$

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\end{array}\right.$$

- 10 components, 9 invariants, 1 ordinary orthotropy, 6 special orthotropies

Elastic damage

- Problem: an initially isotropic layer, when stressed can be damaged and become anisotropic; how much?
- Damage model (Chaboche 1979, Leckie and Onat 1980, Sidoroff 1980, Chow 1987):
  $\mathbb{Q}$: initial stiffness, $\widetilde{\mathbb{Q}}$: damaged stiffness, $\widehat{\mathbb{Q}}$: loss of stiffness

$$\widetilde{\mathbb{Q}} = [(\mathbb{I} - \mathbb{D})\mathbb{Q}]^{Sym} \Rightarrow \widetilde{\mathbb{Q}} = \mathbb{Q} - \widehat{\mathbb{Q}} \quad \text{with } \widehat{\mathbb{Q}} = \frac{\mathbb{Q}\mathbb{D} + \mathbb{D}\mathbb{Q}}{2},$$

- $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ are positive definite, while $\mathbb{D}$ and $\widehat{\mathbb{Q}}$ are semi definite

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- $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ are positive definite, while $\mathbb{D}$ and $\widehat{\mathbb{Q}}$ are semi definite
- Imposing the positive semi definiteness of $\widehat{\mathbb{Q}}$ and the positive definiteness of $\widetilde{\mathbb{Q}}$, gives the bounds on $\mathbb{D}$ and $\widetilde{\mathbb{Q}}$ (the positive semi definiteness of $\widehat{\mathbb{Q}} \Rightarrow$ that of $\mathbb{D}$)

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- $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ are positive definite, while $\mathbb{D}$ and $\widehat{\mathbb{Q}}$ are semi definite
- Imposing the positive semi definiteness of $\widehat{\mathbb{Q}}$ and the positive definiteness of $\widetilde{\mathbb{Q}}$, gives the bounds on $\mathbb{D}$ and $\widetilde{\mathbb{Q}}$ (the positive semi definiteness of $\widehat{\mathbb{Q}} \Rightarrow$ that of $\mathbb{D}$)
- Advantages of the polar formalism:
  - the polar bounds on a fourth-rank tensor concern invariant quantities and are valid for any type of anisotropy $\Rightarrow$ any type of anisotropic damage can be investigated
  - each one of the polar parameters of $\widetilde{\mathbb{Q}}$ depends exclusively upon the corresponding polar parameter of $\mathbb{D} \Rightarrow$ the polar formalism allows for uncoupling the expressions of the parameters of $\widetilde{\mathbb{Q}}$ as functions of those of $\mathbb{D}$:

$$\widetilde{T}_0 = T_0(1-2D_0), \quad \widetilde{T}_1 = T_1(1-4D_1), \quad \widetilde{R}_0 = 2T_0S_0,$$
$$\widetilde{R}_1 = (T_0+2T_1)S_1, \quad \widetilde{\Phi}_0 = \Psi_0 + \frac{\pi}{4}, \quad \widetilde{\Phi}_1 = \Psi_1 + \frac{\pi}{2}.$$

- $D_0, D_1, S_0, S_1, \Psi_0, \Psi_1$: polar parameters of $\mathbb{D}$
- $\widetilde{T}_0, \widetilde{T}_1, \widetilde{R}_0, \widetilde{R}_1, \widetilde{\Phi}_0, \widetilde{\Phi}_1$: polar parameters of $\widetilde{\mathbb{Q}}$

$$\widetilde{T}_0 = T_0(1-2D_0), \quad \widetilde{T}_1 = T_1(1-4D_1), \quad \widetilde{R}_0 = 2T_0S_0,$$
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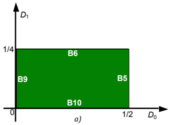
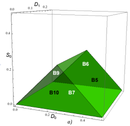
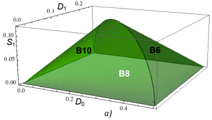
- $D_0, D_1, S_0, S_1, \Psi_0, \Psi_1$: polar parameters of $\mathbb{D}$
- $\widetilde{T}_0, \widetilde{T}_1, \widetilde{R}_0, \widetilde{R}_1, \widetilde{\Phi}_0, \widetilde{\Phi}_1$: polar parameters of $\widetilde{\mathbb{Q}}$
- Minimal set of polar bounds for $\mathbb{D}$ in the completely anisotropic case
  - $2(D_0+S_0)<1$
  - $\frac{\tau_1}{4(1+\tau_1)^2}(1-4D_1)[(1-2D_0)^2-4S_0^2] > S_1^2[1-2D_0+2S_0\cos 4(\Psi_0-\Psi_1)]$
  - $S_0 \geq 0$
  - $S_1 \geq 0$
  - $D_0 \geq S_0$
  - $D_1(D_0^2-S_0^2) \geq \frac{(1+\tau_1)^2}{2\tau_1}S_1^2[D_0-S_0\cos 4(\Psi_0-\Psi_1)]$



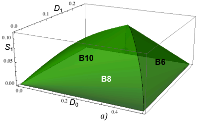
$\widetilde{\mathbb{Q}} \to$ Isotropic



$R_1 = 0$

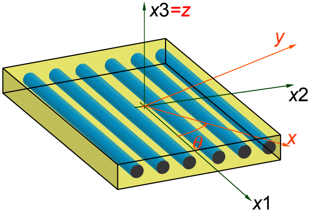


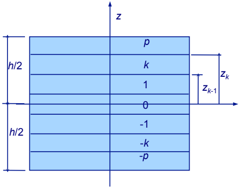
$R_0 = 0$



$r_0 = 0$

Abridged laminate mechanics





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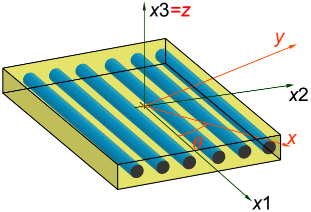

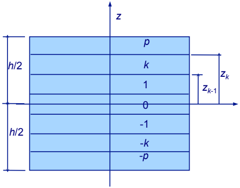

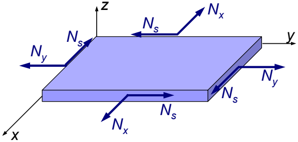

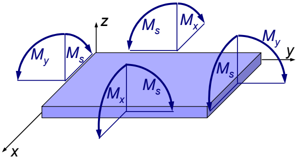

$$\left\{\frac{\mathbf{N}}{\mathbf{M}}\right\} = \left[\begin{array}{c|c} h\mathbb{A} & \frac{h^2}{2}\mathbb{B} \\ \hline \frac{h^2}{2}\mathbb{B} & \frac{h^3}{12}\mathbb{D} \end{array}\right] \left\{\frac{\boldsymbol{\varepsilon}}{\chi}\right\} \quad \Leftrightarrow \quad \left\{\frac{\boldsymbol{\varepsilon}}{\chi}\right\} = \left[\begin{array}{c|c} \frac{1}{h}\mathcal{A} & \frac{2}{h^2}\mathcal{B} \\ \hline \frac{2}{h^2}\mathcal{B}^\top & \frac{12}{h^3}\mathcal{D} \end{array}\right] \left\{\frac{\mathbf{N}}{\mathbf{M}}\right\}$$

The stiffness tensors

$$\mathbb{A} \rightarrow \begin{cases} T_0^A = T_0 \\ T_1^A = T_1 \\ R_0^A e^{4i\Phi_0^A} = R_0 e^{4i\Phi_0}(\xi_1 + i\xi_3) \\ R_1^A e^{2i\Phi_1^A} = R_1 e^{2i\Phi_1}(\xi_2 + i\xi_4) \end{cases}$$

$$\mathbb{B} \rightarrow \begin{cases} T_0^B = 0 \\ T_1^B = 0 \\ R_0^B e^{4i\Phi_0^B} = R_0 e^{4i\Phi_0}(\xi_5 + i\xi_7) \\ R_1^B e^{2i\Phi_1^B} = R_1 e^{2i\Phi_1}(\xi_6 + i\xi_8) \end{cases}$$

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Lamination parameters

$$\begin{cases} \xi_1 + i\xi_3 = \frac{1}{n} \sum_{j=1}^{n} e^{4i\delta_j} \\ \xi_2 + i\xi_4 = \frac{1}{n} \sum_{j=1}^{n} e^{2i\delta_j} \end{cases}$$

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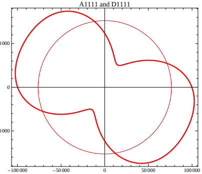
$R_1 e^{2i\Phi_1}$ ↓ material; $(\xi_{10} + i\xi_{12})$ ↓ geometry

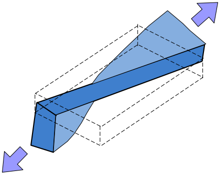
Effects of homogenization on anisotropy

- The definition of the anisotropy behavior and of elastic symmetries can be problematic in laminates:

- Generally speaking, $\mathbb{A} \neq \mathbb{D}$ and $\mathcal{A} \neq \mathcal{D}$

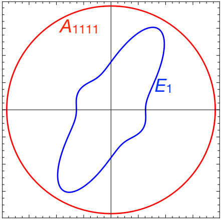
- $\mathbb{B} \neq 0 \rightarrow$coupling



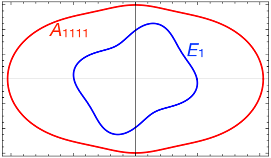


- While $\mathbb{B} = \mathbb{B}^\top$, this is not true for compliance: $\mathcal{B} \neq \mathcal{B}^\top$.

- If $\mathbb{B} \neq 0$, the symmetries of $\mathbb{A}$ and $\mathbb{D}$ are lost for $\mathcal{A}$ and $\mathcal{D}$



[0/60/120]



[0/30/90/150]

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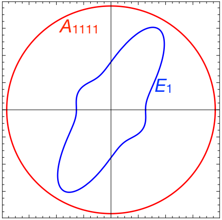
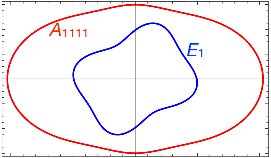
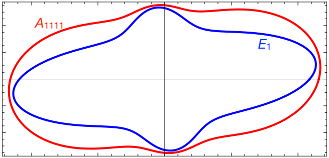


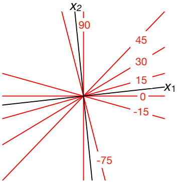
[0/60/120]



[0/30/90/150]

- An elastic symmetry can exist also without any material symmetry. An example with $\mathbb{A}$ orthotropic, $\mathbb{B} = 0$, $\mathbb{D}$ anisotropic: [0/30/-15/15/90/-75/0/45/-75/0/-15/15]

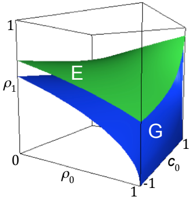




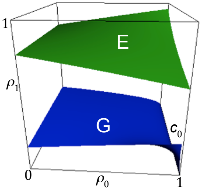
- Geometrical bounds for polar components of a laminate (J Elas, 2013)

$$\rho=\frac{R_0}{R_1},\quad \rho_0=\frac{R_0^{A,D}}{R_0},\quad \rho_1=\frac{R_1^{A,D}}{R_1},\quad \tau_0=\frac{T_0}{R_0},\quad \tau_1=\frac{T_1}{R_1}.$$

$$0\leq\rho_0,\ \ 0\leq\rho_1,\ \ \rho_0\leq 1,\ \ 2\rho_1^2\leq\frac{1-\rho_0^2}{1-(-1)^{K^L}\rho_0\ c_0},\ \ 2\rho_1^2<\rho\tau_0\tau_1\frac{1-\left(\frac{\rho_0}{\tau_0}\right)^2}{1-\frac{\rho_0}{\tau_0}\ c_0}.$$



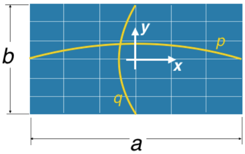
Carbon-epoxy T-300/5208



Braided carbon-epoxy BR45-a

Geometric interactions with anisotropy

- The concept of anisotropic behavior is an absolute one, but that of the anisotropic response no: it depends upon geometry.
- E.g.: the flexural behavior of a rectangular uncoupled orthotropic laminate:



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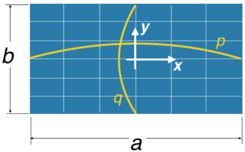

- Compliance:

$$J=\frac{\gamma_{pq}}{p^4h^3(1+\chi^2)^2\sqrt{R_0^2+R_1^2}}\frac{1}{\varphi(\xi_0,\xi_1)};$$

- Buckling load multiplier for the mode pq:

$$\lambda_{pq}=\frac{\pi^2p^2h^3}{12a^2}\frac{(1+\chi^2)^2\sqrt{R_0^2+R_1^2}}{N_x+N_y\chi^2}\varphi(\xi_0,\xi_1);$$

- Frequency of the transversal vibration for the mode pq

$$\omega_{pq}^2=\frac{\pi^4p^4h^3}{12\mu a^4}\left(1+\chi^2\right)^2\sqrt{R_0^2+R_1^2}\varphi(\xi_0,\xi_1).$$

Wavelengths ratio

$$\chi = \frac{a}{b}\frac{q}{p}$$

Isotropy-to-anisotropy ratio

$$\tau = \frac{T_0 + 2T_1}{\sqrt{R_0^2 + R_1^2}}$$

Anisotropy ratio

$$\rho = \frac{R_0}{R_1}$$

$$\varphi(\xi_0, \xi_1) = \tau + \frac{1}{\sqrt{1+\rho^2}}\left[(-1)^k \rho \xi_0 \frac{\chi^4 - 6\chi^2 + 1}{(1+\chi^2)^2} + 4\xi_1 \frac{1-\chi^2}{1+\chi^2}\right]$$

- ξ_0, ξ_1: lamination parameters, defining geometrically the stack

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- If $\varphi(\xi_0, \xi_1) = \tau$ the plate has an isotropic response
- Possible solutions:
  - $\rho = 0,\ \chi = 1$; e.g. $p = q$ on square plates of R_0-orthotropic plies
  - $\xi_0 = 0,\ \chi = 1$; still $p = q$ and stack $[\pm(\pi/8)_{n/4}, \pm(3\pi/8)_{n/4}]$
  - $\rho = \infty,\ \chi = \sqrt{2} \pm 1$; layers with $R_1 = 0$ and $\frac{a}{b} = \frac{p}{q}\left(\sqrt{2} \pm 1\right)$.
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  - more generally, $\xi_0 = \frac{4}{(-1)^K \rho} \frac{\chi^4 - 1}{\chi^4 - 6\chi^2 + 1} \xi_1$

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    - more generally, $\xi_0 = \frac{4}{(-1)^K \rho}\frac{\chi^4 - 1}{\chi^4 - 6\chi^2 + 1}\xi_1$
- In some sense, geometry acts as a filter on the elastic phases

Design of anisotropic laminates

- Design problem → optimization of a cost function: $\min_x f(x)$
  - x: design variables (typically: n, δ_j, thicknesses etc.)
  - usually, the material is chosen *a priori* → the isotropic part is determined by this choice

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- Be: $\mathcal{P} = \{\mathcal{P}_i,\ i = 1, ..., 12\} = \{R_0, R_1, \Phi_0 - \Phi_1, \Phi_1\}_{A,B,D}$
  - $\mathbb{A} = \mathbb{A}(\mathcal{P}_i),\ \mathbb{B} = \mathbb{B}(\mathcal{P}_i),\ \mathbb{D} = \mathbb{D}(\mathcal{P}_i)$, unique correspondence

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  - functions $\mathcal{P}_i = \mathcal{P}_i(\delta_j)$ are not bijective
- Each problem is split into 2 subproblems, linked together and to be solved in sequence:
  - Step 1: the Structure Problem: design of the optimal anisotropy properties with respect to $f(x)$; the problem is formulated in the space of the $\mathcal{P}_i$s;
  - Step 2: the Constitutive Law Problem: determination of a suitable stacking sequence δ_j able to realize a laminate with the optimal $\mathcal{P}_i$s; non-bijectivity ⇒ non-uniqueness.

The advantages of the polar method

- The use of the polar parameters automatically eliminates the redundant design variables, because

$$\mathbb{A} \rightarrow \begin{cases} T_0^A = T_0 \\ T_1^A = T_1 \\ R_0^A e^{4i\Phi_0^A} = R_0 e^{4i\Phi_0} (\xi_1 + i\xi_3) \\ R_1^A e^{2i\Phi_1^A} = R_1 e^{2i\Phi_1} (\xi_2 + i\xi_4) \end{cases} \quad \mathbb{D} \rightarrow \begin{cases} T_0^D = T_0 \\ T_1^D = T_1 \\ R_0^D e^{4i\Phi_0^D} = R_0 e^{4i\Phi_0} (\xi_9 + i\xi_{11}) \\ R_1^D e^{2i\Phi_1^D} = R_1 e^{2i\Phi_1} (\xi_{10} + i\xi_{12}) \end{cases}$$

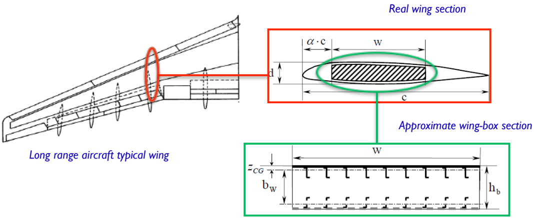
- The size of Step 1 problem is fixed and independent from the number of plies; e.g. for orthotropic laminates made of identical plies, it is of only 3 polar parameters ∀ tensor:

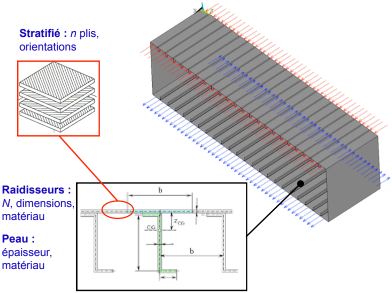
$$R_{0K} = (-1)^K R_0, R_1, \Phi_1$$

- The geometrical constraints are known in explicit form for the polar parameters, ∀ type of anisotropy

Optimization of modular systems
Exemple: an aircraft wing

- Objective: to minimize the weight of a box girder with stiffeners
- Constraint: limit on the buckling load
- Structure made by carbon-epoxy laminates
- Non fixed number of modules (stiffeners)
  (same characteristics, different dimensions and mechanical properties)
- Two-step strategy

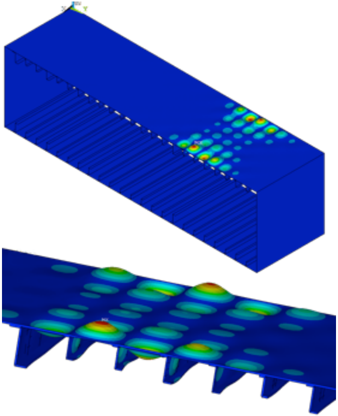




(J Opt Th App, 2012)

Results

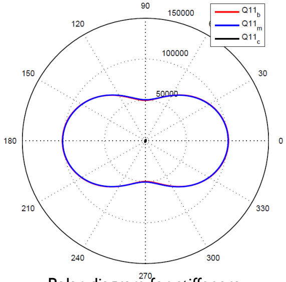
- Optimal solution:
- $W = 620N(-49\%)$
- $\lambda_{cr} = \lambda_{min}$



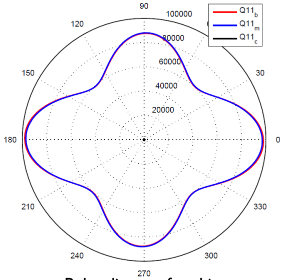
Stiffeners				
ID	t^S [mm]	h^S [mm]	$(R_{0K}^{A^*})^S$ [MPa]	$(R_1^{A^*})^S$ [MPa]
01	2.75	86.5	−7336.27	13651.0
02	4.75	55.5	−9565.0	1970.67
03	2.625	55.0	−1392.96	15991.2
04	2.125	73.5	18888.6	14574.8
05	3.625	46.0	−5404.69	2750.73
06	4.625	49.0	5701.86	13261.0
07	2.125	58.0	5924.73	11249.3
08	2.0	65.0	−8450.64	8847.51
09	4.0	48.0	14876.8	4495.6
10	4.0	43.0	−1578.69	739.0
11	3.0	43.5	1801.56	7574.78
12	3.75	41.5	8042.03	4290.32
13	3.0	59.0	−1095.8	11495.6
14	4.25	52.0	17811.3	1149.56
15	4.375	54.0	10865.1	2832.84
16	4.0	84.0	12536.7	13178.9
17	2.125	48.5	3993.16	10633.4
18	3.125	48.5	12276.6	14349.0
19	3.0	56.0	12610.9	11536.7
20	2.125	56.5	−6333.33	7615.84
21	4.375	43.0	15322.6	8950.15
22	3.375	56.0	17551.3	5994.13
23	3.625	41.0	13242.4	7020.53
Skin				
	t [mm]		$R_{0K}^{A^*}$ [MPa]	$R_1^{A^*}$ [MPa]
	4.0		12945.3	882.70

Step 2 results

N. of layers	Stacking sequence	Residual
Stiffeners		
29	[−8/28/26/ − 45/ − 58/ − 3/55/ − 31/76/34/ − 39/ − 87/ − 7/6/30/ −12/ − 21/ − 51/18/ − 55/49/ − 8/18/12/57/44/ − 27/ − 79/ − 18]	2.996×10^{-4}
Skin		
32	[−81/7/ − 3/ − 12/82/86/ − 87/20/ − 6/76/ − 7/85/ − 6/90/ − 7/87/ 10/ − 82/ − 4/ − 7/ − 82/18/ − 11/ − 84/ − 83/7/70/85/1/ − 12/1/89]	8.445×10^{-5}



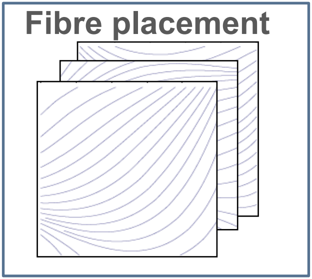
Polar diagram for stiffeners



Polar diagram for skin

Optimal anisotropic fields

- Idea: fibre placement
- Pb: properties (p. ex. $\mathbb{B} = 0$) are local
- Mathematically: optimization of three tensor fields of anisotropy, with local constraints



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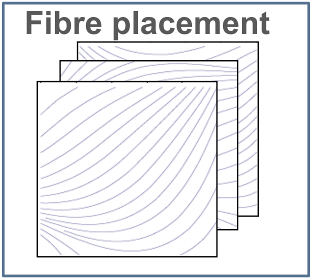
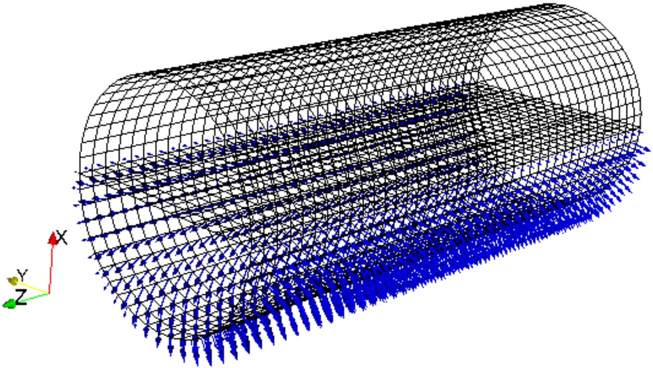
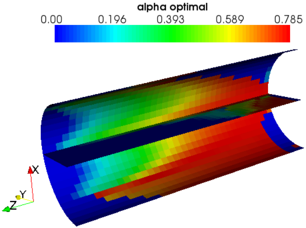


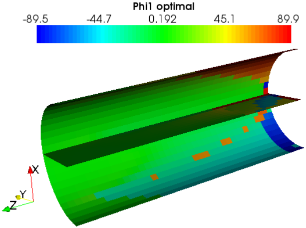
- Problems considered up till now:
    - stiffness optimization
      (PhD theses of C. Julien and A. Jibawy, 2010, Univ P6)
    - stiffness and strength optimization
      (PhD thesis of A. Catapano, 2013, Univ P6)

- Example 1: optimization of an aircraft-like structure
- Objective: minimization of the compliance; angle-ply laminates

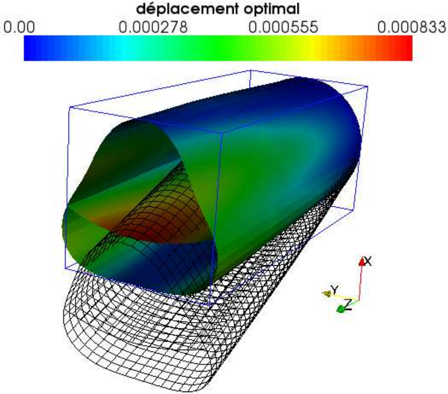




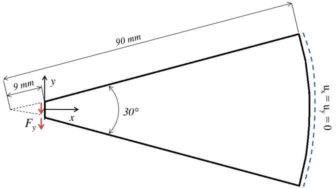
(a) Répartition de l'orientation optimale des fibres α^{opt}

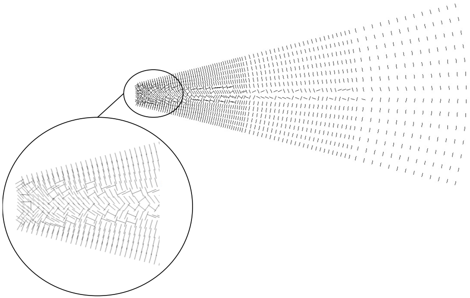


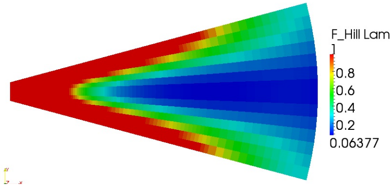
(b) Répartition des valeurs de Φ_1^{opt}



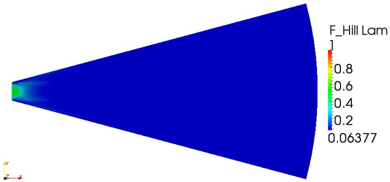
- Example 2: maximize stiffness and strength; for a laminate having the minimal compliance, determine the highest strength (or viceversa).
  A. Catapano: Stiffness and strength optimisation of the anisotropy distribution for laminated structures. PhD thesis, Univ P6, 2013







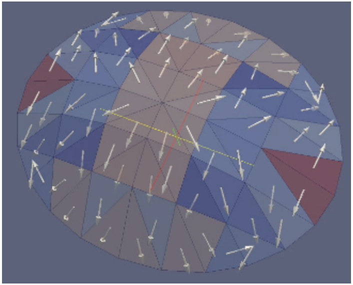
a)



b)

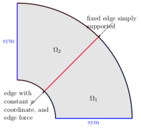
Designing geometry and anisotropy

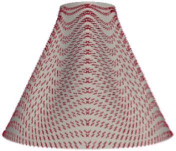
- Industrial research (RENAULT) (PhD thesis of F. Kpadonou, 2017)
- Problem: determine the optimal shape and field of anisotropy for a shell to be designed with respect to a given criterion and constraints
- Such a research touches to the interaction between anisotropy and geometry.

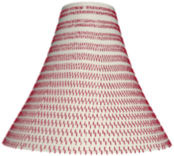


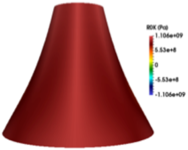
Une structure conique

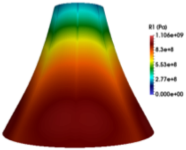
La structure à optimiser est originalement une plaque circulaire trouée chargée transversalement sur le contour interne.

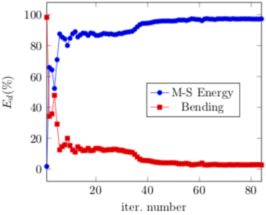




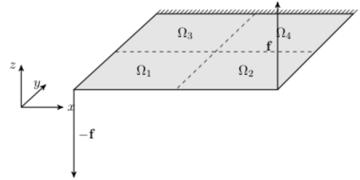


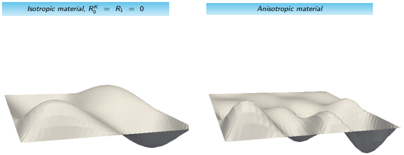


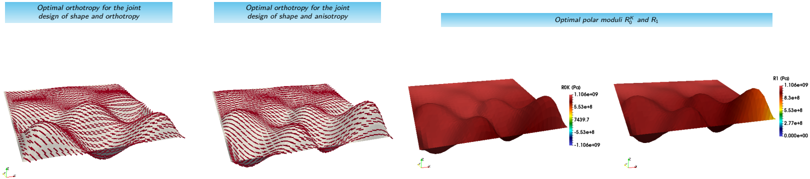




Plaque encastrée soumise à un chargement de torsion







Quelques perspectives

2D

- Pour les stratifiés : bornes géométriques pour $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$
- Bornes géométriques pour d'autres micro structures (treillis, ...)
- Autres lois de comportements
  (piezo-électricité, élasticité de milieux du second gradient, ...)
  - bases d'invariants et de covariants
  - sens mécanique des invariants et covariants
  - optimisation de la distribution de matière/anisotropie

3D ($\mathbb{E}$la)

- Optimisation simultanée de la distribution de matière et d'anisotropie
  Isotropie transverse OK[Thèse N. Ranaivomiarana, 2019]
- Design de microstructure à comportement exotique

$$\begin{pmatrix} \sigma \\ \tau \\ p \end{pmatrix} = \begin{pmatrix} \mathbb{C} & \mathbb{M} & \mathbb{P} \\ & \mathbb{A} & \mathbb{F} \\ & & \mathbb{S} \end{pmatrix} \begin{pmatrix} \varepsilon \\ \eta \\ \upsilon \end{pmatrix} \tag{1}$$

$$S_{(ij)} \qquad P_{(ij)k} \qquad F_{(ij)kl} \qquad C_{\underline{(ij)}\,\underline{(kl)}} \qquad M_{(ij)(kl)m} \qquad A_{\underline{(ij)k}\,\underline{(lm)n}}$$