LaMCoS Unité Mixte de Recherche 5259

Laboratoire de Mécanique des Contacts et des Structures

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Heterogeneous Asynchronous Variational Time Integrators for coupling methods

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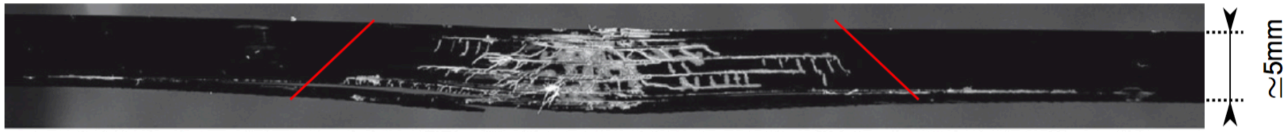
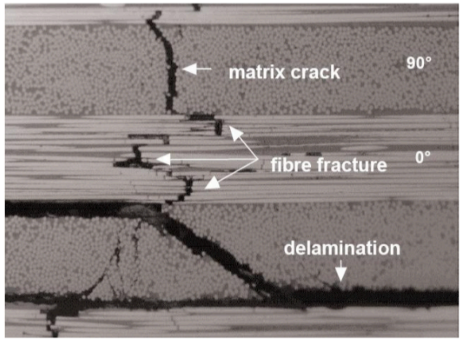
- *Low energy impact on composite panels*

Cone of damage

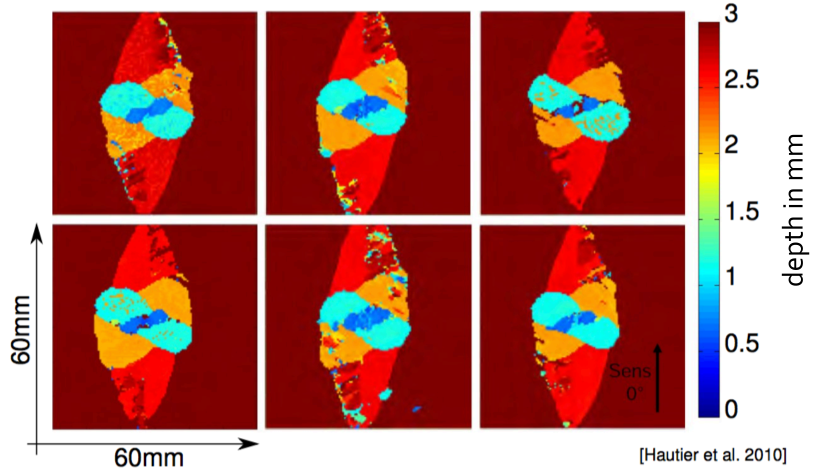


[Lopes et al. 2009]



[Davies et al. 2004]

6 different experiments



[Hautier et al. 2010]

spatial localization of damage and thickness of the ply << size of the structure (helical cracks)

implicit-explicit co-simulation (Zebulon / Europlexus) ***[Chantrait 2014]***

DGA

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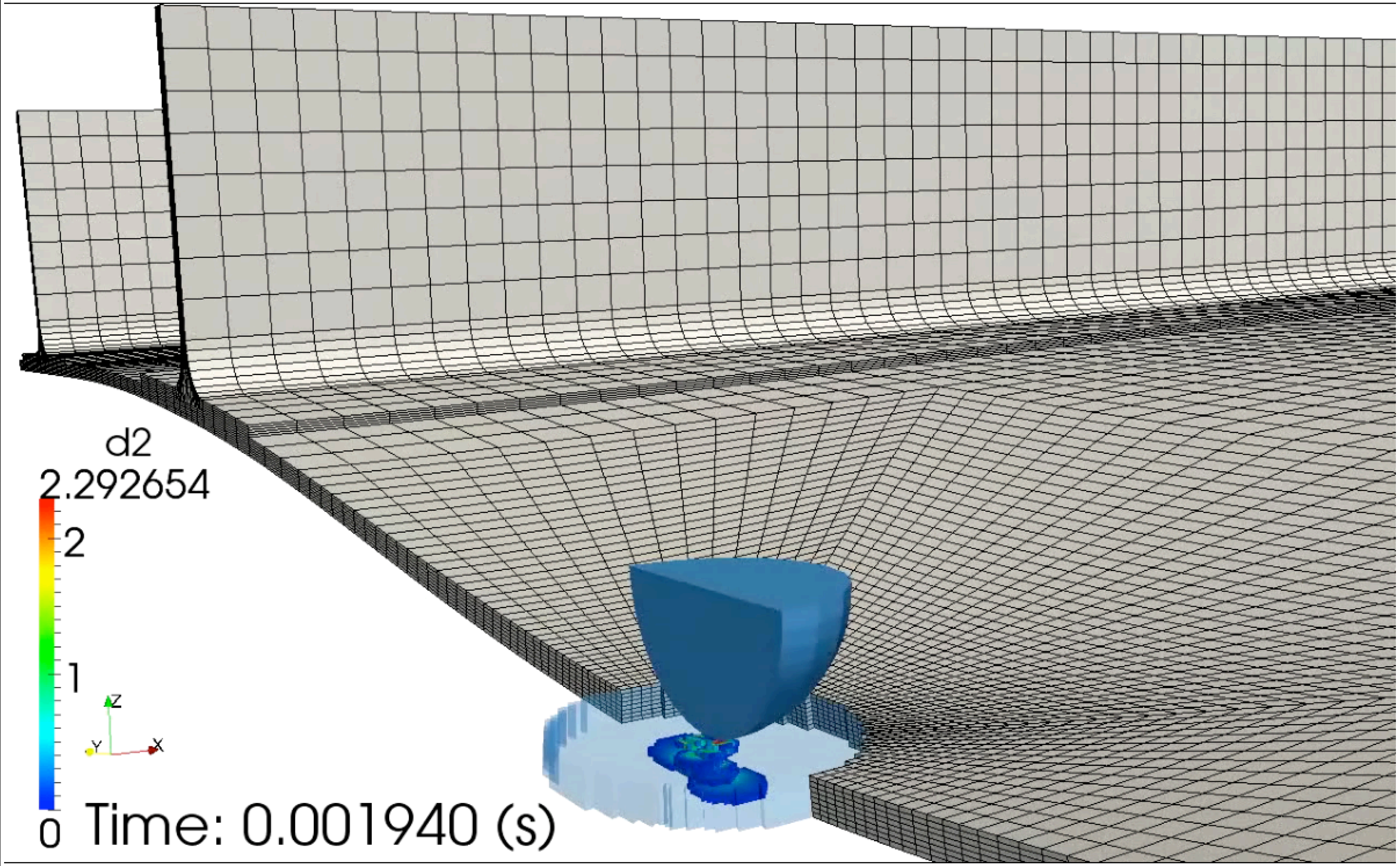
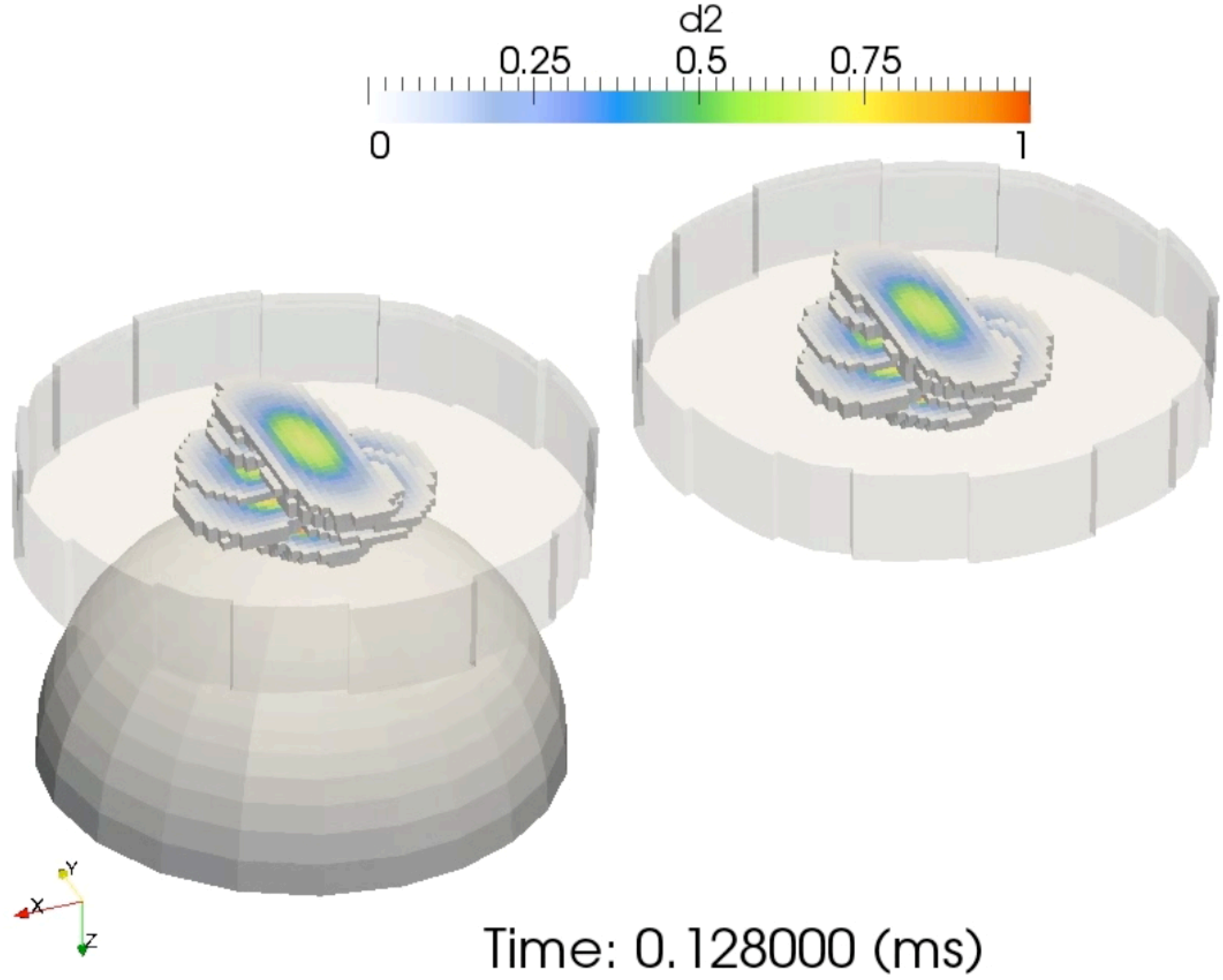
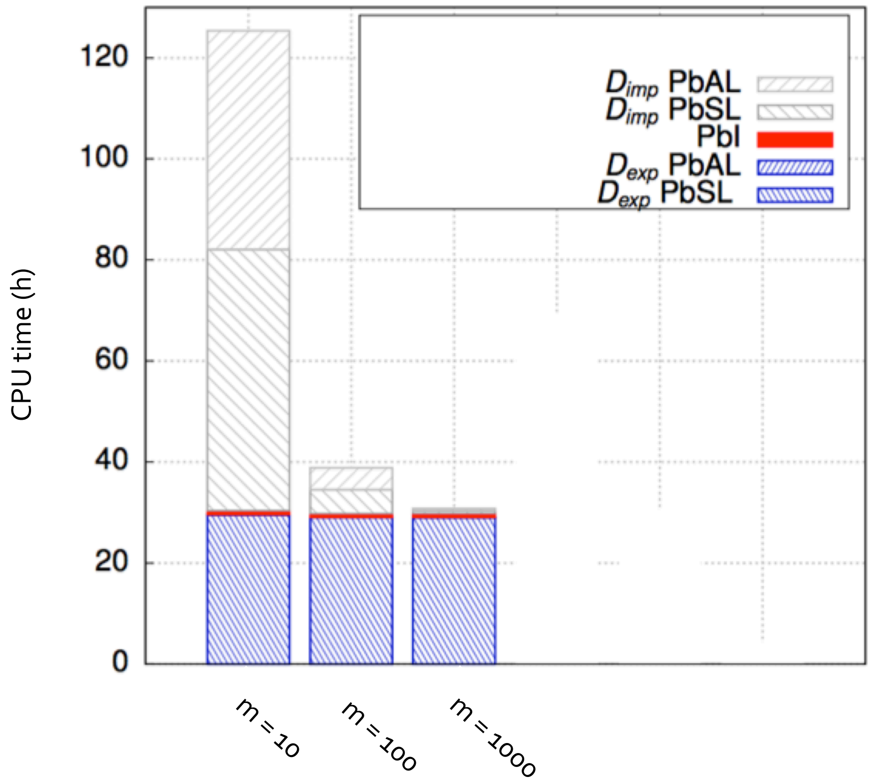

d2
2.292654
2
1
0
Z
Y
X
Time: 0.001940 (s)


d2
0.25
0.5
0.75
0
1
Y
X
Z
Time: 0.128000 (ms)

- *Industrial application of the BGC-micro method (explicit / implicit co-simulation)*
  [Chantrait Rannou Gravouil 2014]

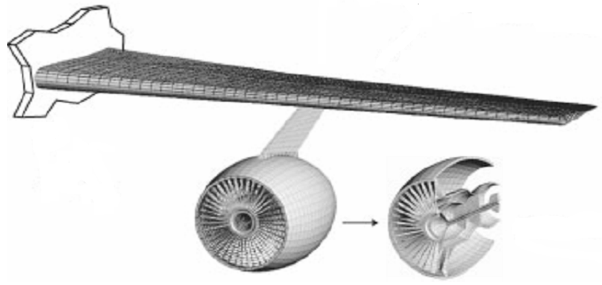


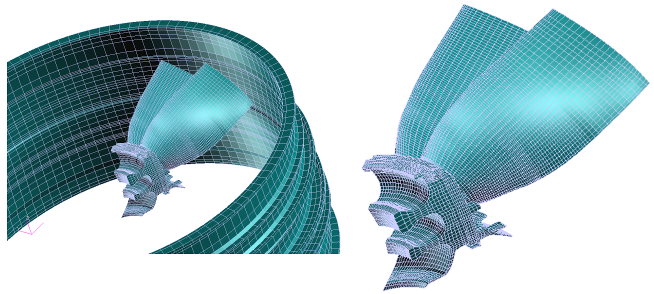
Schur complement
(20 mn)

Significant time saving (dt_{exp} = 1.x10-^{8}s) and main numerical effort in the explicit domain

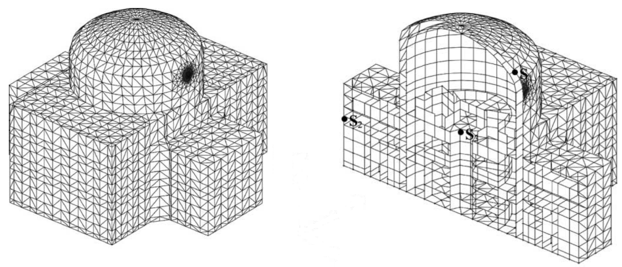
Introduction

- ***Nonlinear transient dynamics***
  - ***Earthquake***
  - ***Crash***
  - ***Impacts on critical components***
  - ***Fluid structure interaction***
  - ***Multi-physics***





- ***Taking into account localized nonlinearities***
  - ***Nonlinear behavior (concrete, steel)***
  - ***Contact, friction***
  - ***Large deformations***



➡ ***Multi-scale problems***

both in space and time

➡ ***Main goal: overcoming limitation of direct time integrators (a unique integrator and a unique time step for the global mesh)***

Outline

1. *A general formalism of direct time integrators combined with Lagrange multipliers*
2. *Stability analysis*
3. *A unified strategy to control « locally » the accuracy and the time step – Application to a bridge crane under earthquake*
4. *Conclusions & prospects*

1

A general formalism of direct time integrators combined with Lagrange multipliers

In a first step, it consists in controlling the ACCURACY

- *5 properties are generally identified:*
  - ***Unconditional stability** for implicit integrators in the linear case*
  - ***Second order accuracy** (Dahlquist barrier for linear multi steps methods (LMS)) [Dahlquist 1963]*
  - ***A unique implicit equation** to solve at each time step*
  - ***« self-starting » integrators***
  - ***Control of the numerical dissipation** in the "high frequency" range [Hilber & Hughes 1978 (overshoot), Hulbert 2004]*

Example 1 (limited to Dahlquist barrier): Newmark integrators, α time integrators (HHT-α, Wood-Bossak-Zienkiewicz (WBZ)-α, Chung-Hulbert (CH)-α) allow to ensure a second order accuracy and numerical dissipation of high frequency spurious oscillations [Newmark 1959, Hilber Hughes 1977, Wood Bossak Zienkiewicz 1980, Wiberg Zeng Li 1992, Safjan Oden 1993, Chung Hulbert 1993, Pegon 2001, Krenk 2005, Masuri Hoitink Zhou Tamma 2009]

Example 2 (not limited by Dahlquist barrier): time discontinuous Galerkin (TDG), time extended finite elements (T-XFEM) allow strong time discontinuities with high order accuracy [Yin & al 2000, Kanapady Tamma 2003, Réthoré & al 2005]; space-time finite elements allow a space-time adaptation for controlling both the accuracy and the time step [Hughes Hulbert 1988 and 1990, Palaniappan & al 2004, Haber & al 2005, Cavin & al 2005, Gopalakrishnan & al 2017] (mapped tent pitching schemes)

- *Historically, the extension to non-linear or non-smooth problems has highlighted the following challenges:*
  - *Loss of unconditional stability for implicit time integrators*
  - *Loss of accuracy (from order 2 to order 1)*
  - *Energy properties to check / enforce*
  - *Overshoot, high frequency numerical behavior*
  - *Numerical integration of internal forces*

➡ *Many contributions for 50 years until today propose to overcome these difficulties:*

- *Integrators ensuring balance of linear momentum, angular momentum, energy, entropy [Simo et Tarnow 1992, Armero et Petocz 1998, Betsch et Steinmann 2001, Botasso et al 2001, Krenk 2006, Romero 2009]*
- *Symplectic integrators (Hamiltonian approaches, conservative problems) [Simo et al 1992, Marsden et West 2001, Chhay Hoarau Hamdouni Sagaut 2011]*
- *Variational and α-variational integrators [Cannarozzi & al 1995, Laursen & Chawla 1997, Kuhl & Crisfield 1999, Armero & Romero 2001, Masuri Hoitink Zhou Tamma 2009]*
- *Variational integrators for discrete mechanics [Marsden & West 2001, Hauret Letallec 2006]*

➡ *For example, symplectic variational integrators can exactly conserve a discrete Lagrangian with a symplectic structure and have better numerical properties over long periods, port-Hamiltonian approach, Q-structures [Hamdouni et al 2011]; applications also to non-smooth dynamics [Moreau 1999, Jean 1999, Pandolfi Kane Marsden Ortiz 2002, Acary 2008, West 2004, Betsch 2011, Renard 2013]*

In a second step, it consists in controlling the TIME STEP

- *3 main conditions are generally identified:*
  - *Frequency content of the structure and the prescribed loading*
  - *Accuracy and stability properties of integrators (conditionally stable explicit schemes)*
  - *Convergence of nonlinear solver for implicit integrators*

Adaptive variable time steps strategies are often used to ensure the automatically the accuracy and efficiency in terms of CPU
[Wood 1990, Zienkiewicz Taylor 1991, Wiberg Zeng Li 1992, Wiberg Li 1999, Schleupen Ramm 2000, Czekanski El-Abbasi Meguid 2001]

A first conclusion: the main limitation of all these time discretizations is in using A UNIQUE TIME INTEGRATOR (homogeneous time integrator) and A UNIQUE TIME STEP for all the finite element mesh (synchronous time integrator).

One obvious improvement is to develop heterogeneous (each part of the mesh has its own integrator) asynchronous (each part of the mesh has its own time step) time integrators.

Construction of a unified formalism for a large class of time integrators with Lagrange multipliers

- *Transient non-linear dynamics equation semi-discretized by the finite element method:*

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{f}_{int}(\mathbf{u}(t)) = \mathbf{f}_{ext}(t) \quad \forall t \in [t_0, t_f]$$

$$\mathbf{u}(t_0) = \mathbf{u}_0, \ \dot{\mathbf{u}}(t_0) = \dot{\mathbf{u}}_0$$

- *Linear case:*

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}_{ext}(t) \quad \forall t \in [t_0, t_f]$$

$$\mathbf{u}(t_0) = \mathbf{u}_0, \ \dot{\mathbf{u}}(t_0) = \dot{\mathbf{u}}_0$$

These expressions have the following drawbacks:

- *Second order hyperbolic in time*
- *Strong formulation in time with restrictive properties of differentiability*
- *Inadequate with time discontinuities (non-smooth dynamics, dynamic crack propagation, shock waves)*

Inadequate starting point for the construction of modern time integrators (in the aim to ensure discretized balance equations in the time domain)

unsuitable for building heterogeneous asynchronous time integrators

We consider as a starting point the simplest weak formulation in time for structural dynamics

- *Variational formulation based on the following well-known action integral:*

- $$\mathcal{A}(\mathbf{u},\dot{\mathbf{u}}) = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{u},\dot{\mathbf{u}})\,\mathrm{d}t \qquad \mathcal{L}(\mathbf{u},\dot{\mathbf{u}}) = \mathcal{T}(\dot{\mathbf{u}}) - \mathcal{V}(\mathbf{u})$$
  $$\mathcal{T}(\dot{\mathbf{u}}) = \frac{1}{2}\dot{\mathbf{u}}^t\mathbf{M}\dot{\mathbf{u}}$$
  $$\mathcal{V}(\mathbf{u}) = \mathcal{V}_{int} - \mathcal{V}_{ext} = \frac{1}{2}\mathbf{u}^t\mathbf{K}\mathbf{u} - \mathbf{F}^t_{ext}\mathbf{u}$$
  $$\delta\mathcal{A} = 0 \quad \forall\delta\mathbf{u} \in \mathcal{U}_0\left(H^1\left(\Omega,[t_0,t_f]\right)\right), \delta\mathbf{u}(.,t_0) = \delta\mathbf{u}\left(.,t_f\right) = 0$$

- $\delta\mathcal{A} = 0 \Leftrightarrow \int_{t_0}^{t_f} \delta\mathbf{u}^t(t)\left(\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) - \mathbf{f}_{ext}(t)\right)\mathrm{d}t = 0$

The previous semi-discretized formulation corresponds to the Euler-Lagrange equation of the variational form above.

The corresponding semi-discretized weak formulation can be considered as a starting point for the construction of time integrators. In this context, it "relaxes" the concept of equilibrium at a given time.

Passage from a second order time differential equation to a system of two first order time differential equations

- *Two field Time action integral* **[Hu Washizu 1969]** *(Legendre transform on the velocity):*

$$\mathcal{A}_{HR}(\mathbf{u},\dot{\mathbf{u}},\mathbf{v}) = \int_{t_0}^{t_f} \mathcal{L}_{HR}(\mathbf{u},\dot{\mathbf{u}},\mathbf{v})\, \mathrm{d}t \qquad \mathcal{L}_{HR}(\mathbf{u},\dot{\mathbf{u}},\mathbf{v}) = \mathcal{T}_{HR}(\dot{\mathbf{u}},\mathbf{v}) - \mathcal{V}(\mathbf{u})$$

$$\delta\mathcal{A}_{HR} = 0 \qquad \mathcal{T}_{HR}(\mathbf{u},\dot{\mathbf{u}},\mathbf{v}) = \mathbf{v}^t\mathbf{M}\dot{\mathbf{u}} - \frac{1}{2}\mathbf{v}^t\mathbf{M}\mathbf{v}$$

$$\Leftrightarrow \int_{t_0}^{t_f} -\delta\mathbf{u}^t(t)\left(\mathbf{M}\dot{\mathbf{v}}(t) + \mathbf{K}\mathbf{u}(t) - \mathbf{f}_{ext}(t)\right)\mathrm{d}t + \int_{t_0}^{t_f} \delta\mathbf{v}^t(t)\,\mathbf{M}\left(\dot{\mathbf{u}}(t) - \mathbf{v}(t)\right)\mathrm{d}t = 0$$

- *Corresponding Euler-Lagrange equation and symplectic structure:* $(\mathbf{p} = \mathbf{M}\mathbf{v})$

$$\begin{aligned} \mathbf{M}\dot{\mathbf{v}}(t) + \mathbf{K}\mathbf{u}(t) &= \mathbf{f}_{ext}(t) \quad \forall t \in [t_0, t_f] \\ \dot{\mathbf{u}}(t) - \mathbf{v}(t) &= 0 \quad \forall t \in [t_0, t_f] \\ \mathbf{u}(t_0) = \mathbf{u}_0,\ \mathbf{v}(t_0) &= \mathbf{v}_0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{u}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} \frac{\partial\mathcal{L}_{\mathbf{p}}}{\partial\mathbf{p}} \\ \frac{\partial\mathcal{L}_{\mathbf{p}}}{\partial\mathbf{u}} \end{bmatrix}$$

The corresponding weak formulation introduces the velocity v as a new variable. The fundamental state vector becomes: $\mathbf{X} \equiv (\mathbf{u},\mathbf{v})$

This weak formulation in time is the basis of many modern time integrators, for instance rigid solid dynamics ***[Cardona Geradin 1989, Simo & Wong 1991, Botasso et al 2001, Krenk & Hogsberg 2005, Krenk 2006]****.*

On the other hand, the stationary principle highlights the central role of the velocity in structural dynamics. In other-words, the good duality bracket is based on velocity and momentum/impulse, not acceleration and force.

Example of kinematic constraint on a limited part of the mesh:

$$\mathbf{L}\dot{\mathbf{u}}\,(t) = 0$$

This condition can be introduced easily into the time action integral with a Lagrange multiplier:

$$\begin{aligned}\tilde{A}\,(\mathbf{u},\dot{\mathbf{u}},\mathbf{v},\boldsymbol{\lambda}) &= \int_{t_0}^{t_f} \tilde{\mathcal{L}}\,(\mathbf{u},\dot{\mathbf{u}},\mathbf{v},\boldsymbol{\lambda})\;\mathrm{d}t \\ &= \int_{t_0}^{t_f} \left(\mathcal{L}_{HR}\,(\mathbf{u},\dot{\mathbf{u}},\mathbf{v}) + \boldsymbol{\lambda}^t\,(t)\,\mathbf{L}\dot{\mathbf{u}}\,(t)\right)\;\mathrm{d}t\end{aligned}$$

- *We obtain the following three fields stationary principle:*

$$\delta\tilde{\mathcal{A}} = 0$$
$$\Leftrightarrow \int_{t_0}^{t_f} -\delta\mathbf{u}^t(t)\left(\mathbf{M}\dot{\mathbf{v}}(t) + \mathbf{K}\mathbf{u}(t) - \mathbf{f}_{ext}(t) - \mathbf{L}^t\Lambda(t)\right)\mathrm{d}t$$
$$+ \int_{t_0}^{t_f} \delta\mathbf{v}^t(t)\,\mathbf{M}\left(\dot{\mathbf{u}}(t) - \mathbf{v}(t)\right)\mathrm{d}t$$
$$+ \int_{t_0}^{t_f} \delta\boldsymbol{\lambda}^t(t)\,\mathbf{L}\dot{\mathbf{u}}(t)\,\mathrm{d}t = 0$$

- *We can then deduce the corresponding Euler-Lagrange equations:*

$$\mathbf{M}\dot{\mathbf{v}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}_{ext}(t) + \mathbf{L}^t\Lambda(t) \quad \forall t \in [t_0, t_f]$$
$$\dot{\mathbf{u}}(t) - \mathbf{v}(t) = 0 \quad \forall t \in [t_0, t_f]$$
$$\mathbf{L}\dot{\mathbf{u}}(t) = 0 \quad \forall t \in [t_0, t_f]$$
$$\mathbf{u}(t_0) = \mathbf{u}_0, \; \dot{\mathbf{u}}(t_0) = \dot{\mathbf{u}}_0$$

→ *This gives the physical meaning of the Lagrange multiplier (interface loading* $\mathbf{L}^t\Lambda$ *)* $(\Lambda = -\dot{\boldsymbol{\lambda}})$ *(generalized moment* **[Benes Matous 2010]***)*

→ *Unlike Ordinary Differential Equations without multipliers (ODE), we now get a system of Differential Algebraic Equations (DAE).*

We can now introduce the time discretization:

$$t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_f \qquad h = t_{n+1} - t_n$$

Now, from the previous three fields time action integral, we assume that the time discretized equilibrium equation is checked only at a given time within h. This time strongly depends on the chosen time integrator. This idea is not new and follows the very general concepts developed in the G-α methods [Kuhl Crisfield 1999, Erlicher Bonaventura Bursi 2002] or more generally by Tamma [Hoitink et al 2008, Masuri et al 2009] (particular cases: HHT-α, WBZ-α, CH-α)

$$\mathbf{M}\dot{\mathbf{v}}_{n+\xi_g} + \mathbf{K}\mathbf{u}_{n+\xi_f} = \mathbf{f}_{n+\xi_f} + \mathbf{L}^t \boldsymbol{\Lambda}_{n+\xi_f} \qquad (\xi_g, \xi_f) \in [0,1]^2$$

$$\int_{t_0}^{t_f} \delta\boldsymbol{\lambda}^t(t)\,\mathbf{L}\dot{\mathbf{u}}(t)\,\mathrm{d}t = 0$$

External, internal forces and multipliers are expressed at time $t_{n+\xi_f}$. As indicated by Tamma, a specific parameter is dedicated to the acceleration in order to find a second order accuracy for acceleration. In fact, many well-known 2nd order time integrators (for displacements and velocity) are only of order 1 for the acceleration [Hulbert Hughes 1988, Erlicher et al 2002, Masuri et al 2009]. Thus, the order 2 for all quantities can be obtained if an equilibrium in a weak sense in time is introduced (all LMS integrators have this specific time for acceleration).

- *The previous equation allows now to introduce a general formalism for direct time integrators with Lagrange multipliers (prerequisite in order to introduce HATI).*

$$\begin{cases} \dot{\mathbf{v}}_{n+\xi_g} = (1-\xi_g)\dot{\mathbf{v}}_n + \xi_g \dot{\mathbf{v}}_{n+1} \\ \dot{\mathbf{u}}_{n+\xi_f} = (1-\xi_f)\dot{\mathbf{u}}_n + \xi_f \dot{\mathbf{u}}_{n+1} \\ \mathbf{u}_{n+\xi_f} = (1-\xi_f)\mathbf{u}_n + \xi_f \mathbf{u}_{n+1} \\ \mathbf{f}_{n+\xi_f} = (1-\xi_f)\mathbf{f}_n + \xi_f \mathbf{f}_{n+1} \\ \mathbf{\Lambda}_{n+\xi_f} = (1-\xi_f)\mathbf{\Lambda}_n + \xi_f \mathbf{\Lambda}_{n+1} \end{cases}$$

- *Closed system of eq.: (9 unknowns 9 eq.):*

$$\begin{cases} \mathbf{u}_{n+1} = \mathbf{u}_n + h\dot{\mathbf{u}}_n + \left(\frac{1}{2}-\beta\right) h^2 \dot{\mathbf{v}}_n + \beta h^2 \dot{\mathbf{v}}_{n+1} \\ \dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + (1-\gamma)h\dot{\mathbf{v}}_n + \gamma h \dot{\mathbf{v}}_{n+1} \end{cases}$$

$$\mathbf{K}^* \Delta\dot{\mathbf{u}}_{n+1} = \mathbf{g}_{n+1} + \mathbf{L}^t \mathbf{\Lambda}_{n+\xi_f}$$

$$\int_{t_0}^{t_f} \delta\boldsymbol{\lambda}^t(t)\,\mathbf{L}\dot{\mathbf{u}}(t)\,\mathrm{d}t = 0$$

$$\mathbf{K}^* = \xi_g \frac{1}{\gamma h}\mathbf{M} + \xi_f \frac{\beta h}{\gamma}\mathbf{K}$$

$$\mathbf{g}_{n+1} = \mathbf{f}_{n+\xi_f} - \mathbf{K}\mathbf{u}_n - \xi_f h \mathbf{K}\dot{\mathbf{u}}_n - \left(1 - \frac{\xi_g}{\gamma}\right)\mathbf{M}\dot{\mathbf{v}}_n - \xi_f \left(\frac{\gamma - 2\beta}{2\gamma}\right) h^2 \mathbf{K}\dot{\mathbf{v}}_n$$

➡ *It may be noted here the role played by the two parameters in the operator K* as a linear combination of the mass and stiffness.*

- ***General compact form with Lagrange multipliers:***

$$\mathbb{K}^* \Delta \mathbb{U}_{n+1} = \mathbb{G}_{n+1} + \mathbb{L}^t \boldsymbol{\Lambda}_{n+\xi_f}$$

$$\mathbb{G}_{n+1} = \mathbb{F}_{n+\xi_f} - \mathbb{N}\,\mathbb{U}_n$$

$$\mathbb{K}^* = \begin{bmatrix} \mathbf{K}^* & \mathbf{0} & \mathbf{0} \\ -\frac{\beta h}{\gamma}\mathbf{I} & \mathbf{I} & \mathbf{0} \\ -\frac{1}{\gamma h}\mathbf{I} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbb{L}^t = \begin{bmatrix} \mathbf{L}^t \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \mathbb{U}_n = \begin{bmatrix} \dot{\mathbf{u}}_n \\ \mathbf{u}_n \\ \dot{\mathbf{v}}_n \end{bmatrix}, \quad \Delta\mathbb{U}_{n+1} = \begin{bmatrix} \Delta\dot{\mathbf{u}}_{n+1} \\ \Delta\mathbf{u}_{n+1} \\ \Delta\dot{\mathbf{v}}_{n+1} \end{bmatrix}$$

$$\mathbb{F}_{n+\xi_f} = \begin{bmatrix} \mathbf{f}_{n+\xi_f} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad \mathbb{N} = \begin{bmatrix} \xi_f h\mathbf{K} & \mathbf{K}\ \xi_f & \left(\frac{\gamma-2\beta}{2\gamma}\right) h^2\mathbf{K} + \left(1 - \frac{\xi_g}{\gamma}\right)\mathbf{M} \\ h\mathbf{I} & \mathbf{0} & \left(\frac{\gamma-2\beta}{2\gamma}\right) h^2\mathbf{I} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{\gamma}\mathbf{I} \end{bmatrix}$$

This compact form contains all the equations required for the complete solving at a given time step (for all known α-integrators), to which we add the weak kinematic constraint in time:

$$\int_{t_0}^{t_f} \delta\boldsymbol{\lambda}^t(t)\,\mathbf{L}\dot{\mathbf{u}}(t)\ \mathrm{d}t = 0$$

- *Special case of CH-α time integrators introduced by Chung and Hulbert 1993:*

$$(1-\alpha_g)\mathbf{M}\dot{\mathbf{v}}_{n+1} + \alpha_g \mathbf{M}\dot{\mathbf{v}}_n + (1-\alpha_f)\mathbf{K}\mathbf{u}_{n+1} + \alpha_f \mathbf{K}\mathbf{u}_n = (1-\alpha_f)\mathbf{f}_{n+1} + (1-\alpha_f)\mathbf{f}_n + \mathbf{L}^t \mathbf{\Lambda}_{n+1-\alpha_f}$$

$$\begin{cases} \xi_g = 1-\alpha_g \\ \xi_f = 1-\alpha_f \end{cases}$$

Scheme	α_g	α_f	γ	β
HHT-α	0	$-\alpha_{HHT} = \frac{1-\rho_\infty}{1+\rho_\infty}$	$\frac{1}{2} - \alpha_{HHT}$	$\frac{1}{4}(1-\alpha_{HHT})^2$
WBZ-α	$\alpha_{WBZ} = \frac{\rho_\infty - 1}{1+\rho_\infty}$	0	$\frac{1}{2} - \alpha_{WBZ}$	$\frac{1}{4}(1-\alpha_{WBZ})^2$
CH-α	$\frac{2\rho_\infty - 1}{1+\rho_\infty}$	$\frac{\rho_\infty}{1+\rho_\infty}$	$\frac{3}{2} - 2\alpha_f$	$(1-\alpha_f)^2$

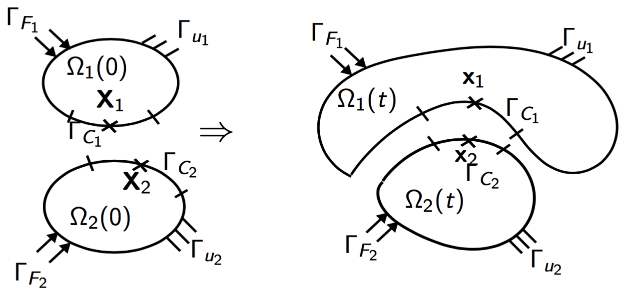
→ *In practice, the four parameters are set to achieve unconditional stability, second order accuracy, control of spurious high frequency oscillations, while minimizing the numerical dissipation at low frequencies. Both α parameters can be defined as functions of the spectral radius to the high frequency limit, in order to control the numerical dissipation rate in this interval.*

- *Particular case of Newmark:* $\xi_g = \xi = \gamma \qquad \xi_f = \xi = \gamma$

→ *No time integrator of Newmark allows the control of the spurious high frequency oscillations while ensuring second-order accuracy.*

→ *This formalism also contains Simo, Krenk, Verlet integrators as special cases with Lagrange multipliers.*

- ***Special case of non-smooth dynamics:***



- ***Hertz-Signorini-Moreau conditions (HSM):***

Impenetrability: $g_N = [(\mathbf{X}_2 + \mathbf{u}_2) - (\mathbf{X}_1 + \mathbf{u}_1)] \cdot \mathbf{n}_1 \geq 0$

Contact pressure: $t_N = \sigma_\alpha \cdot \mathbf{n}_\alpha \cdot \mathbf{n}_\alpha \leq 0 \quad , \quad \alpha = 1, 2$

Complementarity: $g_N \cdot t_N = 0$

- ***Moreau lemma* [Moreau 1978] *(velocity-impulse formulation of HSM):***

$$\begin{cases} g_N \geq 0 \\ t_N \leq 0 \\ g_N \cdot t_N = 0 \end{cases} \iff \begin{cases} \text{if } g_N > 0 \quad \text{then} \quad i_N = 0 \\ \text{if } g_N = 0 \quad \text{then} \quad \begin{cases} \dot{g}_N \geq 0 \\ i_N \leq 0 \\ \dot{g}_N \cdot i_N = 0 \end{cases} \end{cases}$$

- *Four fields time action integral with impact with velocity HSM conditions at time t_c* ***[Cirak and West 2005]***

$$\tilde{A}(\mathbf{u},\dot{\mathbf{u}},\mathbf{v},t_c,\lambda) = \int_{t_0}^{t_c} L(\mathbf{u},\dot{\mathbf{u}},\mathbf{v})\,\mathrm{d}t + \int_{t_c}^{t_f} L(\mathbf{u},\dot{\mathbf{u}},\mathbf{v})\,\mathrm{d}t + \lambda^T(t_c)\mathbf{g}_N(t_c)$$

- *Stationarity of action integral:* $\delta\tilde{A} = 0 \Rightarrow$

Smooth dynamics equation:	$\delta\mathbf{u}$	:	$\mathbf{M}\dot{\mathbf{v}} + \mathbf{F}_{int} = \mathbf{F}_{ext},\ \forall t \in [t_0, t_c^-] \cup [t_c^+, t_f]$
	$\delta\mathbf{v}$	:	$\dot{\mathbf{u}} = \mathbf{v}$
Impact equation:	δt_c	:	$\Big[\mathbf{M}\mathbf{v}(t)\Big]_{t_c^-}^{t_c^+} = \nabla\mathbf{g}_N^T(t_c)\lambda(t_c)$
Kinetic energy balance:	$\delta\mathbf{u}(t_c)$	:	$\Big[(\mathbf{M}\mathbf{v})^T\mathbf{M}^{-1}(\mathbf{M}\mathbf{v})\Big]_{t_c^-}^{t_c^+} = 0$

- *Unified HSM non-smooth dynamics equation* ***[Moreau 1999, Chen 2013, Acary 2014]***

$$d\mathbf{v} = d\mathbf{v}_s + d\mathbf{v}_{ns}$$

$$\mathbf{M}d\mathbf{v} + \mathbf{F}_{int}dt = \mathbf{F}_{ext}dt + d\mathbf{I}$$

$$d\mathbf{v}_s = \dot{\mathbf{v}}dt$$

$$d\mathbf{v}_{ns} = \mathbf{v}(t_c^+) - \mathbf{v}(t_c^-)$$

$$d\mathbf{I}(t) = \begin{cases} 0 & \forall t \in [t_0, t_c^-] \cup [t_c^+, t_f] \\ \mathbf{L}^T(t_c)\lambda(t_c) & \end{cases}$$

$$\mathbf{L} = \nabla\mathbf{g}_N$$

- *A new central difference method with Lagrange multipliers for non-smooth dynamics* ***[Belytschko 2000, Casadei 2009, Heinstein 2010, Geradin and Rixen 2014, Fekak Brun Gravouil 2017] (compatible with the previous general formalism):***

$$\mathbf{M}_{lump}\mathbf{v}_{n+\frac{3}{2}} = \mathbf{M}_{lump}\mathbf{v}_{n+\frac{1}{2}} + \Delta t(\mathbf{F}_{ext,n+1} - \mathbf{F}_{int,n+1} - \mathbf{C}\mathbf{v}_{n+\frac{1}{2}}) + \mathbf{I}_{n+1}$$
$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \mathbf{v}_{n+\frac{1}{2}}$$
$$\mathbf{I}_{n+1} = \mathbf{L}_{n+1}^T \lambda_{n+\frac{3}{2}}$$
$$v_{n+\frac{3}{2}} = \mathbf{L}_{n+1}\mathbf{v}_{n+\frac{3}{2}}$$
$$\begin{cases} \text{if } g_N^l(t_{n+1}) > 0 \text{ then } \lambda^l_{n+\frac{3}{2}} = 0 \\ \text{if } g_N^l(t_{n+1}) \leq 0 \text{ then } \begin{cases} v^l_{n+\frac{3}{2}} \geq 0 \\ \lambda^l_{n+\frac{3}{2}} \geq 0 \\ v^l_{n+\frac{3}{2}}\lambda^l_{n+\frac{3}{2}} = 0 \end{cases} \end{cases} \quad \forall l \in \{1, \ldots, p\}$$

- *Corresponding Delassus or Steklov-Poincaré operator* ***at time $t_{n+3/2}$:***

$$\left(\mathbf{L}_{n+1}\mathbf{M}_{lump}^{-1}\mathbf{L}_{n+1}^T\right)\lambda_{n+\frac{3}{2}} = \mathbf{b}$$

$$\mathbf{b} = v_{n+\frac{3}{2}} - \mathbf{L}_{n+1}\left(\mathbf{v}_{n+\frac{1}{2}} + \Delta t \mathbf{M}_{lump}^{-1}(\mathbf{F}_{ext,n+1} - \mathbf{F}_{int,n+1} - \mathbf{C}\mathbf{v}_{n+\frac{1}{2}})\right)$$

→ ***Comments: acceleration not required, time of impact detection not required***

Two kinds of Time integrators for non-smooth dynamics

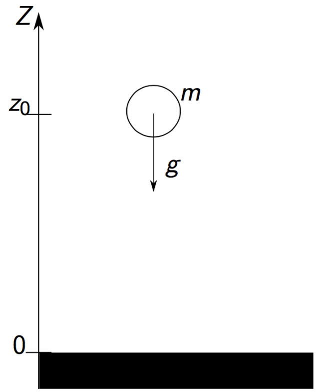
- *Event-tracking time integrators*
  - *Strong formulation in time semi-discretized in space*
  - *Solving impact equation at time tc*
  - *Solving smooth dynamic equation between to events (contact, free movement)*

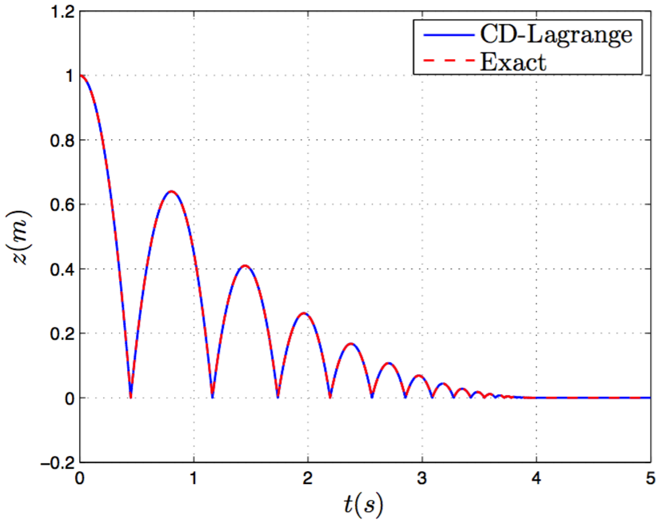
- *Time-stepping time integrators*
  - *Space-time weak formulation for non-smooth dynamics*
  - *Solving a unique non-smooth dynamic equation for one time step*

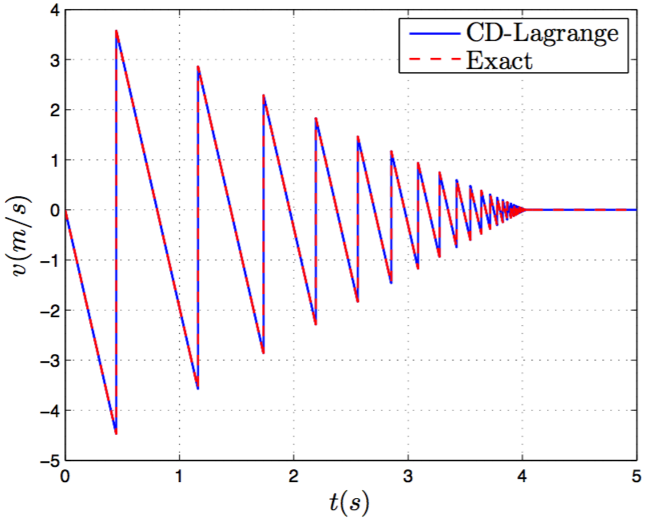
Only Time-stepping integrators converge in time with infinite impacts in a finite time (non-smooth dynamic equation and velocity HSM ensured in a weak sense on a time step).
General proof of convergence with the Hausdorff Measure ***[Acary 2012].***

The new CD-Lagrange is a symplectic explicit time-stepping integrator ***[Fekak 2017]***

- *The bouncing ball example: rigid / rigid analytical solution*
  - *First rebound:* $t_0 = \sqrt{\frac{2h}{g}}$
  - *Nth rebound:* $t_n = t_0\left(2\frac{1-e^n}{1-e} - 1\right)$
  - *Downtime of the mass:* $T = t_0\frac{1+e}{1-e}$

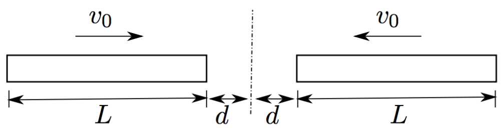




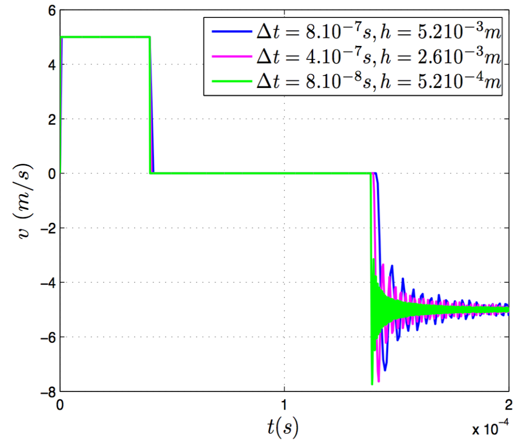


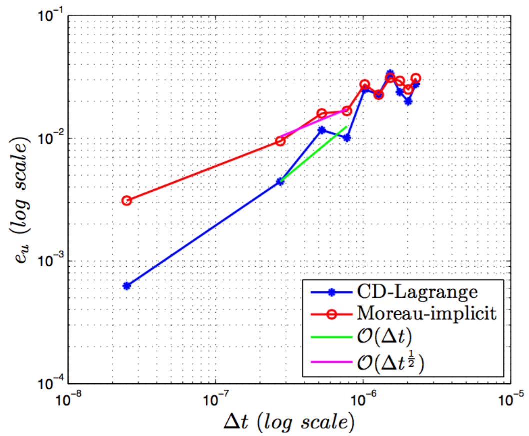
The mass does an infinity of rebounds before it stops in a finite time (Zeno Phenomenon) (restitution coefficient e=0.8)

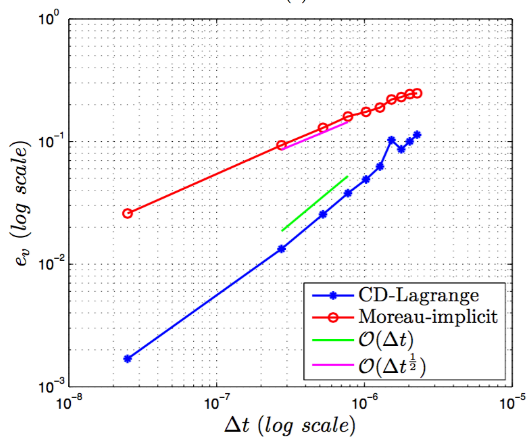
- ***Contact / impact of two identical elastic bars : deformable / deformable analytical solution***



$E = 2.1\ 10^{11}\ Pa$
$\rho = 7847\ kg/m^3$
$L = 0.254\ m$
$d = 0.2\ 10^{-3}\ m$
$v_0 = 5\ m/s$
$A = 0.645\ 10^{-3}\ m^2$



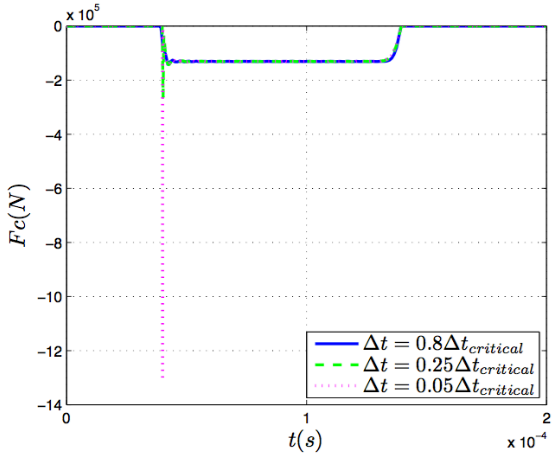




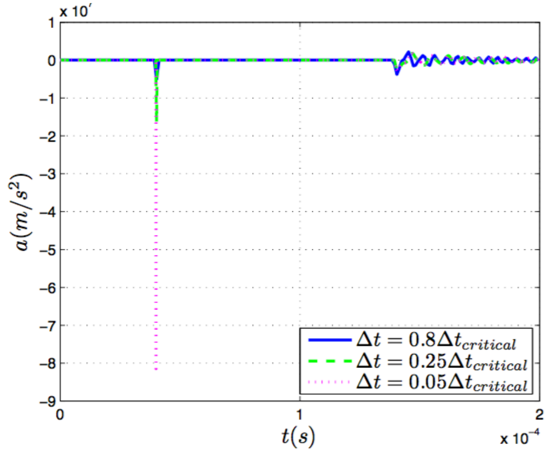
- *Space-time global error indicator (Hausdorff measure)* ***[Acary 2012]***
- *Moreau implicit of order ½, CD-Lagrange of order 1* ***[Fekak 2017]***

- ***Contact / impact of two identical elastic bars : deformable / deformable analytical solution***

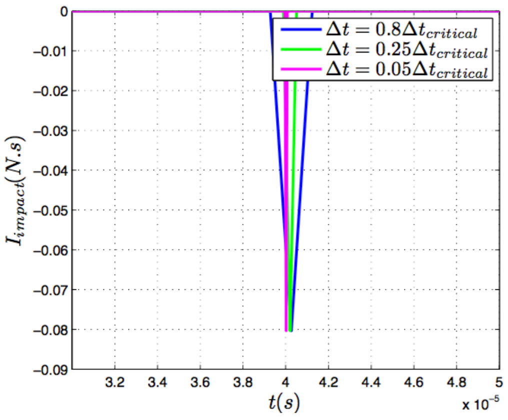
Contact force



Contact acceleration



Impact impulse



At impact time, acceleration and contact forces do not converge, however impulse converges.

2

Stability analysis

- *Numerical analysis of time integrators is essential to understand their advantages and disadvantages for structural dynamics*

→ *Stability, accuracy, numerical dissipation, overshoot are crucial.*

→ *The stability properties may be assessed by appropriate norms on the state vector* $\mathbf{X} \equiv (\mathbf{u}, \mathbf{v})$ *(displacements, velocities and accelerations possibly). They are generally based on the orthogonality property of the eigenvectors (decoupled differential systems, SDOF)* **[Hulbert Hughes 1987, Hulbert 2004]**

→ *Using of the amplification matrix:* $\mathbf{X}(t_{n+1}) - \mathbf{A}\mathbf{X}(t_n) - \mathbf{L}_n = \Delta t \tau(t_n)$

- *A-stability* $\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n \quad |\mathbf{A}| < 1$ *(symplectic)* $\mathbf{A}^t\mathbf{J}\mathbf{A} = \mathbf{J}$
- *consistency* $|\tau(t)| \le c\Delta t^k \,,\ \forall t \in [0, T],\ k > 0$

→ *Convergence: (S+C)* *if* $\Delta t \to 0$ *then* $e(t_n) \to 0 \qquad e(t_{n+1}) = \mathbf{X}(t_{n+1}) - \mathbf{X}_{n+1}$

→ *The majority of time integrators are A-stable (only in the linear case).*
What about transient nonlinear dynamics?

- *A significant example: the HHT-α time integrator can show some inefficiency of the numerical dissipation in the nonlinear case with possible overshoot of the velocity and only first order accuracy for the acceleration* ***[Bauchau Damilano Theron 1995, Hulbert Hughes 1987].***

→ *A better property is L-stability (no overshoot)* ***[Piché 1995].***

→ *L-stability combined with an optimization of the numerical dissipation of high frequency modes illustrates the very good properties of the CH-α integrators* ***[Chung Hulbert 1993].***

→ *L-stability is connected to the high frequency numerical dissipation and it has been shown that this is a critical property for non-linear constraints dynamic problems. Link to Liapunov exponents* ***[Nawrotzki Eller 2000].***

→ *In the non-linear regime, energy-stability is also a norm of great interest* ***[Belytschko Schoeberle 1975, Hughes 1976].***

$$\left[\frac{1}{2}\dot{\mathbf{v}}^t\mathbf{A}\dot{\mathbf{v}} + \frac{1}{2}\dot{\mathbf{u}}^t\mathbf{K}\dot{\mathbf{u}}\right]_n^{n+1} = \frac{1}{h}\Delta\dot{\mathbf{u}}^t\left\{(\mathbf{f}_{n+1} - \mathbf{f}_n)\right\} - \left(\gamma - \frac{1}{2}\right)\left\{\Delta\dot{\mathbf{v}}^t\mathbf{A}\Delta\dot{\mathbf{v}}\right\}$$

$$\Delta E_{kin} + \Delta E_{int} = \Delta E_{ext} + \Delta E_{diss}$$

$$\mathbf{A} = \mathbf{M} + \left(\beta - \frac{\gamma}{2}\right)h^2\mathbf{K}$$

→ *Stability properties by the pseudo-energy method (equivalent to A-stability in the linear case)* ***[Hughes 1987].***

- *Some properties of the stability analysis by the pseudo-energy:*
  - *Equivalent to the A-stability in the linear case*
  - *Does not require any state vector (useful for HATI)*
  - *With Lagrange multipliers, the following pseudo-energy interface is used in the right hand side (ODE -> DAE):*

$$\Delta E_{interface} = \frac{1}{h}\Delta \dot{\mathbf{u}}^t \left\{ \mathbf{L}^t \left(\mathbf{\Lambda}_{n+1} - \mathbf{\Lambda}_n \right) \right\}$$

→ *This method has historically served to validate or invalidate the first attempts to build new HATI.*

- *Difference between the pseudo-energy norm (dedicated to stability) and the discrete energy balance of a time integrator. Example of discretized energy balance of CH-α time integrators introduced by Chung and Hulbert 1993:*

$$\left[\tfrac{1}{2}\dot{\mathbf{u}}^t \mathbf{M}\dot{\mathbf{u}} + \tfrac{1}{2}\mathbf{u}^t\mathbf{K}\mathbf{u} + \left(\beta - \tfrac{\gamma}{2}\right)\tfrac{1}{2}h^2\dot{\mathbf{v}}^t\mathbf{M}\dot{\mathbf{v}}\right]_n^{n+1} = \cdots$$

$$\Delta\mathbf{u}^t\left\{\tfrac{1}{2}\left(\mathbf{f}_{n+1}+\mathbf{f}_n\right) + \left(\gamma - \tfrac{1}{2}\right)\left(\mathbf{f}_{n+1} - \mathbf{f}_n\right)\right\}$$

$$-\left(1-\gamma-\alpha_g\right)\Delta\mathbf{u}^t\mathbf{M}\Delta\dot{\mathbf{v}} - \left(\tfrac{1}{2}-\alpha_f\right)\Delta\mathbf{u}^t\mathbf{K}\Delta\mathbf{u}$$

$$-\left(\gamma-\tfrac{1}{2}\right)\left(\beta-\tfrac{\gamma}{2}\right)h^2\Delta\dot{\mathbf{v}}^t\mathbf{M}\Delta\dot{\mathbf{v}}$$

$$\Delta W_{kin} + \Delta W_{int} + \Delta W_{comp} = \Delta W_{ext} + \Delta W_{diss}$$

3

A unified strategy to control « locally » the accuracy and the time step

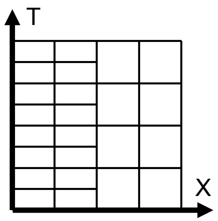
A state of the art on Heterogeneous Asynchronous Time Integrators (HATI)

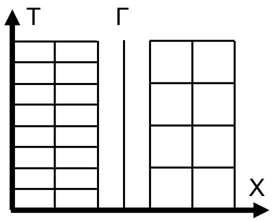
- *In order to go beyond the conventional time integrators in nonlinear computational dynamics, many attempts in the past consisted in developing HATI within the finite element method. We can highlight the pioneering work of Belytschko and co-authors in the 70s.*
  - *Mixed methods (exp / imp) or Multi-time steps (exp / exp or imp / imp) element or node partitions* [Mullen Kennedy Belytschko 1976, Belytschko Mullen 1978, Hughes Liu 1978, Hughes Pister Taylor 1979, Smolinski Belytschko 1988]
  - *Mixed methods and multi-time-step (exp / imp)* [Liu Belytschko 1982]
  - *A first proof of stability by energy method (exp / exp multi-time)* [Smolinski 1992, Smolinski Sleith Belytschko 1996] *and in the case (exp/imp same time)* [Hughes 1987]
  - *An original strategy for element time integration (with a unique time integrator); explicit time integrator* [Casadei Halleux 2009] *variational time integrator* [Ryckman Lew 2012]

Two difficulties: no evidence of stability in the general case (in particular multi-time exp / imp) and difficulties in the explicit case (multi-time exp / exp) with possibilities of instability lower than the CFL condition [Belytschko Lu 1993, Daniel 1997, Daniel 2003]; *statistic stability condition* [Sotelino 1994]

A first remark: it seems that no evidence of stability will be available for these methods because counter-examples exist (unstable multi-time exp / imp)
A second note: all these methods are equivalent to primal domain decomposition techniques (displacement continuity on the interface)

From a theoretical point of view, space-time weak formulations allow to choose displacement, velocity or acceleration interface continuity. Subsequently, we show that ***a velocity dual approach*** *(with Lagrange multipliers) is the (only?) one able to provide a general framework for any time integrator coupling with their own time scale (HATI)*





$$\Omega_A \cap \Omega_B = \emptyset$$
$$h_A = [t_0; t_m]$$
$$h_A = m h_B$$

- *Starting from the three-fields space-time weak formulation previously introduced with a velocity continuity between two subdomains A and B (subsequently be omitted weak relations between* $\dot{\mathbf{u}}$ *and* $\mathbf{v}$ *for clarity):*

$$\delta \tilde{A} = 0$$
$$\Leftrightarrow -\int_{t_0}^{t_m} \delta \mathbf{u}^{A^t}(t)\left(\mathbf{M}_A \dot{\mathbf{v}}^A(t) + \mathbf{K}_A \mathbf{u}^A(t) - \mathbf{f}_A(t) - \mathbf{L}_A^t \Lambda(t)\right) \mathrm{d}t$$
$$- \int_{t_0}^{t_m} \delta \mathbf{u}^{B^t}(t)\left(\mathbf{M}_B \dot{\mathbf{v}}^B(t) + \mathbf{K}_B \mathbf{u}^B(t) - \mathbf{f}_B(t) - \mathbf{L}_B^t \Lambda(t)\right) \mathrm{d}t$$
$$+ \int_{t_0}^{t_m} \delta \lambda^t(t)\left(\mathbf{L}_A \dot{\mathbf{u}}^A(t) + \mathbf{L}_B \dot{\mathbf{u}}^B(t)\right) \mathrm{d}t = 0$$

- *In order to consider a wide class of time integrators, we follow the approach of the first part inspired by Crisfield, Tamma and co-authors. We derive a system of 7 equations with 8 parameters* $\gamma_A, \beta_A, \xi_{A,f}, \xi_{A,g}$ *and* $\gamma_B, \beta_B, \xi_{B,f}, \xi_{B,g}$ *:*

$$\begin{cases} \mathbf{K}_A^* \Delta\dot{\mathbf{u}}_m^A = \mathbf{g}_m^A + \mathbf{L}_A^t \mathbf{\Lambda}_{n+\xi_A,f} \\ \Delta\mathbf{u}_m^A = \frac{\beta_A h_A}{\gamma_A}\Delta\dot{\mathbf{u}}_m^A + h_A \dot{\mathbf{u}}_0^A + \frac{\gamma_A - 2\beta_A}{2\gamma_A} h_A^2 \dot{\mathbf{v}}_0^A \\ \Delta\dot{\mathbf{v}}_m^A = \frac{1}{\gamma_A h_A}\Delta\dot{\mathbf{u}}_m^A - \frac{1}{\gamma_A}\dot{\mathbf{v}}_0^A \end{cases}$$

$$\begin{cases} \mathbf{K}_B^* \Delta\dot{\mathbf{u}}_j^B = \mathbf{g}_j^B + \mathbf{L}_B^t \mathbf{\Lambda}_{j-1+\xi_B,f} \\ \Delta\mathbf{u}_j^B = \frac{\beta_B h_B}{\gamma_B}\Delta\dot{\mathbf{u}}_j^B + h_B \dot{\mathbf{u}}_{j-1}^B + \frac{\gamma_B - 2\beta_B}{2\gamma_B} h_B^2 \dot{\mathbf{v}}_{j-1}^B \\ \Delta\dot{\mathbf{v}}_j^B = \frac{1}{\gamma_B h_B}\Delta\dot{\mathbf{u}}_j^B - \frac{1}{\gamma_B}\dot{\mathbf{v}}_{j-1}^B \\ \forall j \in \{1, m\} \end{cases}$$

$$\int_{t_0}^{t_m} \delta\lambda^t(t)\left(\mathbf{L}_A \dot{\mathbf{u}}^A(t) + \mathbf{L}_B \dot{\mathbf{u}}^B(t)\right) \mathrm{d}t = 0$$

→ ***Each subdomain A and B has its own time scale and time integrator***

- ***With a more compact form:***

$$\begin{cases} \mathbb{K}_A^* \Delta \mathbb{U}_m^A = \mathbb{F}_{\gamma_A}^A - \mathbb{N}_A \mathbb{U}_0^A + \mathbb{L}_A^t \mathbf{\Lambda}_{\xi_A,f} \\ \mathbb{K}_B^* \Delta \mathbb{U}_j^B = \mathbb{F}_{j-1+\gamma_B}^B - \mathbb{N}_B \mathbb{U}_{j-1}^B + \mathbb{L}_B^t \mathbf{\Lambda}_{j-1+\xi_B,f} \ \forall j \in \{1, m\} \\ \int_{t_0}^{t_m} \delta\lambda^t(t) \left(\mathbf{L}_A \dot{\mathbf{u}}^A(t) + \mathbf{L}_B \dot{\mathbf{u}}^B(t)\right) \mathrm{d}t = 0 \end{cases}$$

➡ *The last step consists in discretizing the weak time continuity of the velocity at the interface between A and B.*

➡ *For that purpose, we consider the following pseudo-interface energy between 2 subdomains:*

$$\Delta E_{kin,m}^A + \Delta E_{int,m}^A + \sum_{j=1}^{m} \left\{\Delta E_{kin,j}^B + \Delta E_{int,j}^B\right\} = \cdots$$

$$\left(\Delta E_{ext,m}^A + \sum_{j=1}^{m} \Delta E_{ext,j}^B\right) + \left(\Delta E_{diss,m}^A + \sum_{j=1}^{m} \Delta E_{diss,j}^B\right) + \Delta E_{interface}$$

$$\Delta E_i = \frac{1}{h}_A \Delta \dot{\mathbf{u}}_m^{A\,t} \left\{\mathbf{L}_A^t \left(\mathbf{\Lambda}_m - \mathbf{\Lambda}_0\right)\right\} + \sum_{j=1}^{m} \left\{\frac{1}{h}_B \Delta \dot{\mathbf{u}}_j^{B\,t} \left\{\mathbf{L}_B^t \left(\mathbf{\Lambda}_j - \mathbf{\Lambda}_{j-1}\right)\right\}\right\}$$

- *Historically, the first HATI based on a dual velocity formulation consisted in the introduction of an additional hypothesis on the Lagrange multipliers within the Newmark time integrators: linear time interpolation from the macro to the micro scale* ***[Gravouil 2000, Combescure Gravouil 2001]****:*

$$\boldsymbol{\Lambda}_j = \left(1 - \frac{j}{m}\right)\boldsymbol{\Lambda}_0 + \frac{j}{m}\boldsymbol{\Lambda}_m$$

- *In the context of α methods with Lagrange multipliers (HTTP-α, α-WBZ, CH-α), this expression is a natural extension of the general expression introduced at the beginning of the presentation:*

$$\boldsymbol{\Lambda}_{n+\xi_f} = (1 - \xi_f)\boldsymbol{\Lambda}_n + \xi_f \boldsymbol{\Lambda}_{n+1}$$

→ *Finally we can also write:*

$$\boldsymbol{\Lambda}_j - \boldsymbol{\Lambda}_{j-1} = \frac{1}{m}(\boldsymbol{\Lambda}_m - \boldsymbol{\Lambda}_0)$$

→ *The pseudo-interface energy becomes:*

$$\Delta E_{interface} = \left[\frac{1}{h_A}\Delta\dot{\mathbf{u}}_m^{A\,t}\mathbf{L}_A^t + \sum_{j=1}^{m}\left\{\frac{1}{m h_B}\Delta\dot{\mathbf{u}}_j^{B\,t}\mathbf{L}_B^t\right\}\right](\boldsymbol{\Lambda}_m - \boldsymbol{\Lambda}_0)$$

Two options for canceling the pseudo-interface energy are available (to ensure stability of the overall problem) ***[Brun Gravouil Combescure 2014]:***

$$\mathbf{L}_A \Delta\dot{\mathbf{u}}_m^A + \sum_{j=1}^{m} \mathbf{L}_B \Delta\dot{\mathbf{u}}_j^B = 0 \qquad \frac{1}{m}\mathbf{L}_A \Delta\dot{\mathbf{u}}_m^A + \mathbf{L}_B \Delta\dot{\mathbf{u}}_j^B = 0$$

The first relation defines ***the family of BGC-macro (dual HATI)*** *, because velocity continuity is ensured at the macro time scale between A and B.*

The second relation defines ***the family of BGC-micro (dual HATI)*** *, because velocity continuity is ensured at the micro time scale between A and B.*

- *Some properties for these two classes of HATI:*

  - *They guarantee by definition the overall stability of the coupling of α-type time integrators with their own time scale (proof of stability by Energy method)*
  - ***The BGC-micro family includes as a special case the GC method*** *(HATI micro dedicated to Newmark time integrators)* ***[Gravouil Combescure 2001].***
  - ***The BGC-macro family includes as a special case the PH method*** *(HATI macro dedicated to Newmark time integrators)* ***[Prakash Hjelmstad 2004].***

- ***BGC-macro HATI numerical implementation:***

$$
\left[\begin{array}{ccccc|c|c}
\mathbb{K}_B^* & & & & & & \mathbb{C}^t_{B,1} \\
\mathbb{N}_B & \mathbb{K}_B^* & & & & & -\mathbb{C}^t_{B,2} \\
\mathbb{N}_B & \mathbb{N}_B & \mathbb{K}_B^* & & & & -\mathbb{C}^t_{B,3} \\
 & \ddots & \ddots & \ddots & & & \vdots \\
\mathbb{N}_B & \mathbb{N}_B & \mathbb{N}_B & \mathbb{N}_B & \mathbb{K}_B^* & & -\mathbb{C}^t_{B,m} \\
\hline
 & & & & & \mathbb{K}_A^* & -\mathbb{C}^t_{A,m} \\
\hline
-\mathbb{L}_B & -\mathbb{L}_B & -\mathbb{L}_B & \cdots & -\mathbb{L}_B & -\mathbb{L}_A & \mathbf{0}
\end{array}\right]
\left[\begin{array}{c}
\Delta\mathbb{U}_1^B \\ \Delta\mathbb{U}_2^B \\ \Delta\mathbb{U}_3^B \\ \vdots \\ \Delta\mathbb{U}_m^B \\ \hline \Delta\mathbb{U}_m^A \\ \hline \mathbf{\Lambda}_m
\end{array}\right]
=
\left[\begin{array}{c}
\mathbb{F}^B_{\xi_{B,f}} - \mathbb{N}_B\mathbb{U}_0^B + \mathbb{E}^t_{B,1}\mathbf{\Lambda}_0 \\
\mathbb{F}^B_{1+\xi_{B,f}} - \mathbb{N}_B\mathbb{U}_0^B + \mathbb{E}^t_{B,2}\mathbf{\Lambda}_0 \\
\mathbb{F}^B_{2+\xi_{B,f}} - \mathbb{N}_B\mathbb{U}_0^B + \mathbb{E}^t_{B,3}\mathbf{\Lambda}_0 \\
\vdots \\
\mathbb{F}^B_{m-1+\xi_{B,f}} - \mathbb{N}_B\mathbb{U}_0^B + \mathbb{E}^t_{B,m}\mathbf{\Lambda}_0 \\
\hline
\mathbb{F}^A_{\xi_{A,f}} - \mathbb{N}_A\mathbb{U}_0^A + \mathbb{E}^t_{A,m}\mathbf{\Lambda}_0 \\
\hline
\mathbf{0}
\end{array}\right]
$$

$$
\begin{bmatrix} K & -L^t \\ B & \mathbf{0} \end{bmatrix}
\begin{bmatrix} \Delta U \\ \Lambda \end{bmatrix}
=
\begin{bmatrix} F \\ \mathbf{0} \end{bmatrix}
$$

[Prakash Hjelmstad 2004, Mahjoubi Gravouil Combescure 2009, Brun Gravouil Combescure 2014]

$$
\begin{cases}
K\Delta U_{free} = F \\
H\Lambda = -B\Delta U_{free} \quad \text{with } H = \left[BK^{-1}L^t\right] \\
K\Delta U_{link} = L^t\Lambda
\end{cases}
$$

The BGC-macro family requires the calculation of a single Lagrange multiplier on each macro time step

- *BGC-micro HATI numerical implementation:*

$$\begin{cases} \mathbb{K}_A^* \Delta \mathbb{U}_m^A = \mathbb{F}_{\gamma_A}^A - \mathbb{N}_A \mathbb{U}_0^A + \mathbb{L}_A^t \mathbf{\Lambda}_{\xi_A,f} \\ \mathbb{K}_B^* \Delta \mathbb{U}_j^B = \mathbb{F}_{j-1+\gamma_B}^B - \mathbb{N}_B \mathbb{U}_{j-1}^B + \mathbb{L}_B^t \mathbf{\Lambda}_{j-1+\xi_B,f} \ \forall j \in \{1,m\} \\ \frac{1}{m} \mathbf{L}_A \Delta \dot{\mathbf{u}}_m^A + \mathbf{L}_B \Delta \dot{\mathbf{u}}_j^B = 0 \end{cases}$$

- *The BGC-micro family requires the calculation of m Lagrange multipliers on each micro time step*

$$\mathbf{H} \Delta \mathbf{\Lambda}_j = -\mathbf{b}_j$$

With:

$$\begin{cases} \mathbf{H} = \left[\xi_{A,f} \mathbf{L}_A (\mathbf{K}_A^*)^{-1} \mathbf{L}_A^t + \xi_{B,f} \mathbf{L}_B (\mathbf{K}_B^*)^{-1} \mathbf{L}_B^t \right] \\ \mathbf{b}_j = \frac{1}{m} \mathbf{L}_A \Delta \dot{\mathbf{u}}_{free,m}^A + L_B \Delta \dot{\mathbf{u}}_{free,j}^B - \frac{1}{m} \mathbf{L}_A (\mathbf{K}_A^*)^{-1} \mathbf{L}_A^t \mathbf{\Lambda}_0 - \mathbf{L}_B (\mathbf{K}_B^*)^{-1} \mathbf{L}_B^t \mathbf{\Lambda}_{j-1} \end{cases}$$

- *In practice, we can prove that the BGC-micro family – although stable – can numerically dissipate energy at the interface, which is not the case for the BGC-macro family* ***[Gravouil & al 2001]***

$$\Delta E_{i\,BGC\text{-}macro} = 0$$
$$\Delta E_{i\,BGC\text{-}micro} \leqslant 0$$

- *Recently, the A-stability analysis (amplification matrix) and the consistency analysis have been conducted for the BGC family with the following state vector* ***[Confalonieri et al 2013, Brun Gravouil Combescure 2014]****:*

- ***Stability analysis:***

$$\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n$$

$$\begin{bmatrix} \mathbf{X}^A_{n+1} \\ \mathbf{X}^B_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{AA} & \mathbf{A}^{AB} \\ \mathbf{A}^{BA} & \mathbf{A}^{BB} \end{bmatrix} \begin{bmatrix} \mathbf{X}^A_n \\ \mathbf{X}^B_n \end{bmatrix}$$

$$\mathbf{X}_n = \left[\mathbf{X}^A_n \ \ \mathbf{X}^B_n\right]^T = \left[\mathbf{u}^A_n \ \ \dot{\mathbf{u}}^A_n \ \ h_A \dot{\mathbf{v}}^A_n \ \ h_A \boldsymbol{\lambda}_n \ \ \mathbf{u}^B_n \ \ \dot{\mathbf{u}}^B_n \ \ h_A \dot{\mathbf{v}}^B_n\right]^T$$

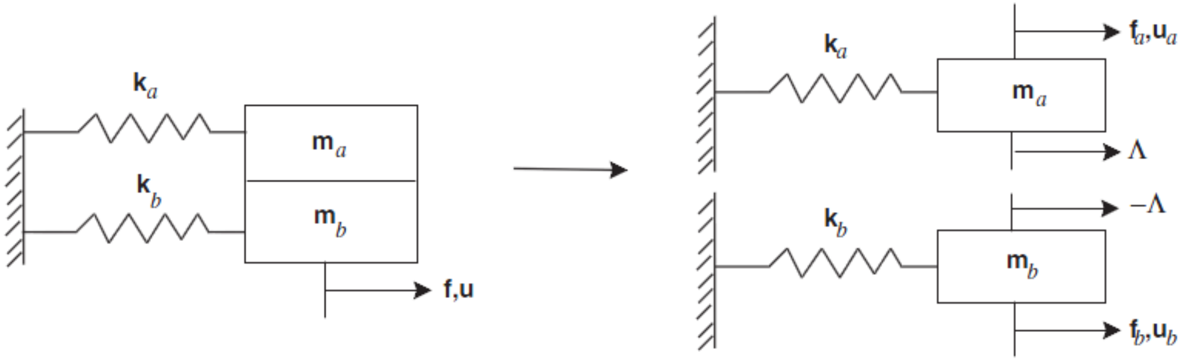
- ***Truncation error anaylsis (consistency)***

$$\tau_n = \mathbf{A}\mathbf{X}(t_n) - \mathbf{X}(t_{n+1}) \qquad \tau_n = \alpha h_A^k + O(h_A^{k+1})$$

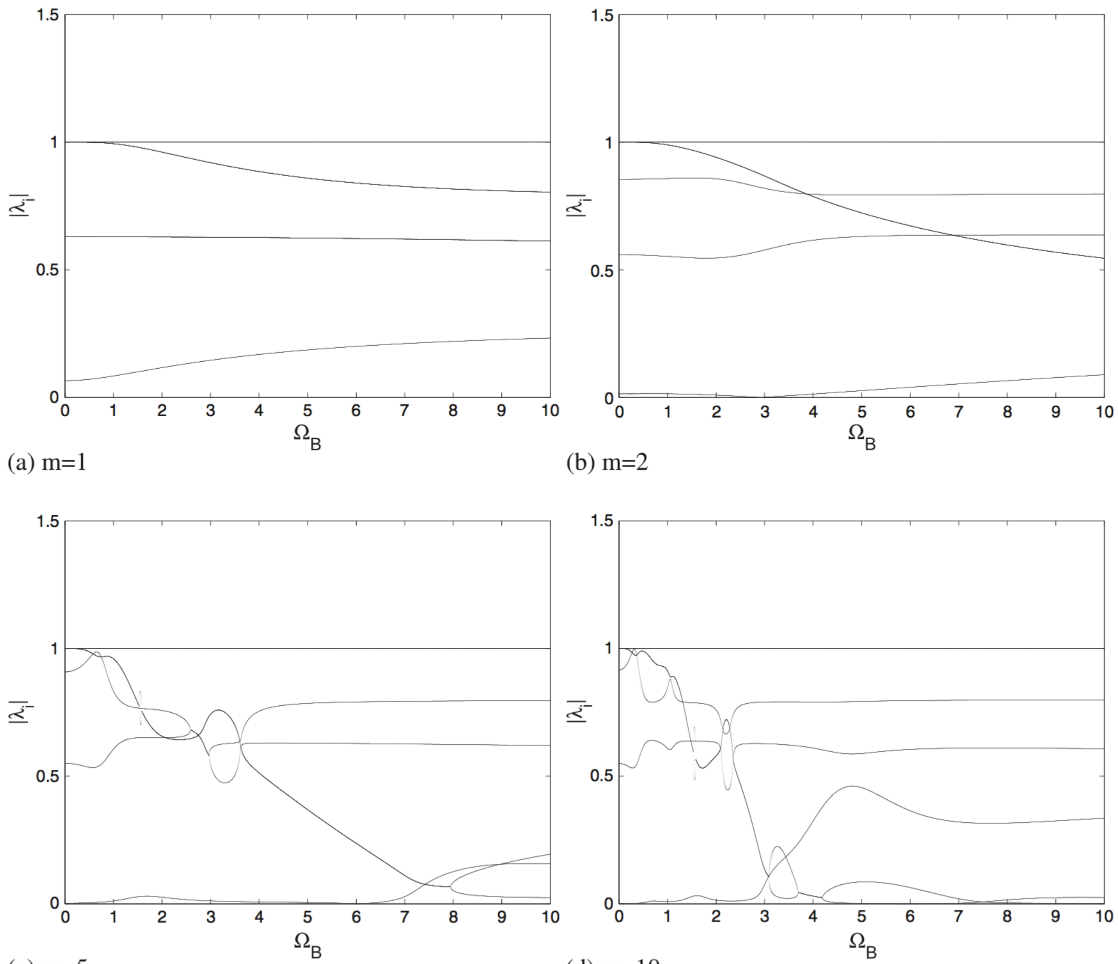
$$\tau^A_u = O\left(h_A^4\right), \tau^A_v = O\left(h_A^3\right), \tau^A_a = O\left(h_A^3\right), \boxed{\tau^A_\lambda = O\left(h_A^3\right)}$$

$$\tau^B_u = O\left(h_A^4\right), \tau^B_v = O\left(h_A^3\right), \tau^B_a = O\left(h_A^3\right)$$

- *Analysis of BGC-macro and BGC-micro methods for a double oscillator*
  [Brun Gravouil Combescure 2014]



- ***Absolute value of eigenvalues of a double oscillator with the BGC-macro (CH-α $\rho_\infty = 0.8$ / CH-α $\rho_\infty = 0.5$) for different ratio of time steps***

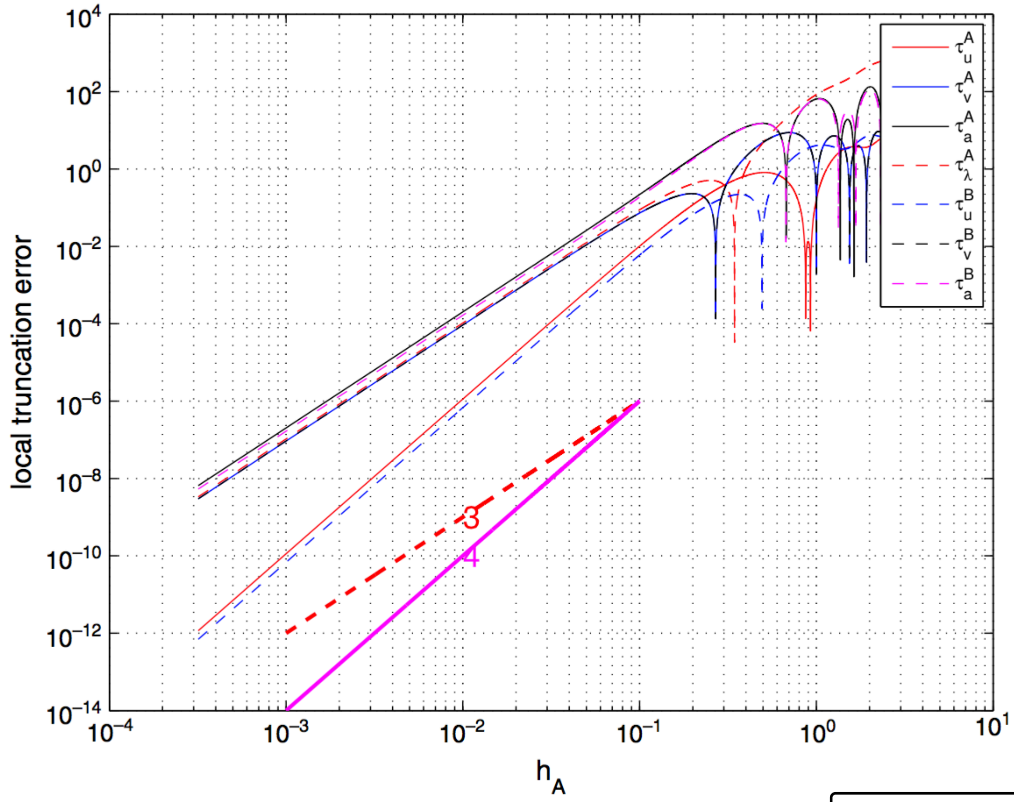


(a) m=1 (b) m=2

(c) m=5 (d) m=10

- *Truncation error τ of a double oscillator with the BGC-macro method (CH-α $\rho_\infty = 0.8$ / CH-α $\rho_\infty = 0.5$) with a ratio of time steps of 2*

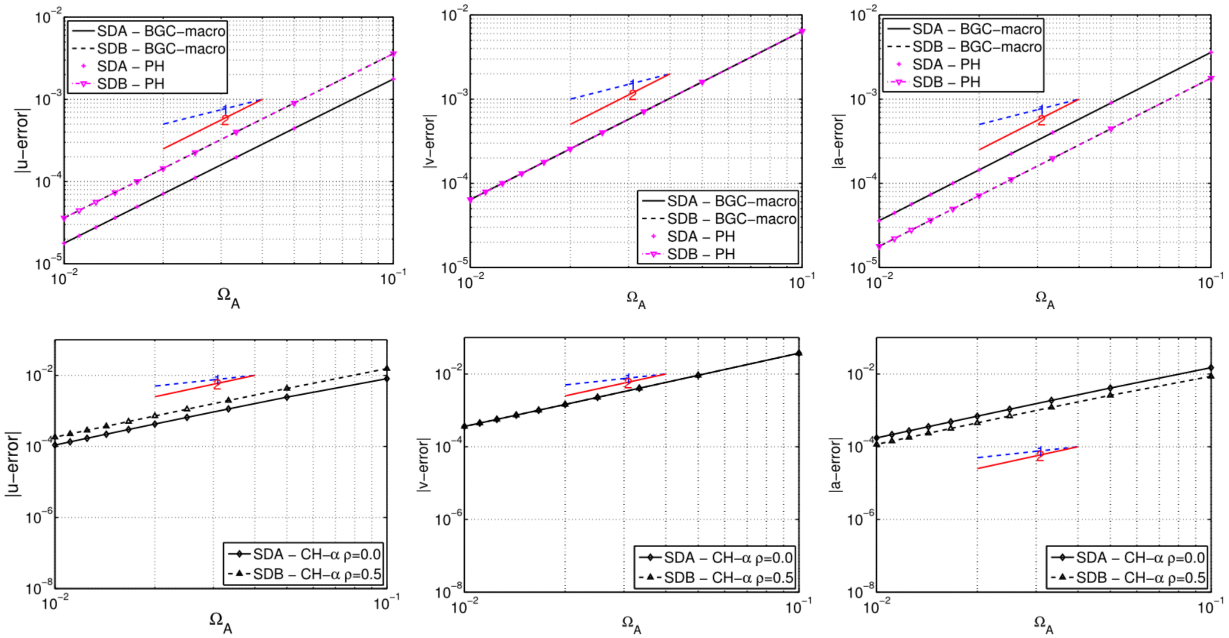
$$k = \frac{\ln(\tau_n(h_{2,A})) - \ln(\tau_n(h_{1,A}))}{\ln(h_{2,A}) - \ln(h_{1,A})}$$



$$\tau_u^A = O\left(h_A^4\right), \tau_v^A = O\left(h_A^3\right), \tau_a^A = O\left(h_A^3\right), \boxed{\tau_\lambda^A = O\left(h_A^3\right)}$$

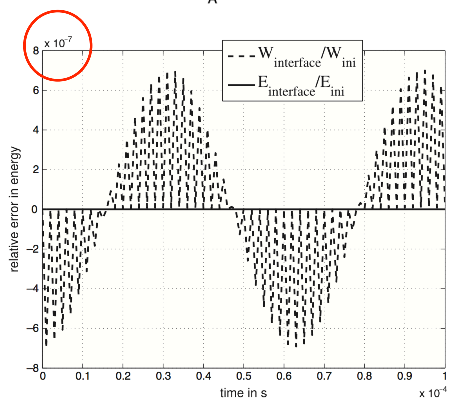
$$\tau_u^B = O\left(h_A^4\right), \tau_v^B = O\left(h_A^3\right), \tau_a^B = O\left(h_A^3\right)$$

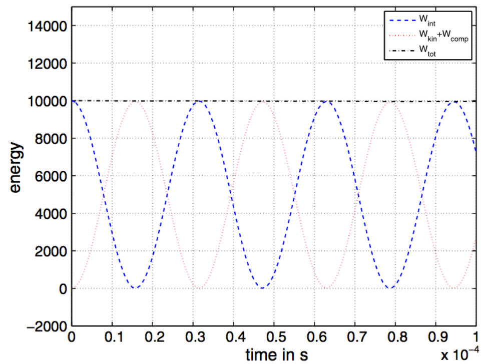
Convergence rate of the BGC-macro with m=20 (globally second order)



AA /CD

CH-α $\rho_\infty = 0.8$ */*
CH-α $\rho_\infty = 0.5$

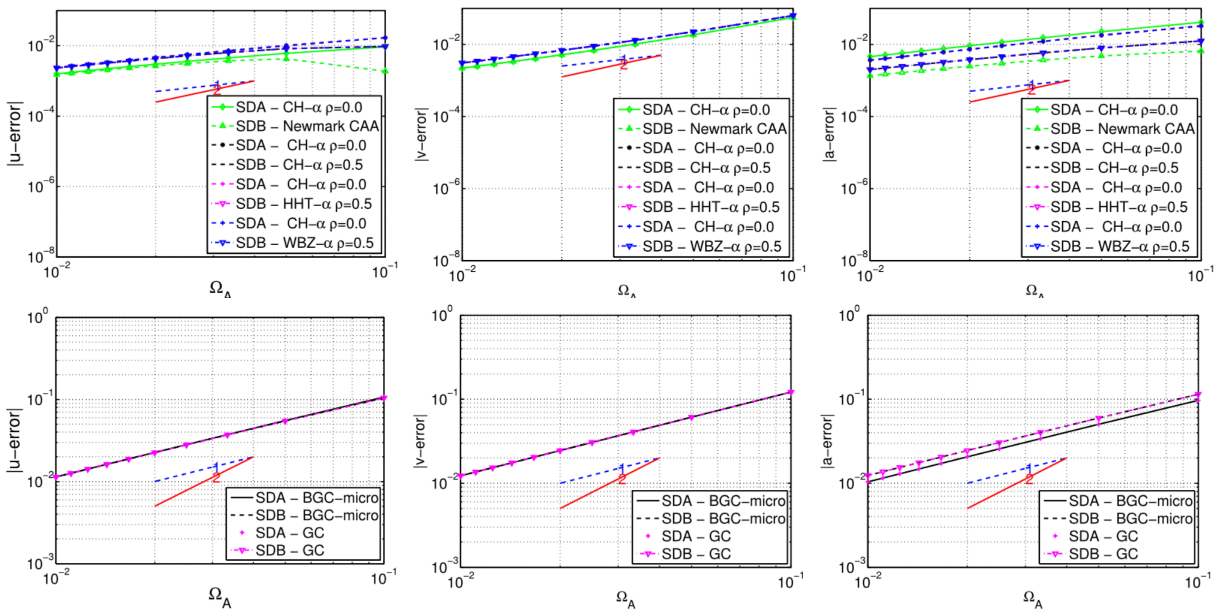




Discretized energy balance

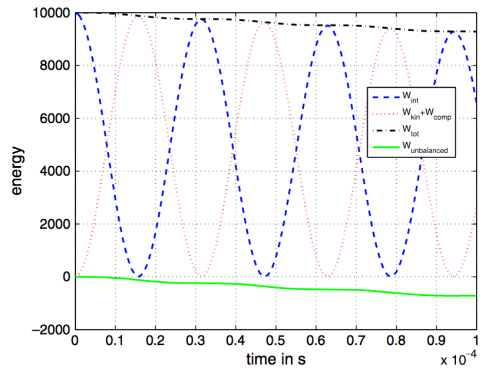
$$\Delta E_{i\,BGC\text{-}macro} = 0$$

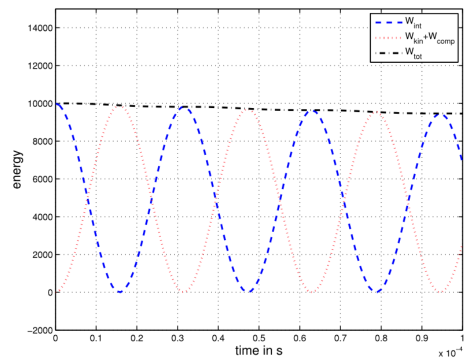
- ***Convergence rate of the BGC-micro with m=20 (globally first order)***



AA /CD

CH-α $\rho_\infty = 0.8$ ***/***
CH-α $\rho_\infty = 0.5$



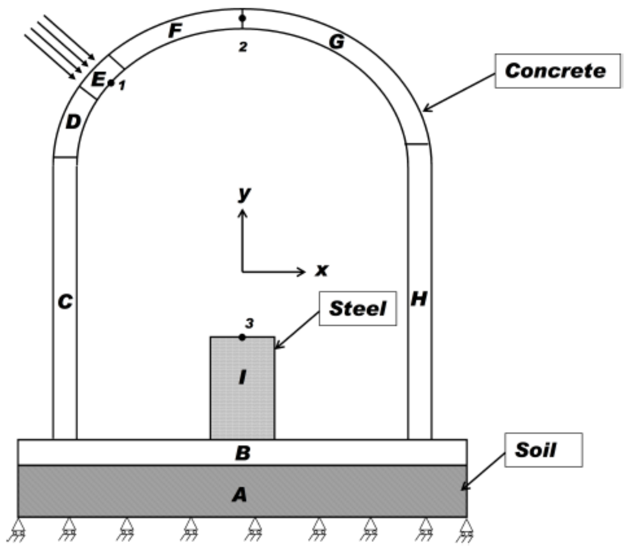


Discretized energy balance

$$\Delta E_{i\,BGC\text{-}micro} \leqslant 0$$

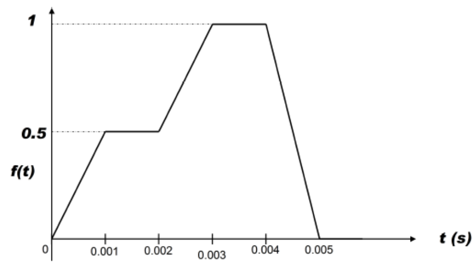
A second example (BGC macro) ***[Mahjoubi et al 2010]***

- Subdomain A : Newmark $\gamma = 0.5$, $\beta = 0.25$
- Subdomain B : Newmark $\gamma = 0.5$, $\beta = 0.66$
- Subdomain C : Krenk $\alpha = 0.25$
- Subdomain D : HHT $\alpha = -0.2$, $\gamma = 0.7$, $\beta = 0.36$
- Subdomain E : Newmark $\gamma = 0.5$, $\beta = 0.0$
- Subdomain F : HHT $\alpha = -0.1$, $\gamma = 0.6$, $\beta = 0.3025$
- Subdomain G : Newmark $\gamma = 0.7$, $\beta = 0.36$
- Subdomain H : Newmark $\gamma = 0.55$, $\beta = 0.2756$
- Subdomain I : Krenk $\alpha = 0.1111$

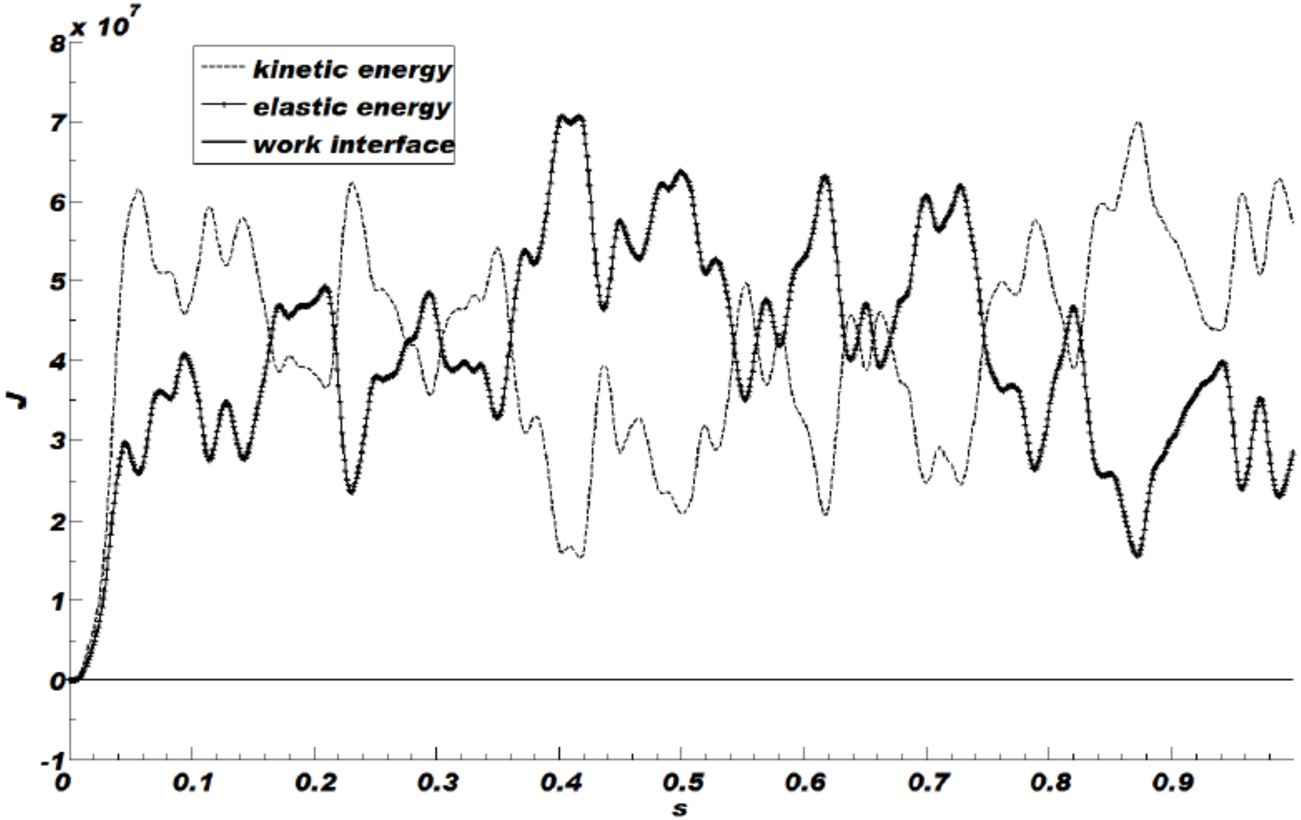


Sous-domaine	A	B	C	D	E	F	G	H	I
m_j	1	4	128	512	1024	256	128	32	16

Ratio between the different time scales



Discretized energy balance



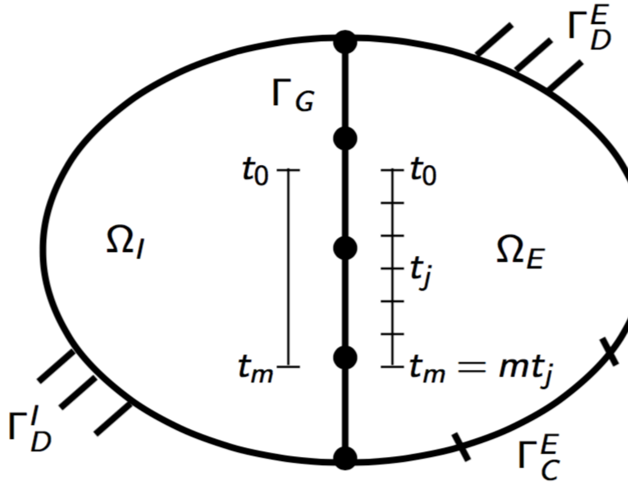
- ***A first conclusion on BGC-type dual HATI***

  Applications in many fields of nonlinear dynamics: simulation of crash tests, impacts on safety structures, earthquakes, incompatible interfaces, experimental / numerical hybrid testing in real time, microsystems, electro-mechanical coupled problems, microelectronics fracture , fluid / structure interaction, co-simulation ***[Gravouil 2000, Combescure et al 2001, Herry et al 2002, Faucher et al 2003, Noels et al 2004, Pinto et al 2004, Bourel et al 2006, Bonnet et al 2008, Bonelli et al 2008, Mahjoubi et al 2009, Abaqus 6.13 users guide 2011, Batti et al 2011, Brun et al 2012, Ghanem et al 2012, Confalonieri et al 2012, Corigliano et al 2013, Bettinotti Allix Malherbe 2013, Li Combescure Leboeuf 2013, Karimi et al 2014, Chantrait Rannou Gravouil 2014, Prakash 2014]***

  → *Co-simulation: cast3m/cast3m; rad2rad; abaqus standard/abaqus explicit; cast3m/europlexus; zébulon/europlexus*

- ***Here we can distinguish the well known 'partitioned' approaches used in fluid / structure interaction and the 'monolithic' approaches. Formally, the approaches developed here are equivalent to monolithic strategies and offer general stability results for any α-time integrator and any time scale.***

- *Heterogeneous Asynchronous Implicit-Explicit integrator for non-smooth dynamics*



Heterogeneous Asynchronous Time Integrator : BGC-micro [Gravouil and Combescure, 2001]

- Space-time Velocity continuity

$$\mathbf{L}_G^E \mathbf{v}^E(t) + \mathbf{L}_G^I \mathbf{v}^I(t) = 0 \quad \forall\ t \in [t_0, t_m]$$

Where $\mathbf{L}_G^k$ $(k = I, E)$ are restriction operators from Ω_k to Γ_G

- ***Space-time weak form for implicit-explicit non-smooth dynamics***

  - Action integral (HATI [Gravouil et al., 2015]) :

$$\tilde{\tilde{A}}(\mathbf{u}^I,\mathbf{v}^I,\mathbf{u}^E,\mathbf{v}^E,t_c) = A^I(\mathbf{u}^I,\mathbf{v}^I)+\tilde{A}^E(\mathbf{u}^E,\mathbf{v}^E,t_c) + \int_{t_0}^{t_m}\left(\mathbf{L}_G^E\mathbf{v}^E(t)+\mathbf{L}_G^I\mathbf{v}^I(t)\right)^T\lambda_G\,\mathrm{d}t$$

  - Stationarity of the Action integral : $\delta\tilde{\tilde{A}}=0 \Rightarrow$
    - In Ω_I :

$$\mathbf{M}^I\dot{\mathbf{v}}^I(t)+\mathbf{F}^I_{int}(t)=\mathbf{F}^I_{ext}(t)+\mathbf{F}^I_{link}(t)$$

    - In Ω_E :

$$\mathbf{M}^E d\mathbf{v}^E+\mathbf{F}^E_{int}dt=\mathbf{F}^E_{ext}dt+\mathbf{F}^E_{link}dt+d\mathbf{I}$$

      $+$ HSM velocity conditions

    - On interface Γ_G :

$$\mathbf{L}_G^E\mathbf{v}^E(t)+\mathbf{L}_G^I\mathbf{v}^I(t)=0$$

- *Time discretization of the implicit-explicit non-smooth dynamics equations*

- Equilibrium equation in Ω_I at macro time step t_{n+m} :

$$\mathbf{M}^I \dot{\mathbf{v}}^I_{n+m} + \mathbf{C}^I \mathbf{v}^I_{n+m} + \mathbf{F}^I_{int,n+m} = \mathbf{F}^I_{ext,n+m} + \mathbf{F}^I_{D,n+m} + \mathbf{F}^I_{link,n+m}$$

- Non-smooth dynamic equation in Ω_E at micro time step t_{n+j} :

$$\mathbf{M}^E_{lump} \mathbf{v}^E_{n+j+\frac{1}{2}} = \mathbf{M}^E_{lump} \mathbf{v}^E_{n+j-\frac{1}{2}} - \Delta t(\mathbf{F}^E_{int,n+j} + \mathbf{C}^E \mathbf{v}^E_{n+j-\frac{1}{2}}) + \Delta t(\mathbf{F}^E_{ext,n+j} + \mathbf{F}^E_{D,n+j} + \mathbf{F}^E_{link,n+j}) + \mathbf{I}_{n+j}$$

- contact equation at micro time step t_{n+j} :

$$\mathbf{H}_c \Lambda_{c,n+j+\frac{1}{2}} = \mathbf{b}_c$$

- coupling equation at micro time step t_{n+j} (GC) :

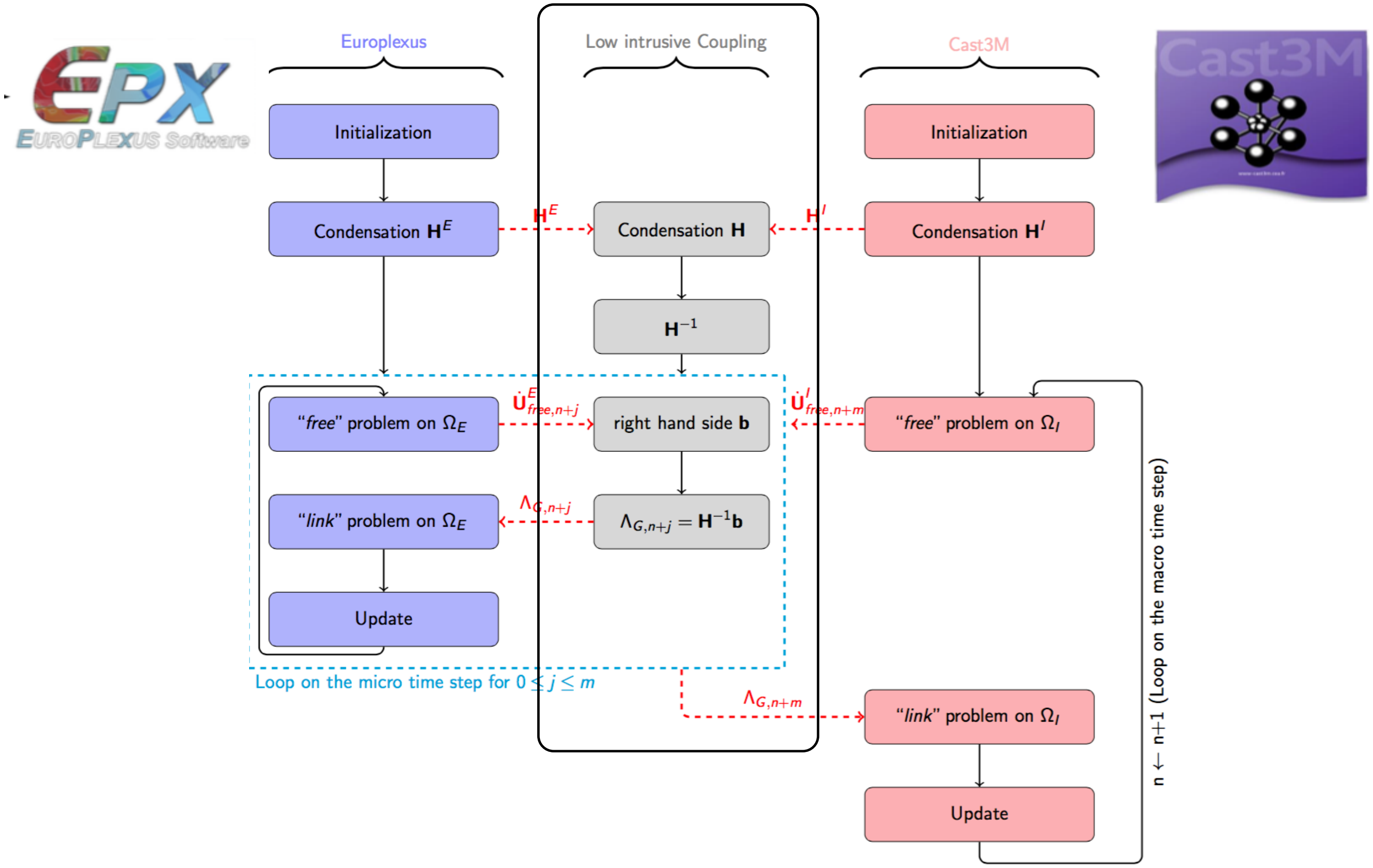
$$\mathbf{L}^E_G \mathbf{v}^E_{n+j} + \mathbf{L}^I_G \mathbf{v}^I_{n+j} = 0 \Rightarrow \; (\mathbf{H}^E + \mathbf{H}^I)\Lambda_{G,n+j} = \mathbf{b}$$

where $\mathbf{H}^E = \frac{\Delta t}{2} \mathbf{L}^E_G (\mathbf{M}^E_{lump})^{-1} (\mathbf{L}^E_G)^T$

and $\mathbf{H}^I = \Delta T \gamma_I \mathbf{L}^I_G \left(\mathbf{M}^I + \Delta T \gamma_I \mathbf{C}^I + \Delta T^2 \beta_I \mathbf{K}^I\right)^{-1} (\mathbf{L}^I_G)^T$

→ *Very efficient for implicit linear / explicit non-linear co-simulation* ***(Hc and H csts)***

- ***Co-simulation for implicit-explicit non-smooth dynamics* [Fekak et al 2017]**

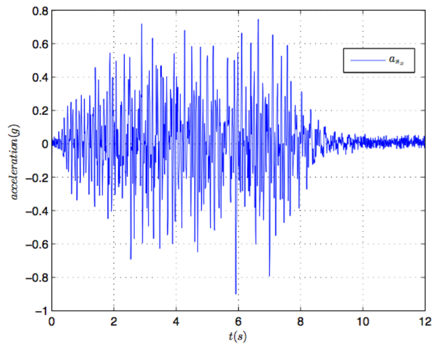


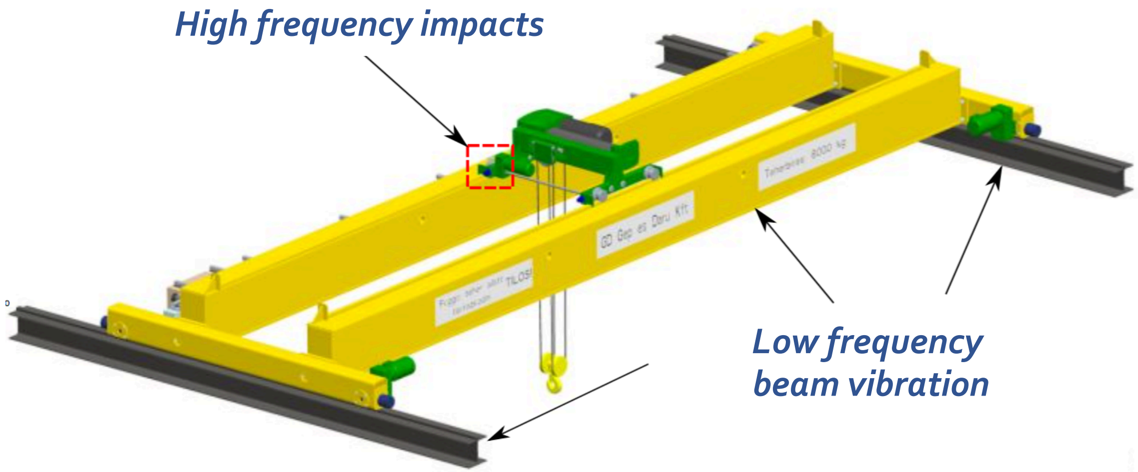
➡ ***Low intrusive coupling and constant H and Hc operators (linear implicit and non-linear non-smooth explicit)***

- *Industrial context*

➡ *Multi-contact, multi-scale in time and space*

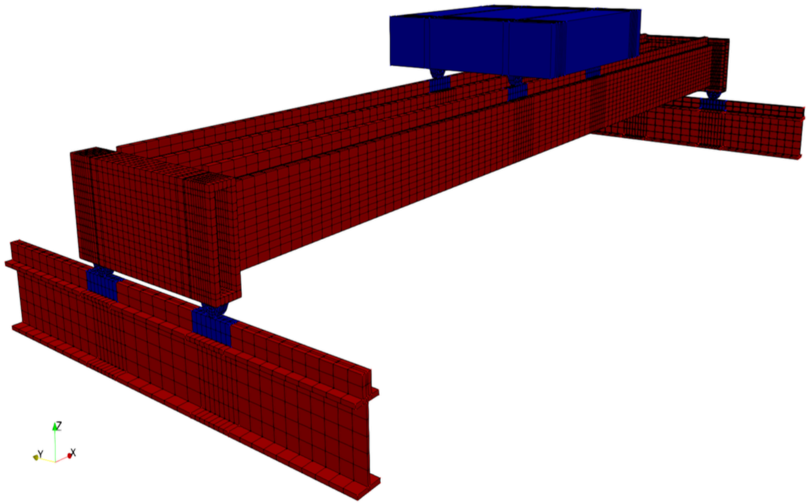
Given Earthquake (10s)



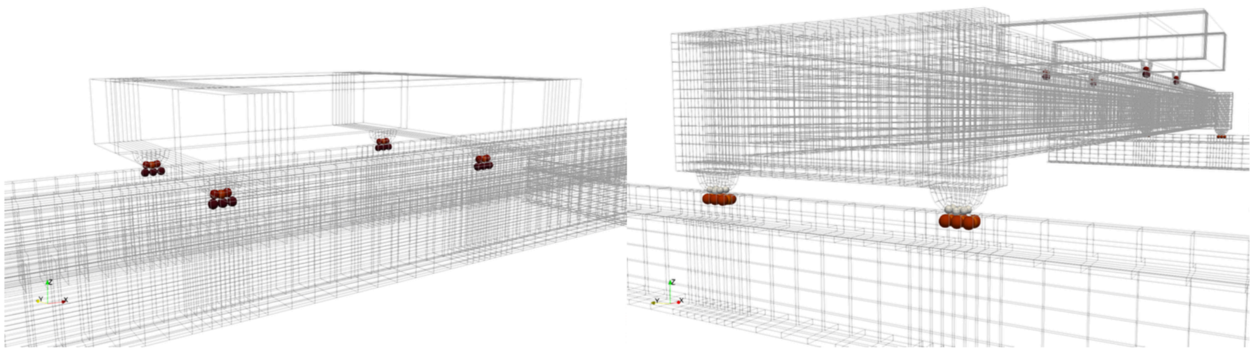


➡ *Application of implicit-explicit co-simulation to accurately model bridge cranes subjected to multiple impacts during an earthquake*

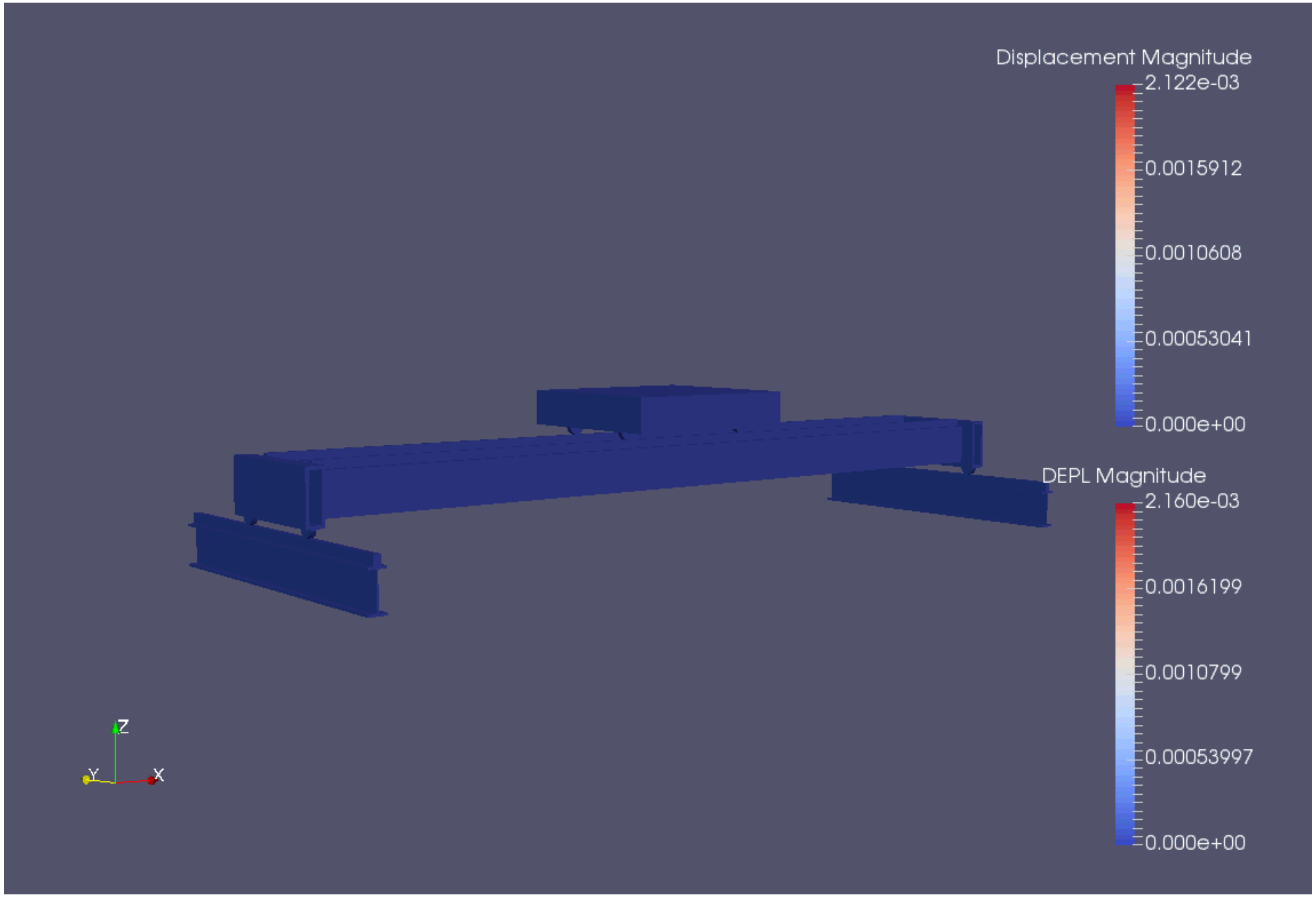
- *Explicit-implicit co-simulation (Europlexus / Cast3M)* ***[Fekak et al 2017]***



➡ *Subdomain decomposition (explicit: blue, implicit: red) according to high frequency area close to contact zones and pinballs* ***[Belytschko 1993, Casadei 2002]***



- *Explicit-implicit co-simulation (Europlexus / Cast3M)* ***[Fekak et al 2017]***



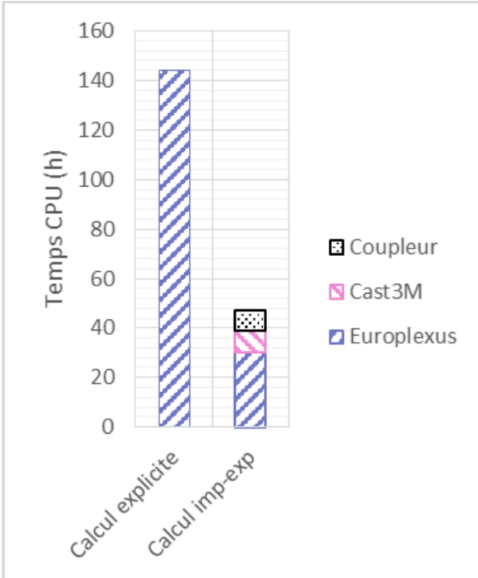
- *Numerical efficiency of co-simulation* **[Fekak et al 2017]**

	explicit SD	implicit SD	Interface
dofs	5088	70344	1296
time step	10^{-6} s	$100 * 10^{-6}$ s	–

Seismic loading of 10 s

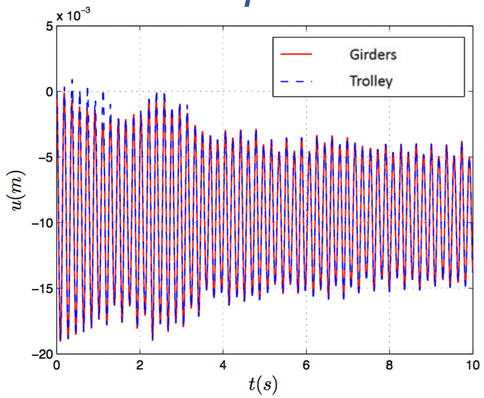
Explicit: 10 millions time steps
Implicit: 100 thousand time steps
m = 100

Optimization of the coupling implementation between Europlexus and Cast3m

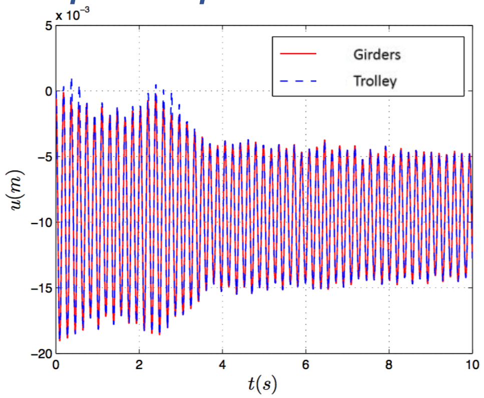


- *Comparison between full explicit and implicit-explicit co-simulation* ***[Fekak et al 2017]***

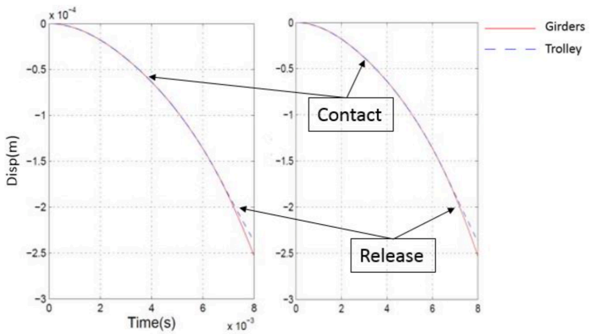
Full explicit



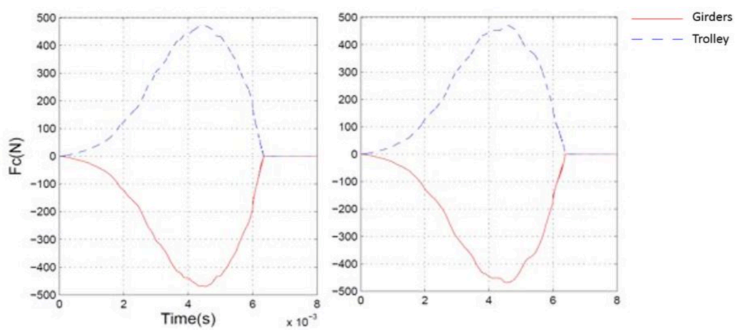
Explicit-implicit co-simulation



First rebound



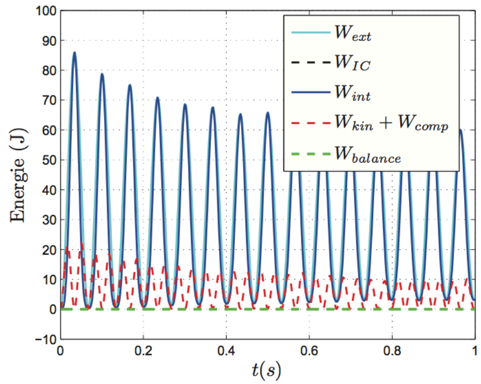
Zoom of the first rebound



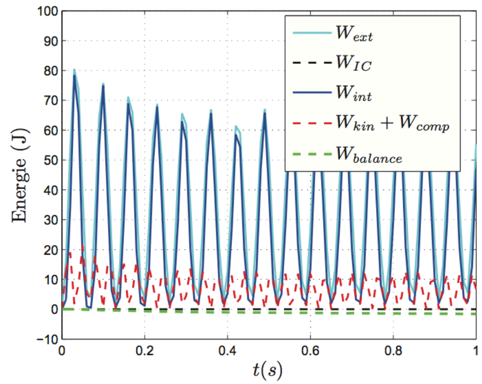
Rebound between trolley and girders are simulated accurately (6.35ms versus 6.39ms for full explicit and co-simulation respectively)

- *Comparison between full explicit and implicit-explicit co-simulation* ***[Fekak et al 2017]***

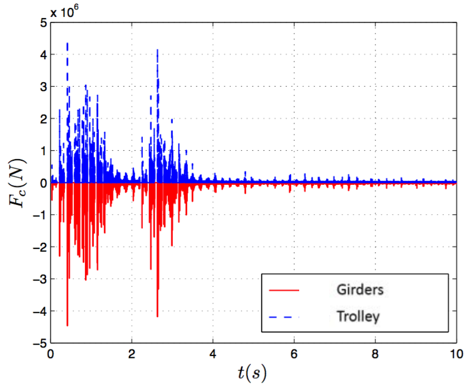
Full explicit

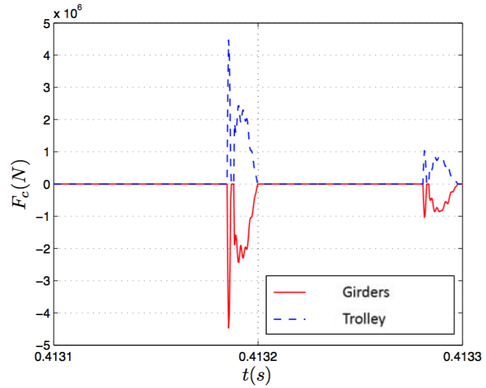


Explicit-implicit co-simulation



→ *Small numerical dissipation at the interface for the co-simulation (in agreement with the stability analysis) (here less than 2% with m=100)*





→ *Successive impacts and contacts between the trolley and girders*

→ *Observed Dirac correspond to impacts followed by smooth contact*

4

Conclusions & prospects

Conclusions and prospects

- *Inspired from the pioneer works of Belytschko & co-authors, we have proposed a new class of dual Heterogeneous Asynchronous Time Integrators (HATI) based on interface velocity continuity: BGC-macro and BGC-micro*
- *A general framework of HATI for G-α integrators (particular cases: HHT-α, WBZ-α, CH-α, Krenk-α)*
- *Wide range of applications: non-linear and non-smooth dynamics, multi-physics, FSI, co-simulation*
- *Development of a BGC-micro HATI with zero numerical dissipation at the space-time interface in progress*

Thank you for your attention