Géométrie généralisée et graduée pour la mécanique

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Global philosophy / religion

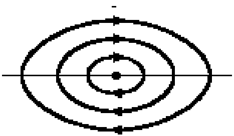
Geometry encodes the mechanics / physics of the system
and this is useful for numerics

Classical classical mechanics (ODE)	symplectic Poisson (almost) Dirac Q-structures	
Modern classical mechanics (PDE)	symmetries multi-symplectic Stokes–Dirac Dirac Q-structures	

Episode 1:

Generalized geometry...

Hamiltonian systems and more

Canonical case: given $H\colon T^*Q \to \mathbb{R}$ $\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$	Symplectic geometry $\omega = \sum_i dp_i \wedge dq^i$ $\iota_{X_H}\omega = \mathrm{d}H$	
More general case: given $H\colon M \to \mathbb{R}$ and an antisymmetric $J(\mathbf{x})$ $\dot{\mathbf{x}} = J(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}}$	Poisson geometry $\{\cdot,\cdot\}$ on : $C^\infty(M)$ $X_H = \{H, \cdot\}$ $\dot{\mathbf{x}} = \{H, \mathbf{x}\}$	
Dissipation, interaction, constraints	Dirac structures	

Courant algebroids, Dirac structures

Let us construct on $E = TM \oplus T^*M$
an *exact Courant algebroid structure*:
– symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \iota_{v'}\eta + \iota_v \eta'$,
– Courant – Dorfman bracket:

$$[v \oplus \eta, v' \oplus \eta']_{CD} = [v, v']_{\mathrm{Lie}} \oplus ((\mathrm{d}\iota_v + \iota_v \mathrm{d})\eta' - \mathrm{d}\eta(v')).$$

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An *almost Dirac structure* $\mathcal{D}$ is a maximally isotropic (Lagrangian) subbundle of an exact Courant algebroid E.
It is a *Dirac structure* iff it is closed w.r.t. $[\cdot,\cdot]_{CD}$

Trivial example: $\mathcal{D} = TM$.

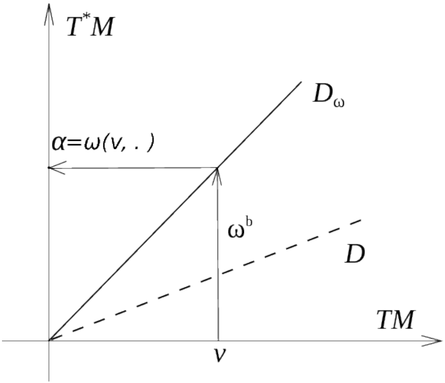
Dirac structures: (pre)symplectic example.

$$\omega = \sum_{i,j} \omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$$

Example. $\mathcal{D} = graph(\omega)$

Isotropy $\Leftrightarrow$
ω_{ij} antisymmetric.

Involutivity $\Leftrightarrow$
ω closed.



$$\mathcal{D}_\omega = \{(v, \iota_v \omega)\}$$

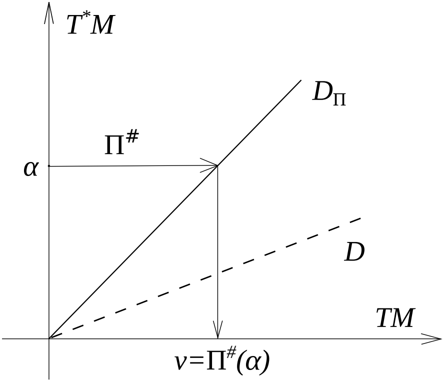
Dirac structures: Poisson example.

$$\Pi = \sum_{i,j} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \qquad \text{where } \pi^{ij}(x) := \{x^i, x^j\}$$

Example. $\mathcal{D} = graph(\Pi^\sharp)$

Isotropy $\Leftrightarrow$
π^{ij} antisymmetric.

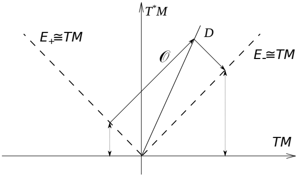
Involutivity $\Leftrightarrow$
Π Poisson.



$$\mathcal{D}_\Pi = \{(\Pi^\sharp \alpha, \alpha)\}$$

Dirac structures: general

Choose a metric on $M \Rightarrow TM \oplus T^*M \cong TM \oplus TM$,
Introduce the eigenvalue subbundles $E_\pm = \{v \oplus \pm v\}$
of the involution $(v, \alpha) \mapsto (\alpha, v)$. Clearly, $E_+ \cong E_- \cong TM$.

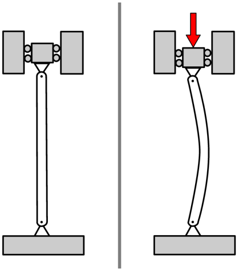


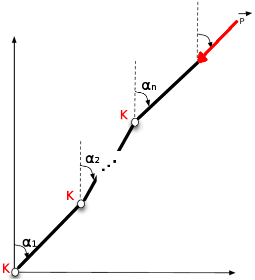
(Almost) Dirac structure – graph of an orthogonal operator $\mathcal{O} \in \Gamma(\mathrm{End}(TM))$:
$(v, \alpha) = ((\mathrm{id} - \mathcal{O})w, g((\mathrm{id} + \mathcal{O})w, \cdot))$
Dirac structure = almost Dirac + (Jacobi-type) integrability condition:

$$g\left(\mathcal{O}^{-1}\nabla_{(\mathrm{id}-\mathcal{O})\xi_1}(\mathcal{O})\xi_2, \xi_3\right) + cycl(1,2,3) = 0$$

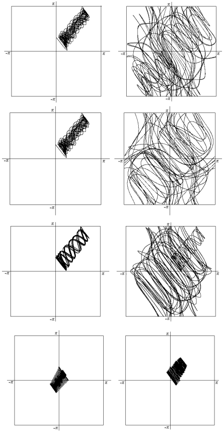
Remark. If the operator $(\mathrm{id} + \mathcal{O})$ is invertible, one recovers D_Π with $\Pi = \frac{\mathrm{id} - \mathcal{O}}{\mathrm{id} + \mathcal{O}}$ (Cayley transform), integrability $\Leftrightarrow [\Pi, \Pi]_{SN} = 0$.

Application 1: Implicit Lagrangian systems / constraints





Ziegler–Bigoni system



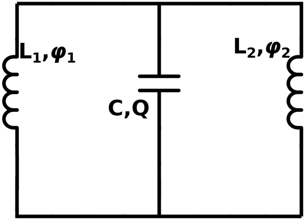
Exercise: Compare with Jean Lerbet!

Details:

- V.S., A.Hamdouni, From modelling of systems with constraints to generalized geometry and back to numerics, ZAMM 2019;
- D. Razafindralandy, V.S., A. Hamdouni, A. Deeb, Some robust integrators for large time dynamics, AMSES, 2019.

Application 2. Port-Hamiltonian systems.

Example: Electric circuit (L_1, L_2, C)



$$\begin{cases} \dot{Q} = \varphi_1/L_1 - \varphi_2/L_2 \\ \dot{\varphi}_1 = -Q/C \\ \dot{\varphi}_2 = Q/C. \end{cases}$$

$$H = \frac{\varphi_1^2}{2L_1} + \frac{\varphi_2^2}{2L_2} + \frac{Q^2}{2C}$$

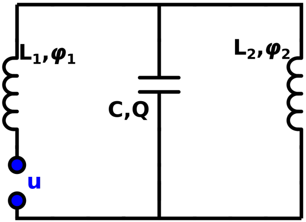
Hamiltonian system: $\dot{\mathbf{x}} = J(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}}$ with

$$\mathbf{x} = \begin{pmatrix} Q \\ \varphi_1 \\ \varphi_2 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$f_s := -\dot{\mathbf{x}}, \quad e_s := \begin{pmatrix} p_Q \\ p_{\varphi_1} \\ p_{\varphi_2} \end{pmatrix} \equiv \begin{pmatrix} Q/C \\ \varphi_1/L_1 \\ \varphi_2/L_2 \end{pmatrix}. \qquad \dot{H} \equiv -e_s^T f_s = 0$$

Application 2. Port-Hamiltonian systems.

Example: Electric circuit (L_1, L_2, C) with a controlled port u



$$\begin{cases} \dot{Q} = \varphi_1/L_1 - \varphi_2/L_2 \\ \dot{\varphi}_1 = -Q/C + u \\ \dot{\varphi}_2 = Q/C. \end{cases}$$

$$H = \frac{\varphi_1^2}{2L_1} + \frac{\varphi_2^2}{2L_2} + \frac{Q^2}{2C}$$

Port: input u, output $e = \varphi_1/L_1$.

Port-Hamiltonian system: $\dot{\mathbf{x}} = J(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}} + g(\mathbf{x})\mathbf{f}$ with

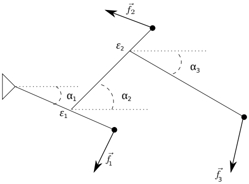
$$\mathbf{x} = \begin{pmatrix} Q \\ \varphi_1 \\ \varphi_2 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{matrix} f = u \\ e = \varphi_1/L_1 \end{matrix}$$

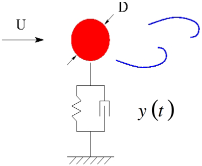
$$f_s := -\dot{\mathbf{x}}, \quad e_s := \begin{pmatrix} p_Q \\ p_{\varphi_1} \\ p_{\varphi_2} \end{pmatrix} \equiv \begin{pmatrix} Q/C \\ \varphi_1/L_1 \\ \varphi_2/L_2 \end{pmatrix}.$$

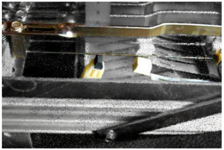
$$\dot{H} \equiv -e_s^T f_s = u\varphi_1/L_1 \Leftrightarrow$$
$$e_s^T f_s + ef = 0$$
$$\Rightarrow \textit{almost Dirac}$$

Port-Hamiltonian systems

A lot of examples
(ask Antoine Falaize) :







Port-Hamiltonian systems

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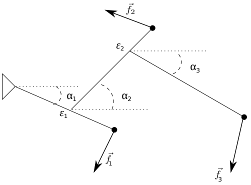
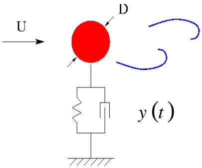
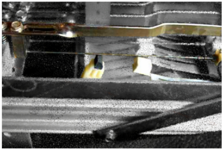




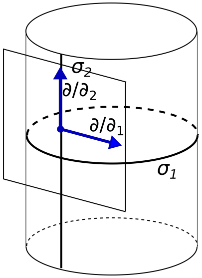


Conjecture (VS): Everything is port-Hamiltonian.
Question: OK for the fun, but does it really help?

Episode 2:

Graded geometry...

Graded manifolds – motivating example



Consider functions on $T[1]\Sigma$.

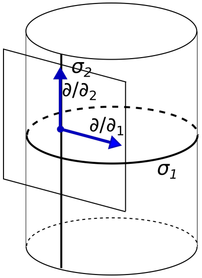
$\sigma^1, \ldots, \sigma^d$ – coordinates on Σ:
$deg(\sigma^\mu) = 0$, $\sigma^{\mu_1}\sigma^{\mu_2} = \sigma^{\mu_2}\sigma^{\mu_1}$.
$deg(h(\sigma^1, \ldots, \sigma^d)) = 0$.

$\theta^1, \ldots, \theta^d$ – fiber linear coordinates:
$deg(\theta^\mu) := 1$, $\theta^{\mu_1}\theta^{\mu_2} = -\theta^{\mu_2}\theta^{\mu_1}$

Arbitrary homogeneous function on $T[1]\Sigma$ of $deg = p$:
$f = \sum f_{\mu_1 \ldots \mu_p}(\sigma^1, \ldots, \sigma^d)\theta^{\mu_1} \ldots \theta^{\mu_p}$.

Graded commutative product: $f \cdot g = (-1)^{deg(f)deg(g)} g \cdot f$

Graded manifolds – motivating example



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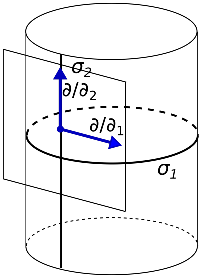

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$\rightarrow$ *"Definition" of a graded manifold*
– manifold with a ($\mathbb{Z}$-)grading defined on the sheaf of functions.

Graded manifolds – details

"...graded manifolds are just manifolds with a few bells and whistles..." (D. Roytenberg)

Graded geometry: definitions (do not read)

- Graded vector space V is a collection of vector spaces $V = \oplus V_i$ ($i \in \mathbb{Z}$ or $i \in \mathbb{Z}_{\geq 0}$); if $v \in V_i$, $deg(v) = i$.
- Homomorphism shifting the grading by p: $(V[p])_i = V_{i-p}$.
- Assume the base to be of degree 0, the dual vector space $(V_i)^*$ is defined as $(V^*)_{-i}$.
- Graded algebra structure $\cdot$: $V \otimes V \to V$, s.t. $V_p \otimes V_q \to V_{p+q}$.
- Graded commutator $[a, b] = ab - (-1)^{deg(a)deg(b)} ba$.
- Graded symmetric algebra over V: $S(V) = \mathrm{Tensor}(V)/[\cdot, \cdot]$

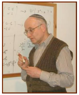
Definition. Graded manifold M is a couple $(M_0, \mathcal{O}_M)$, where M_0 is a smooth manifold and the sheaf of functions $\mathcal{O}_M$ is locally isomorphic to $C^\infty(U_0) \otimes S(V)$, where U_0 is an open subset of M_0.

- Top degree of the generators of $\mathcal{O}_M$ – is called degree of M. Standard abuse of notations: V_k-vector bundle or sheaf of sections.

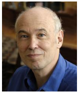
Graded manifolds, super manifolds

History

Felix Berezin



Joseph Bernstein



Pierre Deligne

Dimitry Leites

Philosophy

Dmitry Roytenberg

"...graded manifolds are just manifolds with a few bells and whistles..."

Graded manifolds

D. Roytenberg: "...bells and whistles..."

Prop. (D.Roytenberg) Given a non-negatively graded manifold $(M, \mathcal{O}_M)$ there is a tower of fibrations

$$M = M_n \to M_{n-1} \to \cdots \to M_1 \to M_0,$$

where any M_k is a graded manifold of degree at most k, for $k > 0$ $M_{k+1} \to M_k$ is an affine bundle.

Remark. Gradings can be encoded in the Euler vector field $\epsilon = deg(q^\alpha) q^\alpha \frac{\partial}{\partial q^\alpha}$; V_i corresponds to the i-eigenspace of ϵ.

Remark. Gradings can be encoded in the homogeneity structure h: $\mathbb{R}_+ \times \mathcal{M} \to \mathcal{M}$ such that $(q^1, \ldots, q^N) \mapsto h_t(q^1, \ldots, q^N) \equiv (t^{deg(q^1)} q^1, \ldots, t^{deg(q^N)} q^N)$.

Q-manifolds (DG-manifolds)

Motivating example:
$T[1]\Sigma$, $deg(\sigma^\mu) = 0$, $deg(\theta^\mu) = 1$,
Functions of the form: $f = \sum f_{\mu_1 \dots \mu_p}(\sigma^1, \dots, \sigma^d)\theta^{\mu_1} \dots \theta^{\mu_p}$

Consider a vector field $Q = \sum \theta^\mu \frac{\partial}{\partial \sigma^\mu}$

deg $Q = 1$

$Q(f \cdot g) = (Qf) \cdot g + (-1)^{1 \cdot deg(f)} f \cdot (Qg)$

$[Q, Q] \equiv 2Q^2 = 0$

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Definition. *Q-structure* – vector field Q on a graded manifold, s.t. $deg(Q) = 1$ and it squares to zero.

Poisson manifold $\to$ $(T^*[1]M, Q_\pi)$

Consider a Poisson manifold M,
$\{\cdot,\cdot\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$.

A Poisson bracket can be written as $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$, where $\pi \in \Gamma(\Lambda^2 TM)$ is a bivector field. $\pi^{ij}(x) = \{x^i, x^j\}$.

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Consider $T^*[1]M$ (coords. $x^i(0), p_i(1)$), with a *deg* $= 1$ vector field

$$Q_\pi = \left\{\frac{1}{2}\pi^{ij}p_i p_j, \cdot\right\}_{T^*M} = \pi^{ij}(x)p_j \frac{\partial}{\partial x^i} - \frac{1}{2}\frac{\partial \pi^{jk}(x)}{\partial x^i} p_j p_k \frac{\partial}{\partial p_i}$$

Jacobi identity for π: $Q_\pi^2 = 0 \Leftrightarrow$

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \Leftrightarrow$$

$$\frac{\partial \pi^{ij}(x)}{\partial x^l}\pi^{lk}(x) + \frac{\partial \pi^{ki}(x)}{\partial x^l}\pi^{lj}(x) + \frac{\partial \pi^{jk}(x)}{\partial x^l}\pi^{li}(x) = 0$$

Derived bracket construction

Let $(\mathcal{M}, Q)$ be a Q-manifold, and
$\mathcal{G}$ be degree -1 vector fields ε on $\mathcal{M}$.
Define the <u>Q-derived bracket</u>: $[\varepsilon, \varepsilon']_Q := [\varepsilon, [Q, \varepsilon']]$.

Remark. Good for equivariant Q-cohomology.
V.S. "Graded geometry in gauge theories and beyond", JGP, 2015.

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Example 1. $(T^*[1]M, Q_\pi)$
$\varepsilon = \varepsilon_i(x)\frac{\partial}{\partial p_i} \leftrightarrow \varepsilon_i(x)dx^i \in \Omega^1(M)$.
If ε is exact, i.e. $\varepsilon_i dx^i = \epsilon_{,i}dx^i$, then $[\varepsilon, \varepsilon']_Q = \{\epsilon, \epsilon'\}_{,i}\frac{\partial}{\partial p_i}$

Example 2. Dirac structures.

Example from physics. (Part of) the Standard Model

Quarks

$SU(3)$ symmetry

$\Rightarrow \Rightarrow$

$\Rightarrow \Rightarrow$

Gauging

$\Rightarrow \Rightarrow$

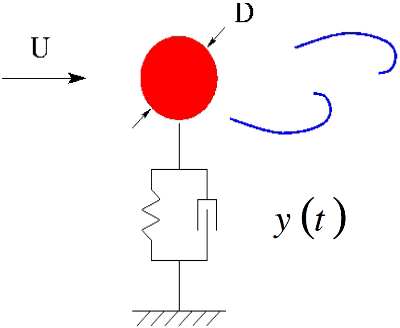
$\Rightarrow \Rightarrow$

8 connection 1-forms

Gluons

(j/w Thomas Strobl – Lyon, Alexei Kotov – Hradec Králové)

Example: vortex induced vibrations



Simplified model of fluid–structure interaction (cf. T. Leclercq, E. de Langre, Journal of Fluids and Structures, 80:2018)

The phenomenon is modelled by a harmonic oscillator coupled to the Van der Pol system:

$$\begin{aligned} \ddot{y} &+ y = m\Omega^2 q \\ \ddot{q} &- \varepsilon(1-q^2)\dot{q} + \Omega^2 q = A\ddot{y} \end{aligned}$$

Oscillator

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + m\Omega q\end{aligned}$$

Port-Hamiltonian framework: $H = \frac{1}{2}(x_1^2 + x_2^2)$,

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$f_s = -\dot{X}, \quad e_s = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad f = M\Omega^2 q, \quad e = \dot{x}_1,$$

Energy evolution: $\dot{H} = -e_s^T f_s = ef$.

Associated Dirac structure

Consider a manifold M, with coordinates (x_1, x_2),
then $f_s \in \Gamma(TM), e_s \in \Gamma(T^*M)$,
and the inputs–outputs: $f \in \Gamma(\mathcal{F})$, $e \in \Gamma(\mathcal{F}^*)$.
$\mathbb{TM} = (TM \times \mathcal{F}) \oplus (T^*M \times \mathcal{F}^*)$, where one considers $\mathcal{F}$ as a bundle over a point The (almost) Dirac is defined by $e_s^T f_s + ef = 0$ – a subbundle of rank 3, given by

$$\begin{pmatrix} f_s \\ f \\ e_s \\ e \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 - M\Omega^2 q \\ m\Omega^2 q \\ x_1 \\ x_2 \\ x_2 \end{pmatrix}$$

Van der Pol system

Rewrite the second equation as

$$\ddot{q} - \varepsilon(1 - q^2)\dot{q} + (\Omega^2 - Am\Omega^2)q = -Ay,$$

where $(\Omega^2 - Am\Omega^2) =: \tilde{\Omega}^2$ and $\varepsilon(1 - q^2) =: a$.

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= -\tilde{\Omega}^2 q_1 + a q_2 - Ay, \end{aligned}$$

Port-Hamiltonian structure: $H = \frac{1}{2}(\tilde{\Omega} q_1^2 + q_2^2)$,

$$Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$f_s = -\dot{Q}, e_s = \begin{pmatrix} \tilde{\Omega}^2 q_1 \\ q_2 \end{pmatrix}, \quad f_i = -Ay, e_i = \dot{q}_1, \quad f_d = -\dot{q}_1, e_d = -a\dot{q}_1.$$

Energy evolution: $\dot{H} = -e_s^T f_s = e_i f_i + e_d f_d$.

Associated Dirac structure

By abuse of notation, a manifold M with coordinates (q_1, q_2);
$f_s \in \Gamma(TM), e_s \in \Gamma(T^*M)$. Inputs–outputs: $(f_i, f_d) \in \Gamma(\mathcal{F})$,
$(e_i, e_d) \in \Gamma(\mathcal{F}^*)$.
$\mathbb{TM} = (TM \times \mathcal{F}) \oplus (T^*M \times \mathcal{F}^*)$, almost Dirac structure
$e_s^T f_s + e_i f_i + e_d f_d = 0$ – a subbundle of rank 4, given by

$$\begin{pmatrix} f_s \\ f_i \\ f_d \\ e_s \\ e_i \\ e_d \end{pmatrix} = \begin{pmatrix} -q_2 \\ \tilde{\Omega}^2 q_1 - a q_2 + Ay \\ -Ay \\ -q_2 \\ \tilde{\Omega}^2 q_1 \\ q_2 \\ q_2 \\ -a q_2 \end{pmatrix}$$

Coupling.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + m\Omega q\\ \dot{q}_1 &= q_2\\ \dot{q}_2 &= -\tilde{\Omega}^2 q_1 + a q_2 - Ay\end{aligned}$$

Port–Hamiltonian formalism: $H = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}(\tilde{\Omega} q_1^2 + q_2^2)$,

$$X = \begin{pmatrix} x_1\\ x_2\\ q_1\\ q_2\end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}, \quad g = \begin{pmatrix} 0\\ m\Omega^2 q_1\\ 0\\ a(q_1)q_2 - Ax_1\end{pmatrix},$$

$$f_s = -\dot{X}, e_s = \begin{pmatrix} x_1\\ x_2\\ \tilde{\Omega}^2 q_1\\ q_2\end{pmatrix}, \quad e_i = 1 \in \mathbb{R}^1, f_i = m\Omega^2 q_1 x_2 - A x_1 q_2 + a(q_1) q_2^2.$$

Graded description

Manifold M, with coordinates: (x_1, x_2, q_1, q_2),
$f_s \in \Gamma(TM), e_s \in \Gamma(T^*M)$,
inputs–outputs: $f_i \in \Gamma(\mathcal{F})$, $e_i \in \Gamma(\mathcal{F}^*)$.
$\mathbb{TM} = (TM \times \mathcal{F}) \oplus (T^*M \times \mathcal{F}^*)$.
The (almost) Dirac structure is given by $e_s^T f_s + e_i f_i$ – a subbundle of rank 5, or by

$$\begin{pmatrix} f_s \\ f_i \end{pmatrix} = D \begin{pmatrix} e_s \\ e_i \end{pmatrix},$$

where $D\colon T_X^*M \times \mathcal{F}^* \to T_X M \times \mathcal{F}$ – a bivector, in components:

$$D = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -m\Omega^2 q_1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -a(q_1)q_2 + Ax_1 \\ 0 & m\Omega^2 q_1 & 0 & aq_2 - Ax_1 & 0 \end{pmatrix}.$$

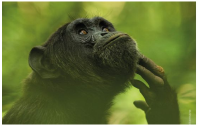
Q-structure

For the *graded description* consider the graded manifold $T^*[1]\mathcal{M}$ with coordinates x^i (of degree 0), et p_i (of degree 1). The degree 1 vector field constructed from D is

$$
\begin{aligned}
Q = -p_2\frac{\partial}{\partial x^1} + (p_1 - p_5 m\Omega^2 x^3)\frac{\partial}{\partial x^2} - p_4\frac{\partial}{\partial x^3} + \\
+(p_3 - p_5 a(x^3)x^4 - Ax^1)\frac{\partial}{\partial x^4} + \\
+(p_2 m\Omega^2 x^3 + p_4 a(x^3)x^4 - Ax^1)\frac{\partial}{\partial x^5} + \\
+Ap_4p_5p_5\frac{\partial}{\partial p_1} + (-m\Omega^2 p_2p_5 + 2\varepsilon x^3x^4p_4p_5)\frac{\partial}{\partial p_3} - a(x^3)p_4p_5\frac{\partial}{\partial p_4}
\end{aligned}
$$

Global philosophy / religion

Q-structure



~~Geometry~~ encodes the physics of the system

Classical classical mechanics (ODE)	Poisson symplectic (almost) Dirac Q-structures	
Modern classical mechanics (PDE)	DEC multi-symplectic Stokes–Dirac Dirac	

... and this is still useful for numerics

Trugarez deoc'h evit bezañ bet o selaou ac'hanon!