Elements of Geometry for Continuum Physics (Example of vacuum spacetime)

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Space and **time** are the most fundamental concepts of mechanics / physics.

1. **Aristotle paradigm :** There was an absolute time, absolute space, **and** an absolute referential frame, the earth;

2. **Newton paradigm (1686):** There was absolute 1-continuum time (same for all), absolute **Euclidean** 3-spatial continuum, **but no** absolute rest frame;

3. **Einstein paradigm (1915) :** Space and time fused to form a **Riemanian** 4-continuum spacetime, **not an inert entity**,; **Gravitation is a manifestation of spacetime geometry**

1. **Elasticity** is based on Euclidean geometry (metric = angle and length) with **zero curvature**
2. **Gravitation** lies on the geometry of Riemannian spacetime with **non zero curvature** (2^{nd} derivatives of metric)
3. **Electromagnetism** *a priori* assumes the introduction of **electric** and **magnetic** field as primal variables

Basic motivation : Could the electromagnetic variables (and then continuum physics) be deduced from spacetime geometry ?

"Geometrization" of electromagnetism

Historical :

- Weyl approach (non metricity) : Weyl 1918
- Kaluza-Klein theory (Kaluza 1921, Klein 1926)
- Non symmetric metric (Einstein 1925)
- Teleparallel gravitation, (Einstein 1928)
- Finslerian geometry (contact geometry) : e.g. Asanov 1985, Voicu 2011

Approach with Riemann-Cartan geometry:

- Roots : Reichenbach 1929 (after Cartan geometry)
- Some recent references : e.g. Poplawski 2010, Giglio et al 2012, Hehl & Obukhov 2013, R 2017, Hammond (1987) 2018, ...

Planning

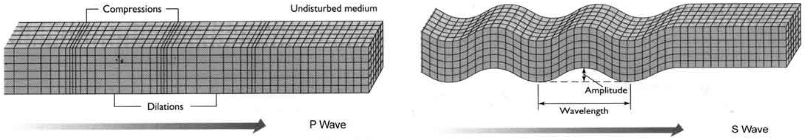
1. Wave equation in homogeneous continuum physics
2. Examples of **N**on **H**omogeneous **C**ontinuum (NHC)
3. Covariant models for NHC
4. **Continuum physics vs. spacetime geometry**
5. Concluding remarks

I. Wave equation in "homogeneous" continuum physics :

Main Interest : wave analysis in continuum

- caracterisation (NDT)
- geophysical prospection and engineering
- models in physics, ...

Elastic waves in solids



Longitudinal wave $\Phi(x^\mu)$ with celerity $c_\ell := \sqrt{(\lambda + 2\mu)/\rho}$, shearing wave $\mathbf{A}(x^\mu)$ with celerity $c_s := \sqrt{\mu/\rho}$.

Wave propagation equation (from **Navier**'s equation):

$$\begin{cases} \partial_0^2 \Phi - \hat{\Delta}\Phi = 0 & (x^0 := c_\ell t) \\ \partial_0^2 \mathbf{A} - \hat{\Delta}\mathbf{A} = 0 & (x^0 := c_s t) \end{cases}$$

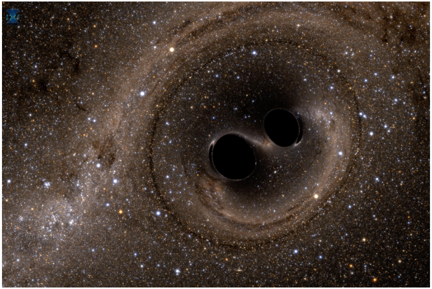
with $\mathbf{u} := \hat{\nabla}\Phi + \hat{\nabla} \times \mathbf{A}$ (Helmholtz decomposition)

Electric **E**, magnetic **B** fields, celerity $c := \sqrt{\mu_0 \epsilon_0}^{-1}$
From electromagnetic potential $A^\nu = (A^0 := \Phi, A^1, A^2, A^3)$ we have:

$$\mathbf{E} := -c\ \partial_0 \mathbf{A} - \hat{\nabla}\phi, \qquad \text{and} \qquad \mathbf{B} := \hat{\nabla} \times \mathbf{A}$$

Wave propagation equation ($x^0 := ct$) (from **Maxwell**'s equation):

$$\begin{cases} \partial_0^2 \Phi - \hat{\Delta}\Phi & = & 0 \\ \partial_0^2 \mathbf{A} - \hat{\Delta}\mathbf{A} & = & 0 \end{cases}$$



Perturbed metric $\tilde{\mathbf{g}} = \mathbf{g} + 2\varepsilon$, traceless part $\overline{\varepsilon} := \varepsilon - (1/2)\mathrm{Tr}(\varepsilon)\ \mathbf{g}$

Wave propagation equation ($x^0 := ct$) (from **Einstein**'s equation):

$$g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta\ \overline{\varepsilon} = 0$$

where $g^{\alpha\beta} := \mathrm{Diag}\{+1, -1, -1, -1\}$ is the Minkowskian metric.

Generic wave equation in "homogeneous" continuum

Wave equation: unknowns : $u := \{\Phi, \mathbf{A}, \cdots\}$, and $(x^0 = ct, x^1, x^2, x^3)$

$$\boxed{g^{\alpha\beta}\ \overline{\nabla}_\alpha \overline{\nabla}_\beta\ u\,(x^\mu) = 0}$$

$$g^{\alpha\beta} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \overline{\nabla}_\alpha := \begin{pmatrix} \partial_0 \\ \overline{\nabla}_1 \\ \overline{\nabla}_2 \\ \overline{\nabla}_3 \end{pmatrix}$$

Models are based on :

- Geometric structure : **metric $g^{\alpha\beta}$** et **connection $\overline{\nabla}_\alpha$**
- Material or spacetime property : celerity c
- Physics field : u (pressure, shear, electric, magnetic, spacetime perturbation ...)

II. Examples of Non Homogeneous Continuum (NHC)

Motivation: Hypothesis of continuum "homogeneity" (and smoothness) is not always acceptable !

NHC : set of cells in an alloy

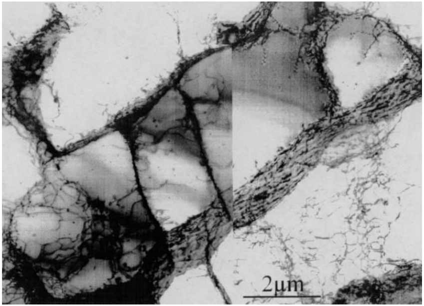


Figure: Aluminium cell structure : Cell formation closely connected with the cross-slip of screw dislocations $\boxed{\ell_c = 2\mu m}$ (from Püschl, 2002).

Continuum at micro-metric scale with **sharp gradients**

NHC : set of layers (following indentation)

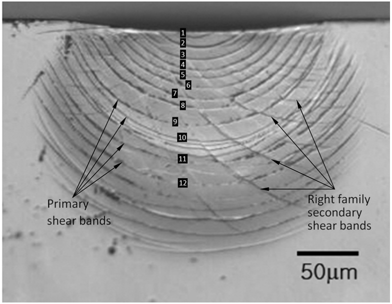


Figure: Primary Shear Band (PSB) and secondary (SSB) in metallic glass $\ell_c = 50\mu m$. (from Chen & Lin, 2010)

Continuum at micro-metric scale with **sharp gradients**

NHC : geophysics (set of layers)

Figure: Rock layers (mainly from Cretacea) near Lulworth Cove, Dorset, Devon, $\ell_C \simeq 1m$ (from Smith 2014)

Continuum at metric scale with **sharp gradients**

NHC : Universe filament structure

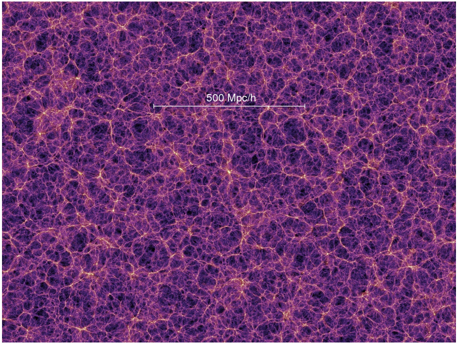
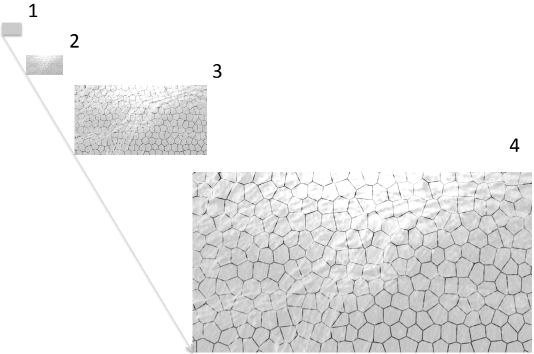


Figure: Filament structure of the universe $\ell_C \simeq 1,5 \times 10^{25} m$: fiber structures represent matter (galaxies) and empty regions (dark) cosmic voids.

(https://www.universetoday.com/135954/largest-scales-milky-way-galaxy-middle-nowhere/)



All materials are non homogeneous / defects!

Ariadne's thread : Extension of connection $\overline{\nabla}_\alpha$ to account for sharp gradients and hopefully **electromagnetism** !

Choice of vacuum spacetime as toy model

"Biggest" question: More than 2000 years before admitting that **gravitation** and **electromagnetic** occur in a **vacuum spacetime**.

Mechanical waves	Electromagnetic waves	Gravitational waves
Need matter	Do not need matter	Do not need matter

$\underbrace{\text{Electromagnetic waves: Do not need matter} \quad \text{Gravitational waves: Do not need matter}}_{\text{Physics in Vacuum spacetime}}$

Physics in Vacuum spacetime

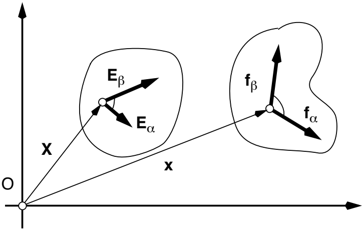
Vacuum spacetime continuum :

- Empty (classical mechanics)
- Curved (relative gravitation)
- Contains gravitons - hypothetical - (quantum mechanics)

III. Covariant models of NH Continuum Physics

Motivation : How to account for NH and particularly sharp gradients (defects) in a continuum physics ?

Deformation : "Simple Material" continuum (Noll, 1958)



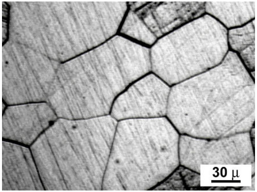
- **Metric** $\mathbf{g}$ measures local deformation : **(length & angle)** :

$$g_{\alpha\beta} = \mathbf{g}\left(\mathbf{f}_\alpha, \mathbf{f}_\beta\right) := \mathbf{f}_\alpha \cdot \mathbf{f}_\beta \quad \Longrightarrow \quad \left\{ \begin{array}{rcl} \|\mathbf{f}_\alpha\| & = & \sqrt{\mathbf{f}_\alpha \cdot \mathbf{f}_\alpha} \\ \cos\left(\mathbf{f}_\alpha, \mathbf{f}_\beta\right) & = & \dfrac{\mathbf{f}_\alpha \cdot \mathbf{f}_\beta}{\|\mathbf{f}_\alpha\| . \|\mathbf{f}_\beta\|} \end{array} \right.$$

- **Triads** $F^i_\alpha(\mathbf{X})$ defines local mapping $\mathbf{f}_\alpha = F^i_\alpha\, \mathbf{E}_i$
- **Strain** defined by metric : $\varepsilon = (1/2)\left(g_{\alpha\beta} - \delta_{\alpha\beta}\right)$

Deformation : NHC (set of microcosms)

Microcosms move **relatively each other (discontinuity).**



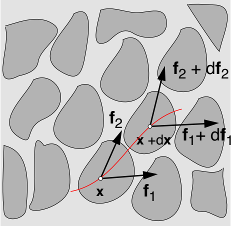
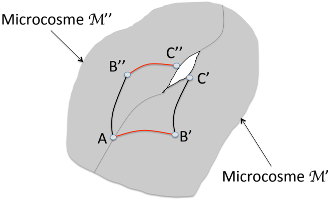


Figure: NHC := {**Microcosms**} (Gonseth, 1929),
$d\mathbf{f}_\beta = \nabla_\alpha \mathbf{f}_\beta dx^\alpha$ and $\nabla_\alpha \mathbf{f}_\beta := \Gamma^\gamma_{\alpha\beta} \mathbf{f}_\gamma$

- **Microcosm** deformation is defined by tetrads : $\mathbf{f}_\alpha := F^i_\alpha \; \mathbf{E}_\alpha$
- **Relative motion** of microcosms is defined by **connection** ∇_α
  with its coefficients : $\Gamma^\gamma_{\alpha\beta} := \left(F^i_\gamma\right)^{-1} \; \partial_\alpha F^i_\beta.$

How to model sharp gradients (relative motions) ? [1]



Non integrabilty of F^i_α (or **Jump** of scalar field) $\Longrightarrow$ **Torsion**

$$\aleph^\gamma_{\alpha\beta} := \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} \neq 0$$

Non integrability of $\nabla_\beta F^i_\alpha$ (or **Jump** of vector field) $\Longrightarrow$ **Curvature**

$$\mathfrak{R}^\kappa_{\alpha\beta\lambda} := \left(\partial_\alpha \Gamma^\kappa_{\beta\lambda} + \Gamma^\xi_{\beta\lambda}\Gamma^\kappa_{\alpha\xi}\right) - \left(\partial_\beta \Gamma^\kappa_{\alpha\lambda} + \Gamma^\xi_{\alpha\lambda}\Gamma^\kappa_{\beta\xi}\right) \neq 0$$

[1] Bilby et al, 1955, e.g. R 1997, 2003; Maugin 2005, Kleinert 2008.

NHC mathematical model is (often) defined by:

1. Lagrangian $\mathscr{L}$ (extension of Reichenbach model, 1929):

$$\mathscr{S} := \int_{\mathscr{B}} \mathscr{L}(g_{\alpha\beta},\ \Gamma^{\gamma}_{\alpha\beta},\ \partial_{\lambda}\Gamma^{\gamma}_{\alpha\beta})\ \omega_n$$

2. Conservation laws (variational calculus):

$$\delta\mathscr{S} = \delta \int_{\mathscr{B}} \mathscr{L}\ \omega_n = 0$$

Covariance

A model is **covariant** if its governing equation keeps the **same shape** following an arbitrary change of coordinate system (diffeomorphism).

In mechanics/physics, any model should be covariant !

Covariance of Lagrangian $\mathscr{L}$ (Antonio & R, 2011)

Theorem

The NHC model defined by the Lagrangian

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \Gamma^{\gamma}_{\alpha\beta},\ \partial_\lambda \Gamma^{\gamma}_{\alpha\beta})$$

is **covariant** *if and only if*

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \aleph^{\gamma}_{\alpha\beta},\ \mathfrak{R}^{\gamma}_{\alpha\beta\lambda})$$

Remarks :

- **Primal variables** are metric $g_{\alpha\beta}$, torsion $\aleph^{\gamma}_{\alpha\beta}$, and curvature $\mathfrak{R}^{\gamma}_{\alpha\beta\lambda}$ (**Continuum physics** : elasticity, fluid mechanics, gravitation, electromagnetism (?), ...)
- This theorem extends **Cartan** (1922) and **Lovelock-Rund** (1971, 1975) theorems from Riemann to **Riemann-Cartan continuum**.

Examples of Lagrangian

- **Elasticity**

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta})$$

where $g_{\alpha\beta} := \mathbf{g}\left(\mathbf{f}_\alpha, \mathbf{f}_\beta\right)$ form the Cauchy-Green tensor as :

$$\begin{aligned} \mathscr{L} &:= (\rho/2)\|\partial_t \mathbf{u}\|^2 - (\lambda/2)\mathrm{Tr}^2\varepsilon - \mu\mathrm{Tr}\left(\varepsilon^2\right) \\ \varepsilon_{\alpha\beta} &:= (1/2)\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right) \end{aligned}$$

- **Gravitation**

$$\mathscr{L} = \mathscr{L}(g_{\alpha\beta},\ \mathfrak{R}^{\gamma}_{\alpha\beta\lambda}) \qquad \text{as} \qquad \mathscr{L} = (1/2\chi)\ g^{\alpha\beta}\ \mathfrak{R}^{\lambda}_{\lambda\alpha\beta}$$

where spacetime is a NH (second gradient) continuum !

IV. Continuum physics vs. spacetime geometry

$$\mathscr{L} = \mathscr{L}(\mathbf{g}_{\alpha\beta}, \aleph^{\gamma}_{\alpha\beta}, \Re^{\lambda}_{\alpha\beta\gamma})$$

Figure: Spacetime allowing gravitation and electromagnetic fields

Motivation: "Geometrization" of gravitation and electromagnetism in vacuum spacetime

1. **Gravitation (Hilbert-Einstein action)** alone :

$$\mathscr{L}_{HE} := \frac{1}{2\chi} g^{\alpha\beta} \Re^{\lambda}_{\lambda\alpha\beta} \qquad \Longrightarrow \qquad \Re^{\lambda\rho} - \frac{\mathcal{R}}{2} g^{\lambda\rho} = 0$$

depending on the curvature (2^{nd} GC) with $\chi := 8\pi G/c^3$.

2. **Electromagnetism (Yang-Mills action)** alone :

$$\mathscr{L}_{YM} := -\frac{1}{4} \mathcal{F}^{\mu\nu} \, \mathcal{F}_{\mu\nu} \qquad \Longrightarrow \qquad \nabla_{\nu} \mathcal{F}^{\mu\nu} = 0$$

with $\mathcal{F}_{\alpha\beta} := \nabla_{\alpha} A_{\beta} - \nabla_{\beta} A_{\alpha}$ and $A_{\alpha} := (\Phi, A_1, A_2, A_3)$.

$$\mathcal{F}_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B^3 & -B^2 \\ E_2 & -B^3 & 0 & B^1 \\ E_3 & B^2 & -B^1 & 0 \end{bmatrix}$$

What would be the interaction of gravitation and electromagnetism ?

1. **Gravitation & electromagnetism** in coupling :

$$\mathscr{L} := \underbrace{(1/2\chi)\ g^{\alpha\beta}\Re^{\lambda}_{\lambda\alpha\beta}}_{\text{gravitation}} - \underbrace{(1/4)\ \mathcal{F}^{\mu\nu}\ \mathcal{F}_{\mu\nu}}_{\text{electromagnetism}}$$

2. **Variational calculus** $\delta\mathscr{S} = \delta \int_{\mathscr{M}} \mathscr{L}\ \omega_n = 0$ (R 2017, 2018)

$$\begin{aligned}
\delta\mathscr{S} \quad &= \quad \int_{\mathscr{M}} \nabla_\nu \mathcal{F}^{\mu\nu}\ \delta A_\mu\ \omega_n + \int_{\mathscr{M}} \Big[(1/2\chi)\left(\Re^{\lambda\rho} - (\mathcal{R}/2)\ g^{\lambda\rho}\right) \\
&+ \quad (1/8)\ \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\ g^{\lambda\rho} + (\mathcal{F}_{\mu\nu}/4)\left(g^{\mu\lambda}\mathcal{F}^{\rho\nu} + \mathcal{F}^{\mu\rho}g^{\lambda\nu}\right)\Big]\ \delta g_{\lambda\rho}\ \omega_n \\
&- \quad \int_{\mathscr{M}} \underbrace{\left[\left(\mathcal{F}^{\mu\nu} - \mathcal{F}^{\nu\mu}\right) A_\lambda + (1/\chi)\ g^{\rho\nu}\ \aleph^{\mu}_{\lambda\rho}\right]}_{\text{NEW DUAL TERMS}}\ \underbrace{\delta\Gamma^{\lambda}_{\mu\nu}}_{\text{N.H.}}\ \omega_n
\end{aligned}$$

Coupled governing PDE

System of 3 coupled equations (instead of 2 usual equations) :

1. Maxwell equations (electromagnetic fields) (δA_μ)

$$\nabla_\nu \mathcal{F}^{\mu\nu} = 0$$

2. Einstein-Maxwell equations (gravitation + EM) ($\delta g_{\lambda\rho}$)

$$\underbrace{\Re^{\lambda\rho} - \frac{\mathcal{R}}{2} g^{\lambda\rho}}_{\text{Einstein}} = \underbrace{-\frac{\chi}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}\, g^{\lambda\rho} + \frac{\chi\, \mathcal{F}_{\mu\nu}}{2} \left(g^{\mu\lambda} \mathcal{F}^{\nu\rho} + \mathcal{F}^{\rho\mu} g^{\lambda\nu} \right)}_{-\text{Minkowski energy momentum}}$$

T^{00} = energy, $T^{0i} = \mathbf{E} \times \mathbf{H}$; $T^{i0} = \mathbf{D} \times \mathbf{B}$; T^{ij} = Maxwell tensor

3. Relation of torsion with electromagnetism ($\delta \Gamma^\lambda_{\mu\nu}$)

$$g^{\rho\nu}\, \aleph^\mu_{\lambda\rho} = -\chi \left(\mathcal{F}^{\mu\nu} - \mathcal{F}^{\nu\mu} \right) A_\lambda$$

- **Maxwell's wave equations** : From $\nabla_\nu \mathcal{F}^{\mu\nu} = 0$ and $\mathcal{F}_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu$ we obtain :

$$-\underbrace{g^{\nu\beta}\nabla_\nu \nabla_\beta A^\mu}_{\text{classic Maxwell}} - \underbrace{g^{\mu\alpha}\aleph^\gamma_{\nu\alpha}\nabla_\gamma A^\nu + g^{\mu\alpha}\Re_{\alpha\gamma}A^\gamma}_{\text{"configurational" forces}} = 0$$

with $(x^0 := ct, x^1, x^2, x^3)$ and with Lorenz gauge $\nabla_\beta A^\beta \equiv 0$.

- **Remarks** :
  - Same wave equation as for elastic NH wave equation
  - Torsion and curvature influence EM waves
  - Torsion $\longrightarrow$ dispersion, Curvature $\longrightarrow$ "breathing"
  - Bending, twisting of light (e.g. Leonhardt & Philbin 2006)

(3) Relation of torsion with electromagnetism

- **Relation of torsion with electromagnetism** (R 2017):

$$g^{\rho\nu}\ \aleph^{\mu}_{\lambda\rho} = -\chi\ (\mathcal{F}^{\mu\nu} - \mathcal{F}^{\nu\mu})\ A_{\lambda}$$

  - **Electromagnetism is related to torsion** of spacetime (e.g. de Andrade & Pereira 1999, Hammond 2018, 2019)
  - Giglio & Rodrigues assumed similar relation but using contortion tensor $\mathscr{T}^{\mu}_{\lambda\rho} := \Gamma^{\mu}_{\lambda\rho} - \overline{\Gamma}^{\mu}_{\lambda\rho}$ (e.g. Giglio et al. 2012)

- **Spacetime "new" paradigm**: Space and time fused to a **Riemann-Cartan** 4-continuum and where :
  - Curvature "geometrizes" Gravitation,
  - **Torsion "geometrizes" Electromagnetism**.

Physical interpretation : 2^{nd} member $(\mathcal{F}^{\mu\nu} - \mathcal{F}^{\nu\mu})\, A_\lambda$ is exactly the 4-dim expression of the **Spin Angular Momentum** $\mathbf{L}_{\text{spin}}$ (optics, ...).

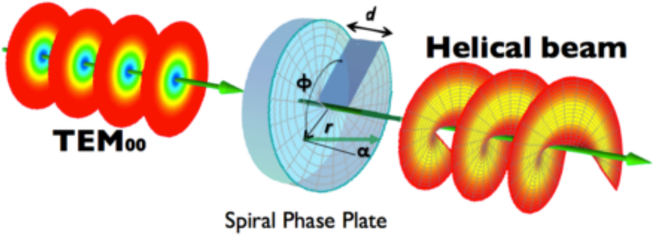


Figure: Transverse ElectroMagnetic wave (green axis) :
Moment of Poynting vector $\int_{\mathscr{M}} \mathbf{r} \times (\mathbf{D} \times \mathbf{B})\, dv := \mathbf{L}_{\text{orbital}} + \mathbf{L}_{\text{spin}}$
(Allen et al. 1992, Barnett 2002, Padgett et al 2004, Hammond 2018)

Question : Can gravitation and electromagnetic **forces** be defined solely with geometric elements of the spacetime ?

Method (classic) : Analyze gap vector ξ between two geodesics

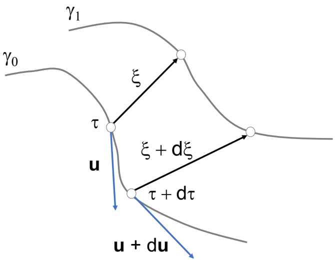


Figure: The gap vector ξ separating autoparallel timelike curves γ_0 and γ_1, defined by $\nabla_{\mathbf{u}}\mathbf{u} = 0$, with orthogonality condition $\mathcal{L}_{\mathbf{u}}\xi = 0$.

Geometry induces gravitation and electromagnetic forces

- **Einstein gravitation** (Levi-Civita 1927, Synge 1934)

$$\frac{D^2\xi}{D\tau^2} = \Re\left(\mathbf{u}, \xi, \mathbf{u}\right)$$

where for classic gravitation : $\Re^a_{b00} = \nabla^a \nabla_b \Phi$, 2^{nd} derivatives of Newtonian potential Φ

- **Einstein gravitation with electromagnetism** (R 2019)

$$\frac{D^2\xi}{D\tau^2} = \Re\left(\mathbf{u}, \xi, \mathbf{u}\right) + \underbrace{\aleph\left(\frac{D\xi}{D\tau}, \mathbf{u}\right) + \nabla_{\mathbf{u}}\aleph\left(\xi, \mathbf{u}\right)}_{\text{electromagnetic forces}}$$

Similar results from physical approach by directly extending **Lorentz force F** $:= q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$ (e.g. Balakin et al. 2000).

Classification of fundamental forces in physics

Proposition : Gravitation AND Electromagnetic forces could be also considered as **geometric forces** of spacetime vacuum

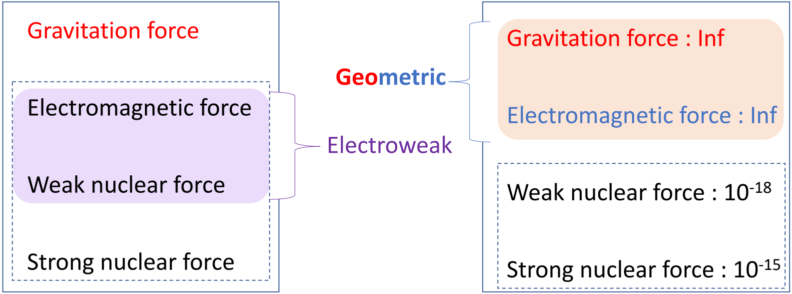


Figure: **Classification of fundamental forces** :
(Left) State-of-the-Art : Gravitation + Unified nuclear forces;
(Right) Sketch : Geometric forces (G & EM) + Nuclear forces

V. Concluding remarks

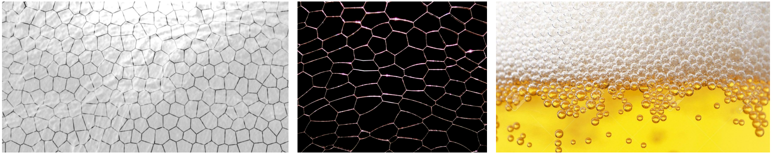


Figure: **Left**: NH Continuum (Al alloy, set of **microcosms**); **Middle**: Spacetime in Loop Quantum Gravitation (set of "**quanta**"); **Right**: Beer foam (for fun, set of "happy hours")

Main conclusion Continuum geometry links :

- Elasticity $\longleftrightarrow$ metric (length, angle) $g_{\alpha\beta}$,
- Electromagnetism $\longleftrightarrow$ torsion $\aleph^{\gamma}_{\alpha\beta}$,
- Gravitation $\longleftrightarrow$ curvature $\Re^{\gamma}_{\alpha\beta\lambda}$.

Other concluding remarks

- **Riemann-Cartan** candidate model of (spacetime) continuum with sharp gradients (as a mosaic of little **microcosms of spacetime**).
- Electromagnetic and Gravitation are **"geometric forces"** for the vacuum spacetime, described by **torsion and curvature**.
- **Spacetime model** extended to **matter continuum physics** (acoustic, elasticity, electromagnetism, gravitation, ...)
- Link with **experimental** tedious (hopefully not impossible) namely **boundary conditions** for $\aleph$ and $\Re$ are (still) difficult to derive.

Aknowledgements

Aknowledgements : IRMAR, Région de Bretagne (ARED), ESIROI (Université de La Réunion)