GDR-GDM

Sorbonne Université 2023

Critères limites d'élasticité anisotropes: apport de la théorie des groupes

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24 Novembre 2023.

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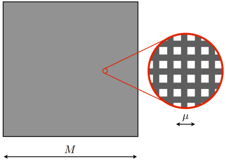
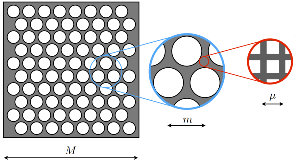
Outline

Architectured materials: Definition

Architectured materials are materials which have between their macrostructure and microstructure a mesostructure [1].



(a) Standard material: macrostructure and microstructure ($\mu \ll M$).



(b) An architectured material presenting 3 scales of organisations.

Examples[2]:

(a) Gyroid structure

(b) Honeycomb lattice

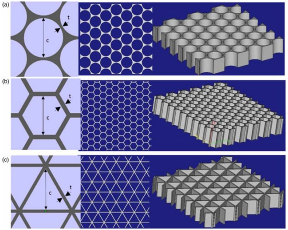
(c) Spongy bone

[1] M. Poncelet et al. "An experimental evidence of the failure of Cauchy elasticity for the overall modeling of a non-centro-symmetric lattice under static loading". In: *International Journal of Solids and Structures* (2018)

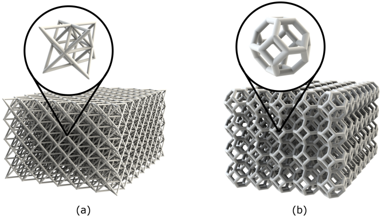
[2] T. Dassonville. "Experimental approach of the numerical homogenization". PhD thesis. 2020

LATTICE MATERIALS[3][4]

- A spatially periodic network of structural elements, such as rods, beams, plates, or shells.
- Main characteristics: a unit cell (periodicity).



(a) 2D Lattice materials (or honeycombs).



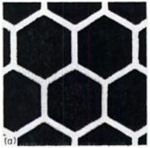
(b) Rod-based lattice structures with (a) octet-truss and (b) Kelvin unit cell.

[3] John Weeks et al. “Effect of Topology on Transient Dynamic and Shock Response of Polymeric Lattice Structures”. In: *Journal of Dynamic Behavior of Materials* (2022)

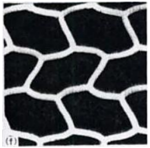
[4] Y. Yap et al. “Shape recovery effect of 3D printed polymeric honeycomb”. In: *Virtual and Physical Prototyping* (2015)

Lattice materials: Interests and disadvantages

- Interests [5]:
  - A good stiffness/weight ratio;
  - Low thermal expansion;
  - Controlling wave path;
  - ...
- Lattice materials have two modes of ruins:
  - Plasticity/brittle in tension;
  - Buckling in compression.



(a) Undeformed.



(b) Elastic buckling.

Defining a versatile limit criterion for lattice materials is important for **topology optimisation**.

[5] A. Phani et al. *Dynamics of Lattice Materials*. Wiley, 2017

Objectives

Definition of a generalised criterion for lattice materials that can be:

- dissymmetric in tension/compression,
- pressure dependent,
- anisotropic.

Hypothesis: our study is restricted to 2D considering linear elastic behaviour.

Outline

THE STRESS TENSOR (2D): DEFINITION

- $\underset{\sim}{\sigma}$ is a symmetric 2nd order tensor

$$[\underset{\sim}{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}_{\mathcal{B}}, \qquad \underset{\sim}{\sigma} \in S^2(\mathbb{R}^2),$$

$S^2(\mathbb{R}^2)$ is the space of 2nd order symmetric tensors.

- $\underset{\sim}{\sigma}$ decomposes as:

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{(2)} + \underset{\sim}{\sigma}^{(0)}, \quad \text{where} \begin{cases} \underset{\sim}{\sigma}^{(2)} = \underset{\approx}{\mathrm{P}}^2 : \underset{\sim}{\sigma}, & \underset{\approx}{\mathrm{P}}^2 = \underset{\approx}{\mathrm{I}} - \underset{\approx}{\mathrm{P}}^0, \\ \underset{\sim}{\sigma}^{(0)} = \underset{\approx}{\mathrm{P}}^0 : \underset{\sim}{\sigma}, & \underset{\approx}{\mathrm{P}}^0 = \frac{1}{2} \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}}. \end{cases}$$

where $\underset{\approx}{\mathrm{P}}^0 = \frac{1}{2} \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}}$ and $\underset{\approx}{\mathrm{P}}^2 = \underset{\approx}{\mathrm{I}} - \underset{\approx}{\mathrm{P}}^0$.

- $\dim \underset{\sim}{\sigma}^{(2)} = 2$ and $\dim \underset{\sim}{\sigma}^{(0)} = 1$

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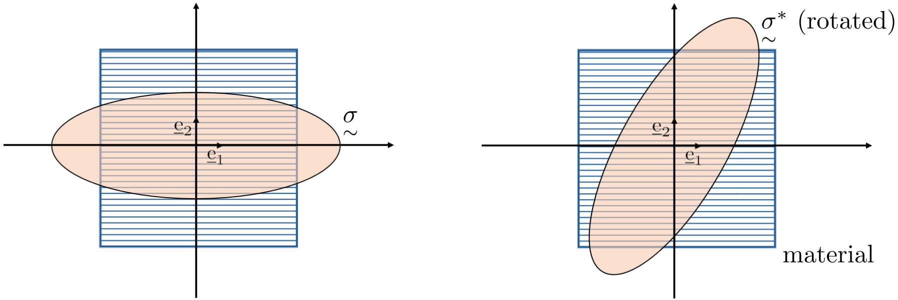
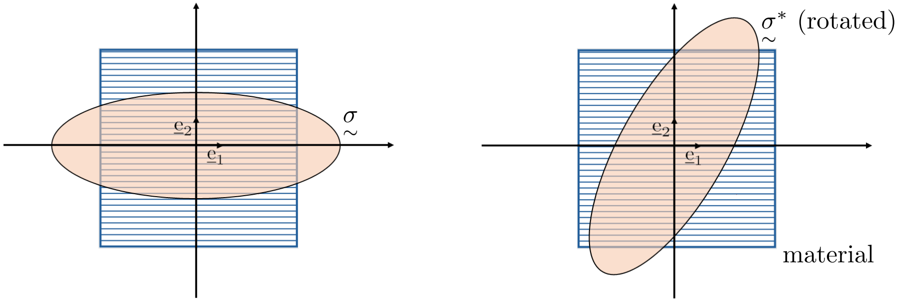
ROTATING $\underset{\sim}{\sigma}$: ACTIVE POINT OF VIEW



How the stress tensor changes when it is rotated ?

Rotating $\underset{\sim}{\sigma}$: active point of view



How the stress tensor changes when it is rotated ?

Orbit of the stress tensor

- $\mathrm{O}(2)$ is the group of invertible transformations of $\mathbb{R}^2$ satisfying $\mathbf{g}^{-1} = \mathbf{g}^T$ generated by:

$$\mathbf{r}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{with } 0 \leq \theta < 2\pi \quad \text{and} \quad \boldsymbol{\pi}(\underline{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

- The $\mathrm{O}(2)$-action on $S^2(\mathbb{R}^2)$ is the classical tensorial action:

$$(\underset{\sim}{\sigma}^*)_{ij} = (\mathbf{g} \star \underset{\sim}{\sigma})_{ij} = g_{ik} g_{jl} \sigma_{kl}.$$

Orbit of the stress tensor

We define the orbit of the stress tensor $\underset{\sim}{\sigma}$ by:

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Polar parametrisation of the orbit[6]: a picture in $\mathbb{R}^3$

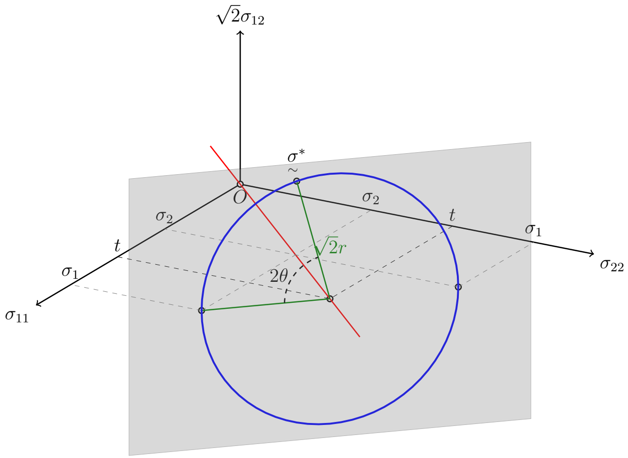


Figure: Visualisation of the orbit in Kelvin basis.

$$\underbrace{\mathcal{K} = \begin{cases} \hat{\underset{\sim}{e}}_1 = \underline{e}_1 \otimes \underline{e}_1 \\ \hat{\underset{\sim}{e}}_2 = \underline{e}_2 \otimes \underline{e}_2 \\ \hat{\underset{\sim}{e}}_3 = \frac{1}{\sqrt{2}}(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1). \end{cases}}_{\text{Kelvin Basis}}$$

$$\underbrace{\begin{cases} t = \frac{1}{2}\,\mathrm{tr}(\underset{\sim}{\sigma}) \\ r = \frac{1}{\sqrt{2}}\left(\underset{\sim}{\sigma}^{(2)} : \underset{\sim}{\sigma}^{(2)}\right)^{\frac{1}{2}}. \end{cases}}_{\text{Polar parameters}}$$

$$\{\underset{\sim}{\sigma}^*\} = \begin{pmatrix} t + r\cos(2\theta) \\ t - r\cos(2\theta) \\ \sqrt{2}\,r\sin(2\theta) \end{pmatrix}_{\mathcal{K}}$$

[6] Paolo Vannucci. "Plane Anisotropy by the Polar Method*". In: *Meccanica* (2005)

Harmonic basis: O(2)-orbit

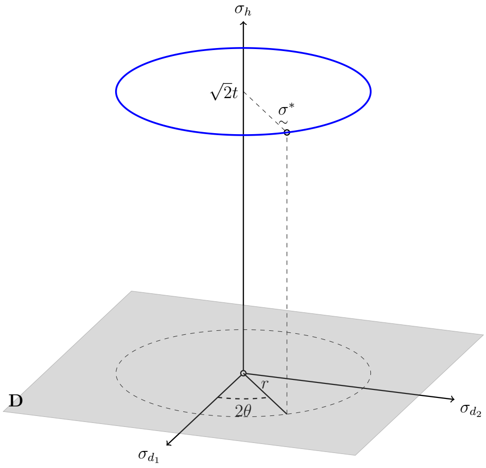


Figure: Harmonic basis (Cylindrical coordinates).

$$\underbrace{\mathcal{H} = \begin{cases} \underset{\sim}{\hat{f}}_1 = \dfrac{1}{\sqrt{2}}(\underset{\sim}{\hat{e}}_1 - \underset{\sim}{\hat{e}}_2) \\ \underset{\sim}{\hat{f}}_2 = \underset{\sim}{\hat{e}}_3 \\ \underset{\sim}{\hat{f}}_3 = \dfrac{1}{\sqrt{2}}(\underset{\sim}{\hat{e}}_1 + \underset{\sim}{\hat{e}}_2) \end{cases}}_{\text{Harmonic basis}} .$$

$$\{\underset{\sim}{\sigma}^*\} = \begin{pmatrix} \sigma^*_{d_1} \\ \sigma^*_{d_2} \\ \sigma^*_h \end{pmatrix}_{\mathcal{H}} = \begin{pmatrix} r\cos(2\theta) \\ r\sin(2\theta) \\ \sqrt{2}t \end{pmatrix}_{\mathcal{H}}$$

θ is the spatial orientation (anisotropy).

Spherical parametrisation: O(3)-orbit

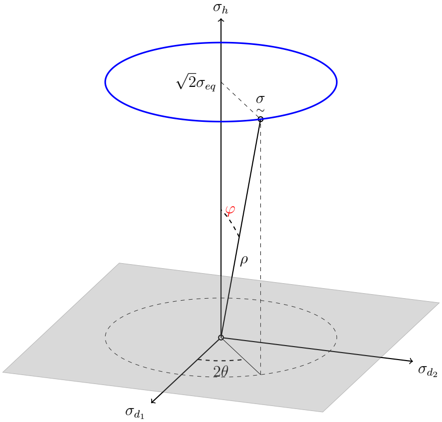


Figure: Representation of the harmonic basis (spherical coordinates).

$$\begin{pmatrix} \sigma_{d_1} \\ \sigma_{d_2} \\ \sigma_h \end{pmatrix}_{\mathcal{H}} = \begin{pmatrix} r' \sin(\varphi)\cos(2\theta) \\ r' \sin(\varphi)\sin(2\theta) \\ r' \cos(\varphi) \end{pmatrix}_{\mathcal{H}}$$

θ = physical orientation.
φ = loading angle:

- $\varphi = 0$: positive hydrostatic stress;
- $\varphi = \frac{\pi}{2}$: deviatoric
- $\varphi = \pi$: compressive hydrostatic state.

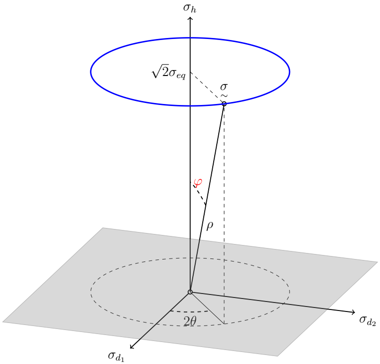
SYNTHESIS

Within the spherical parametrisation of $\underset{\sim}{\sigma}^*$:

- θ is the physical orientation of the stress tensor ($\mathrm{O}(2)$-Orbit).
- Any φ loading angle $\mathrm{O}(3)$-Orbit (knowing that $S^2(\mathbb{R}^2) \simeq \mathbb{R}^3$).

The passage from tension to compression (or vice versa):

$$\underset{\sim}{\sigma} \longrightarrow -\underset{\sim}{\sigma},$$



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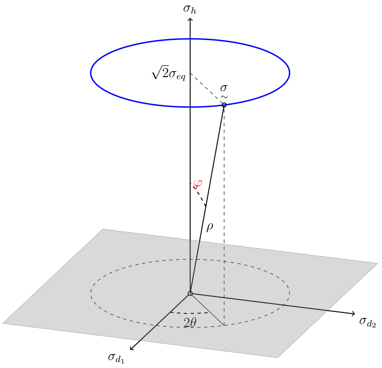


OUTLINE

Criterion function: definition

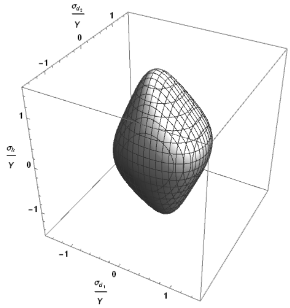
- A threshold function is an application F defined as follows:

$$F : S^2(\mathbb{R}^2) \mapsto \mathbb{R}^+,$$
$$\underset{\sim}{\sigma} \longrightarrow F(\underset{\sim}{\sigma}),$$

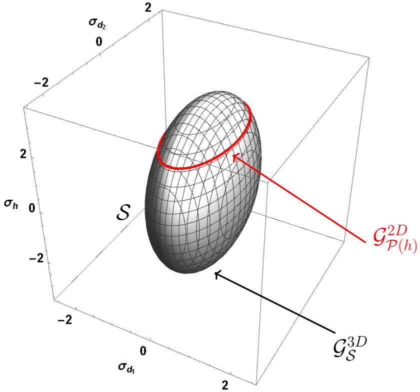
- The threshold surface $\mathcal{S}$ in the stress space is defined as:

$$\mathcal{S} = \left\{ \underset{\sim}{\sigma} \in S^2(\mathbb{R}^2), F(\underset{\sim}{\sigma}) - \sigma_{lim} = 0 \right\}.$$

$\sigma_{lim} \in \mathbb{R}^{*+}$ is the *threshold stress*.



SURFACE SYMMETRIES

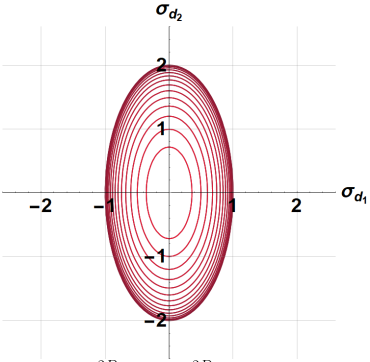


$\mathcal{G}^{3D}_{\mathcal{S}}$ is the symmetry group of the surface in $\mathbb{R}^3$

$\mathcal{G}^{2D}_{\mathcal{P}(h)}$, it is the symmetry group (in $\mathbb{R}^2$) of the threshold surface with the plane $\mathcal{P}(h)$.

Surface symmetries

$\mathcal{G}^{2D}_{\mathcal{S}} = \bigcap_h \mathcal{G}^{2D}_{\mathcal{P}(h)}$, it is the intersection of the symmetry group of all the cross sections of $\mathcal{S}$ by parallel deviatoric planes.



$\mathcal{G}^{2D}_{\mathcal{S}} = \bigcap_h \mathcal{G}^{2D}_{\mathcal{P}(h)}$

Symmetry groups in 2D

Up to conjugacy, symmetry groups in $\mathbb{R}^2$ are:

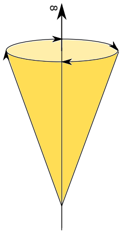
- $\mathrm{Z}_k (k \geq 2)$ the cyclic group with k elements, generated by $\mathbf{r}(2\pi/k)$;
- $\mathrm{D}^{\mathrm{n}}_{k}$ $(k \geq 2)$ is dihedral group with $2k$ elements generated by $\mathbf{r}(2\pi/k)$ and a mirror of normal $\underline{\mathrm{e}}_2$;
- $\mathrm{SO}(2)$: the rotation group in $\mathbb{R}^2$;
- $\mathrm{O}(2)$: the orthogonal group in $\mathbb{R}^2$;

Z_6

D_6

Symmetry groups in 3D: Three types

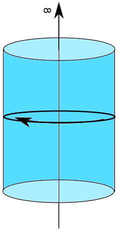
Type I
(Chiral)



$\mathrm{SO}(2)$-invariant

Only rotations

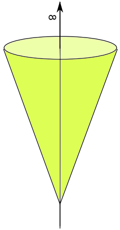
Type II
(Centrosymmetric)



$\mathrm{SO}(2) \otimes \mathrm{Z}_2^c$-invariant

All operations

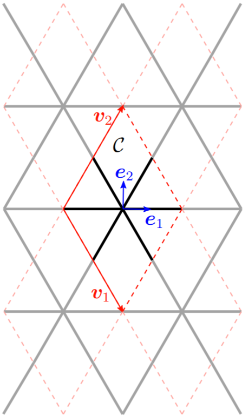
Type III



$\mathrm{O}(2)^-$-invariant

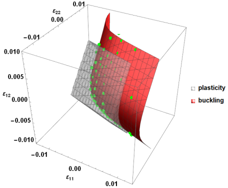
All operations but inversion

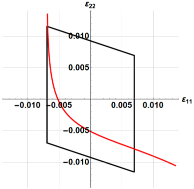
Example: Triangular 2D lattice[7] (PhD, V.Jeanneau)



Hypothesis

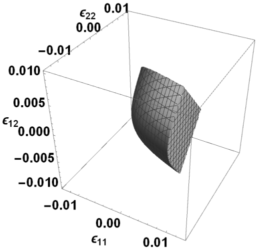
- Considering plasticity and buckling instabilities
- Periodicity
- Symmetry of the geometry: $G_M = D_6$

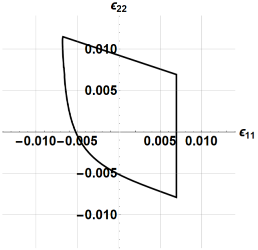


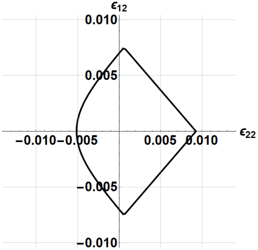


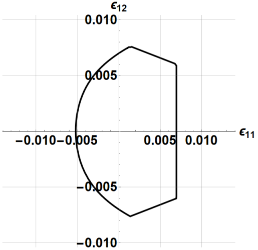
[7] V. Jeanneau et al. “Comportement effectif et limite de linéarité d’un matériau architecturé 2D périodique à celulles triangulaires”. In: (CFM 2022)

EXAMPLE: TRIANGULAR 2D LATTICE (SPATIAL BASIS)

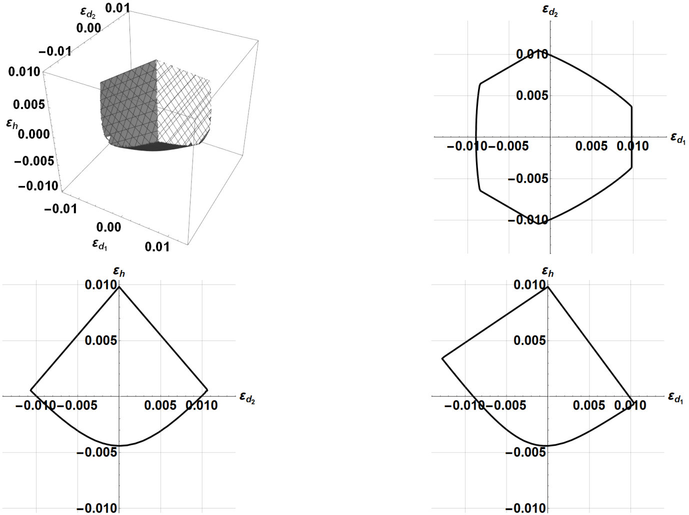




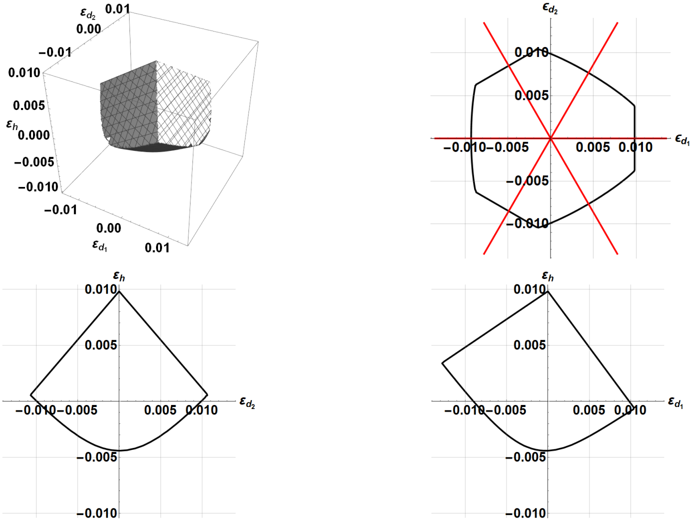




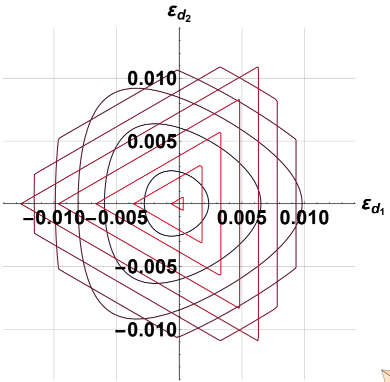
Example: Triangular 2D lattice (harmonic basis)

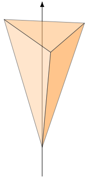


EXAMPLE: TRIANGULAR 2D LATTICE (HARMONIC BASIS)



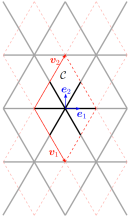
Example: Triangular 2D lattice (deviatoric plane)



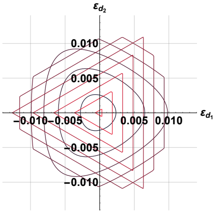


- $\mathcal{G}_{\mathcal{S}}^{2D} = \mathrm{D}_3$ in $\mathrm{O}(2)$ (it implies material symmetry = D_6)
- $\mathcal{G}_{\mathcal{S}}^{3D} = \mathrm{D}_3^v$ in $\mathrm{O}(3)$.

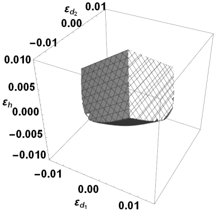
SYNTHESIS



D_6



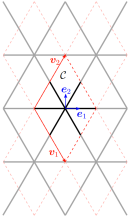
D_3



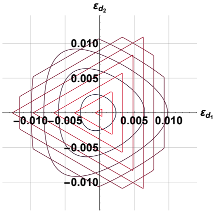
D_3^{v}

How to model an adapted threshold function ?

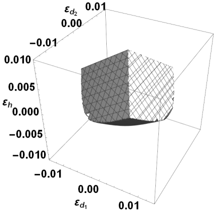
SYNTHESIS



D_6



D_3



D_3^v

How to model an adapted threshold function ?

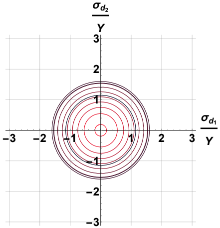
OUTLINE

Criterion function: isotropic and anisotropic

1. Isotropic functions:

$$F(\underset{\sim}{\sigma}) = F(\mathbf{g} \star \underset{\sim}{\sigma}), \quad \forall \mathbf{g} \in \mathrm{O}(2).$$

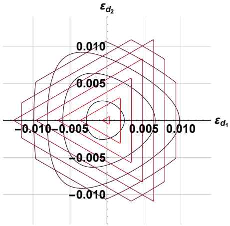
$\Longrightarrow$ The function is constant over the orbit.



2. Anisotropic functions :

$$F(\underset{\sim}{\sigma}) = F(\mathbf{g} \star \underset{\sim}{\sigma}), \quad \forall \mathbf{g} \in \mathrm{H} < \mathrm{O}(2).$$

$\Longrightarrow$ The function is not constant over the orbit.



How to establish an anisotropic function ?

Criterion function: isotropic and anisotropic

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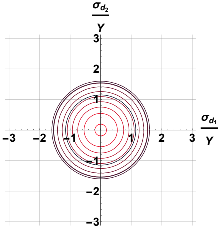

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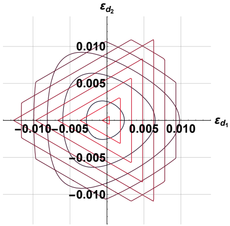


How to establish an anisotropic function ?

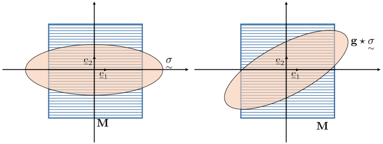
REPRESENTATION THEOREM[8]

Representation Theorem

Considering the variable $\underset{\sim}{\sigma}$ and a structure tensor $\mathbf{M}$ ($\mathrm{G}_{\mathbf{M}} = \mathrm{H} < \mathrm{O}(2)$), an anisotropic function of $\underset{\sim}{\sigma}$ under a group H can be reformulated as an isotropic function of $\underset{\sim}{\sigma}$ and structure tensors $\mathbf{M}$.

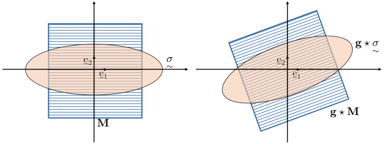
1. the definition anisotropic criterion function:

$$\forall \mathbf{g} \in \mathrm{H}, \quad F(\mathbf{g} \star \underset{\sim}{\sigma}, \mathbf{M}) = F(\underset{\sim}{\sigma}, \mathbf{M})$$



2. the function F is isotropic of $\underset{\sim}{\sigma}$ and $\mathbf{M}$ means:

$$\forall \mathbf{g} \in \mathrm{O}(2), \quad F(\mathbf{g} \star \underset{\sim}{\sigma}, \mathbf{g} \star \mathbf{M}) = F(\underset{\sim}{\sigma}, \mathbf{M})$$



[8] J. P. Boehler. *Applications of Tensor Functions in Solid Mechanics*. Springer, Vienna, 1987

Structure tensor model: harmonic tensors

Definition

Let $\mathbb{K}^n$ be the space of nth-order harmonic tensors in 2D, its elements are:

1. *n-th order tensors*
2. *Totally symmetric*
3. *traceless*

$$\dim \mathbb{K}^n = \begin{cases} 2, & n \geq 1; \\ 1, & n = 0. \end{cases}$$

Theorem

Let $\mathfrak{I}(\mathbb{K}^n)$ denotes the set of all isotropy classes associated to $\mathbb{K}^n$. The symmetry classes of $\mathbb{K}^n$ are:

$$\mathfrak{I}(\mathbb{K}^n) = \begin{cases} n \geq 1, & \{[\mathrm{D}_n], [\mathrm{O}(2)]\} \\ n = 0, & \{[\mathrm{O}(2)]\} \end{cases}.$$

STRUCTURE TENSOR MODEL: HARMONIC TENSORS

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INTEGRITY BASIS ($\mathcal{IB}$)

Theorem

Let $\mathbb{V}$ be a real vector space, there exists a finite set $\mathcal{IB} = \{I_k\}$ of $\mathrm{O}(2)$-invariant polynomials, such that any $\mathrm{O}(2)$-invariant polynomial P on $\mathbb{V}$, is a polynomial with respect to the elements of $\mathcal{IB}$. The set $\mathcal{IB}$ is the integrity basis of $\mathbb{V}$ for the $\mathrm{O}(2)$-action.

Example

Considering $\underset{\sim}{\sigma} \in \mathbf{Inv}(S^2(\mathbb{R}^2))$, a minimal integrity basis for $\mathbf{Inv}(S^2(\mathbb{R}^2), \mathrm{O}(2))$ is:

$$\{I_1 = \mathrm{tr}\, \underset{\sim}{\sigma},\ J_2 = \underset{\sim}{\sigma}^{(2)} : \underset{\sim}{\sigma}^{(2)}\}$$

ESTABLISHING D_n-INVARIANT POLYNOMIAL USING INTEGRITY BASIS

1. Let $\mathbb{V} = S^2(\mathbb{R}^2) \oplus \mathbb{K}^n$, the space model representing $\underset{\sim}{\sigma} \in S^2(\mathbb{R}^2)$ and $\mathbf{M} \in \mathbb{K}^n$.
2. Compute the $\mathrm{O}(2)$-integrity basis $\mathcal{IB}$ of $\mathbb{V}$. In $\mathbb{R}^2$, we have a general algorithm to determine such a basis [9];
3. A D_k-invariant polynomial of degree n is a linear combination of monomials of degree n obtained from the elements of $\mathcal{IB}$.

[9] B. Desmorat et al. "Computation of minimal covariants bases for 2D coupled constitutive laws". In: *International Journal of Engineering Science* (2023)

EXAMPLE: HEXATROPIC CRITERION FUNCTION

1) Consider:

$$\underset{\sim}{\sigma} \in S^2(\mathbb{R}^2) \quad , \quad \underset{\approx}{\mathrm{K}} \propto \underline{\mathrm{e}}_1^{\otimes 6} - 15\left(\underline{\mathrm{e}}_1^{\otimes 4} \otimes \underline{\mathrm{e}}_2^{\otimes 2}\right)^s + 15\left(\underline{\mathrm{e}}_1^{\otimes 2} \otimes \underline{\mathrm{e}}_2^{\otimes 4}\right)^s - \underline{\mathrm{e}}_2^{\otimes 6} \in \mathbb{K}^6$$

($\mathrm{G}_{\underset{\approx}{\mathrm{K}}} = \mathrm{D}_6$)

2) Integrity Basis:

$$\mathcal{IB} = \{I_1 = \mathrm{tr}(\underset{\sim}{\sigma}),\ I_2 = \underset{\sim}{\sigma}^{(2)} : \underset{\sim}{\sigma}^{(2)}, I_3 = \underset{\approx}{\mathrm{K}} \overset{6}{\cdot} \left(\underset{\sim}{\sigma}^{(2)} \otimes \underset{\sim}{\sigma}^{(2)} \otimes \underset{\sim}{\sigma}^{(2)}\right)\}$$

3) The combination of monomial depending on the polynomial degree:

Degree	Monomials	Dimension
1	I_1	1
2	$I_1^2,\ I_2$	2
3	$I_1^3,\ I_1 I_2,\ I_3$	3
4	$I_1^4,\ I_1^2 I_2,\ I_1 I_3,\ I_2^2$	4

SYNTHESIS

- Using representation theorem and integrity basis computation;
- Considering polynomial threshold function in $\underset{\sim}{\sigma}$;

we have seen how to obtain a D_n-invariant polynomial function in $\underset{\sim}{\sigma}$ from a suitable structure tensor.

Example of a hexatropic (D_6) polynomial threshold function:

Degree	Monomials	Dimension
1	I_1	1
2	$I_1^2,\ I_2$	2
3	$I_1^3,\ I_1 I_2,\ I_3$	3
4	$I_1^4,\ I_1^2 I_2,\ I_1 I_3,\ I_2^2$	4

OUTLINE

Generalised Tsai-Wu threshold function TW4

- Proposed threshold function [10] :

$$F(\underset{\sim}{\sigma}) = \underset{\approx\approx}{A} \overset{8}{\cdot} (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underset{\approx}{\underset{\approx}{B}} \overset{6}{\cdot} (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underbrace{\underset{\approx}{C} :: (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underset{\sim}{D} : \underset{\sim}{\sigma}}_{\text{Tsai-Wu criterion}} .$$

where $\underset{\approx\approx}{A}$, $\underset{\approx}{\underset{\approx}{B}}$, $\underset{\approx}{C}$ and $\underset{\sim}{D}$ are tensors of order 8, 6, 4 et 2 respectively.

- Index symmetry:

$$\underbrace{\left(\underset{\approx\approx}{A}\right)_{\underline{(ij)}\,\underline{(kl)}\,\underline{(mn)}\,\underline{(op)}}}_{S^4(\mathbb{K}^2 \oplus \mathbb{K}^0)} \qquad \underbrace{\left(\underset{\approx}{\underset{\approx}{B}}\right)_{\underline{(ij)}\,\underline{(kl)}\,\underline{(mn)}}}_{S^3(\mathbb{K}^2 \oplus \mathbb{K}^0)} \qquad \underbrace{\left(\underset{\approx}{C}\right)_{\underline{(ij)}\,\underline{(kl)}}}_{S^2(\mathbb{K}^2 \oplus \mathbb{K}^0)} \qquad \underbrace{\left(\underset{\sim}{D}\right)_{(ij)}}_{(\mathbb{K}^2 \oplus \mathbb{K}^0)}$$

[10] S. Tsai et al. “A General Theory of Strength for Anisotropic Materials”. In: *Journal of Composite Materials* (1971)

Harmonic structure

The harmonic structure is the decomposition of $\mathbb{T}^n$ into a direct sum of $\mathrm{O}(2)$-irreducible subspaces $\mathbb{K}$:

$$\mathbb{T}^n = \bigoplus_k \alpha_k \mathbb{K}^k.$$

Lemma:

$$S^n\left(\mathbb{K}^2 \oplus \mathbb{K}^0\right) \simeq \bigoplus_{k=0}^{n} S^k\left(\mathbb{K}^2\right) \quad ; \quad \begin{cases} S^{2n}\left(\mathbb{K}^p\right) \simeq \bigoplus_{k=0}^{n} \mathbb{K}^{2kp}, \\ S^{2n+1}\left(\mathbb{K}^p\right) \simeq \bigoplus_{k=0}^{n} \mathbb{K}^{(2k+1)p} \end{cases}$$

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Harmonic structure of each tensor

Proposition

The harmonic structure of tensor spaces of $\mathbb{TW}_4$ *are as follows:*

$\underset{\sim}{D} \in \mathbb{W}_1 \simeq \mathbb{K}^2 \oplus \mathbb{K}^0,$	$\dim(\mathbb{W}_1) = 3,$
$\underset{\approx}{C} \in \mathbb{W}_2 \simeq \mathbb{K}^4 \oplus \mathbb{K}^2 \oplus 2\mathbb{K}^0,$	$\dim(\mathbb{W}_2) = 6,$
$\underset{\approx\!\!\!\sim}{B} \in \mathbb{W}_3 \simeq \mathbb{K}^6 \oplus \mathbb{K}^4 \oplus 2\mathbb{K}^2 \oplus 2\mathbb{K}^0,$	$\dim(\mathbb{W}_3) = 10,$
$\underset{\approx\!\!\!\approx}{A} \in \mathbb{W}_4 \simeq \mathbb{K}^8 \oplus \mathbb{K}^6 \oplus 2\mathbb{K}^4 \oplus 2\mathbb{K}^2 \oplus 3\mathbb{K}^0,$	$\dim(\mathbb{W}_4) = 15.$

As a result, the number of coefficients:

$$\dim(\mathbb{TW}_4) = 34$$

EXPLICIT HARMONIC DECOMPOSITION

$$F(\underset{\sim}{\sigma}) = \underset{\approx}{\overset{8}{A}} \cdot (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underset{\approx}{\overset{6}{B}} \cdot (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underbrace{\underset{\approx}{C} :: (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underset{\sim}{D} : \underset{\sim}{\sigma}}_{\text{Tsai-Wu criterion}} .$$

21 harmonic tensors in the decomposition :

$$F(\underset{\sim}{\sigma}) = \varphi(\underset{\sim}{\sigma}, \underset{\approx}{E}^{8,4}, \underset{\approx}{S}^{8,3}, \underset{\approx}{S}^{6,3}, \underset{\approx}{H}^{8,4}, \underset{\approx}{H}^{8,2}, \underset{\approx}{H}^{6,2}, \underset{\approx}{H}^{4,2}, \underset{\sim}{h}^{8,3}, \underset{\sim}{h}^{8,1}, \underset{\sim}{h}^{6,3}, \underset{\sim}{h}^{6,1},$$

$$\underset{\sim}{h}^{4,1}, \underset{\sim}{h}^{2,1}, \alpha^{8,4}, \alpha^{8,2}, \alpha^{8,0}, \alpha^{6,2}, \alpha^{6,0}, \alpha^{4,2}, \alpha^{4,0}, \alpha^{2,0}).$$

where $\underset{\approx}{E}^{p,q} \in \mathbb{K}^8$; $(\underset{\approx}{S}^{p,q}) \in (\mathbb{K}^6)$; $(\underset{\approx}{H}^{p,q}) \in \mathbb{K}^4$, $(\underset{\sim}{h}^{p,q}) \in \mathbb{K}^2$; $(\alpha^{p,q}) \in \mathbb{K}^0$.

Considering $\mathbf{T}^{q,p} \in \mathbb{K}^n$:

- q means that $\mathbf{T}$ is derived from a tensor of order q.
- p is related to the monomial where the deviatoric part of stress tensor occurs p times.

Explicit harmonic decomposition

$$F(\underset{\sim}{\sigma}) = \underset{\approx\approx}{\mathrm{A}} \overset{8}{\cdot} (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underset{\approx}{\mathrm{B}} \overset{6}{\cdot} (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underbrace{\underset{\approx}{\mathrm{C}} :: (\underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma}) + \underset{\sim}{\mathrm{D}} : \underset{\sim}{\sigma}}_{\text{Tsai-Wu criterion}}.$$

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$$\mathrm{h}^{4,1}, \mathrm{h}^{2,1}, \alpha^{8,4}, \alpha^{8,2}, \alpha^{8,0}, \alpha^{6,2}, \alpha^{6,0}, \alpha^{4,2}, \alpha^{4,0}, \alpha^{2,0}).$$

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21 harmonic tensors in the decomposition [11]:

$$F(\underset{\sim}{\sigma}) = \varphi(\underset{\sim}{\sigma}, \underset{\approx\approx}{\mathrm{E}}^{8,4}, \underset{\approx}{\mathrm{S}}^{8,3}, \underset{\approx}{\mathrm{S}}^{6,3}, \underset{\approx}{\mathrm{H}}^{8,4}, \underset{\approx}{\mathrm{H}}^{8,2}, \underset{\approx}{\mathrm{H}}^{6,2}, \underset{\approx}{\mathrm{H}}^{4,2}, \underset{\sim}{\mathrm{h}}^{8,3}, \underset{\sim}{\mathrm{h}}^{8,1}, \underset{\sim}{\mathrm{h}}^{6,3}, \underset{\sim}{\mathrm{h}}^{6,1}, \underset{\sim}{\mathrm{h}}^{4,1}, \underset{\sim}{\mathrm{h}}^{2,1}, \alpha^{8,4}, \alpha^{8,2}, \alpha^{8,0}, \alpha^{6,2}, \alpha^{6,0}, \alpha^{4,2}, \alpha^{4,0}, \alpha^{2,0}).$$

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[11] N. Auffray et al. “Explicit Harmonic Structure Of Bidimensional Linear Strain-Gradient Elasticity”. In: *European Journal of Mechanics - A/Solids* (2020)

About harmonic tensors

In a general case all harmonic tensors are considered:

Terms degree	$\mathbb{K}^8$	$\mathbb{K}^6$	$\mathbb{K}^4$	$\mathbb{K}^2$	$\mathbb{K}^0$
4	$\underset{\approx\approx}{\mathrm{E}}^{8,4}$	$\underset{\approx\approx}{\mathrm{S}}^{8,3}$	$\underset{\approx}{\mathrm{H}}^{8,4}$, $\underset{\approx}{\mathrm{H}}^{8,2}$	$\underset{\sim}{\mathrm{h}}^{8,3}$, $\underset{\sim}{\mathrm{h}}^{8,2}$	$\alpha^{8,4}$, $\alpha^{8,2}$, $\alpha^{8,0}$
3		$\underset{\approx\approx}{\mathrm{S}}^{6,3}$	$\underset{\approx}{\mathrm{H}}^{6,2}$	$\underset{\sim}{\mathrm{h}}^{6,3}$, $\underset{\sim}{\mathrm{h}}^{6,2}$	$\alpha^{6,0}$, $\alpha^{6,2}$
2			$\underset{\approx}{\mathrm{H}}^{4,2}$	$\underset{\sim}{\mathrm{h}}^{4,1}$	$\alpha^{4,2}$, $\alpha^{4,0}$
1				$\underset{\sim}{\mathrm{h}}^{2,1}$	$\alpha^{2,0}$

About harmonic tensors: hexatropic case

$$\mathcal{G}_{\mathcal{S}}^{2D} = \mathrm{D}_3 \qquad (\mathrm{G}_{\mathcal{M}} = \mathrm{D}_6)$$

Terms degree	$\mathbb{K}^8$	$\mathbb{K}^6$	$\mathbb{K}^4$	$\mathbb{K}^2$	$\mathbb{K}^0$
4	$\underset{\approx}{\mathrm{E}}^{8,4}$	$\underset{\approx}{\mathrm{S}}^{8,3}$	$\underset{\approx}{\mathrm{H}}^{8,4}, \underset{\approx}{\mathrm{H}}^{8,2}$	$\underset{\sim}{\mathrm{h}}^{8,3}, \underset{\sim}{\mathrm{h}}^{8,2}$	$\alpha^{8,4}, \alpha^{8,2}, \alpha^{8,0}$
3		$\underset{\approx}{\mathrm{S}}^{6,3}$	$\underset{\approx}{\mathrm{H}}^{6,2}$	$\underset{\sim}{\mathrm{h}}^{6,3}, \underset{\sim}{\mathrm{h}}^{6,2}$	$\alpha^{6,0}, \alpha^{6,2}$
2			$\underset{\approx}{\mathrm{H}}^{4,2}$	$\underset{\sim}{\mathrm{h}}^{4,1}$	$\alpha^{4,2}, \alpha^{4,0}$
1				$\underset{\sim}{\mathrm{h}}^{2,1}$	$\alpha^{2,0}$

Previously (integrity basis):

Degree	Monomials	Dimension
1	I_1	1
2	$I_1^2,\ I_2$	2
3	$I_1^3,\ I_1 I_2,\ I_3$	3
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About harmonic tensors: hexatropic case

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Terms degree	$\mathbb{K}^8$	$\mathbb{K}^6$	$\mathbb{K}^4$	$\mathbb{K}^2$	$\mathbb{K}^0$
4	$\underset{\approx}{\underset{\approx}{\mathrm{E}}}^{8,4}$	$\underset{\approx}{\mathrm{S}}^{8,3}$	$\underset{\approx}{\mathrm{H}}^{8,4}, \underset{\approx}{\mathrm{H}}^{8,2}$	$\underset{\sim}{\mathrm{h}}^{8,3}, \underset{\sim}{\mathrm{h}}^{8,2}$	$\alpha^{8,4}, \alpha^{8,2}, \alpha^{8,0}$
3		$\underset{\approx}{\mathrm{S}}^{6,3}$	$\underset{\approx}{\mathrm{H}}^{6,2}$	$\underset{\sim}{\mathrm{h}}^{6,3}, \underset{\sim}{\mathrm{h}}^{6,2}$	$\alpha^{6,0}, \alpha^{6,2}$
2			$\underset{\approx}{\mathrm{H}}^{4,2}$	$\underset{\sim}{\mathrm{h}}^{4,1}$	$\alpha^{4,2}, \alpha^{4,0}$
1				$\underset{\sim}{\mathrm{h}}^{2,1}$	$\alpha^{2,0}$

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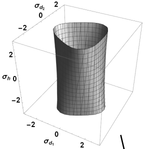
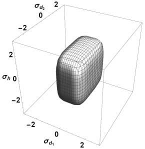
Symmetry classes of TW4's threshold surface

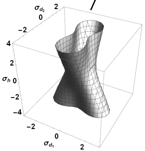


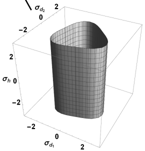


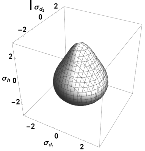
Proposition

The symmetry classes of $\mathbb{TW}_4$ *are*

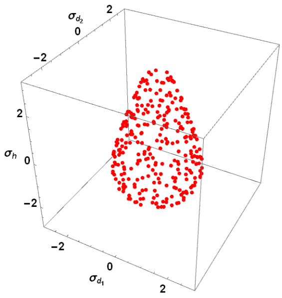
$$\mathfrak{I}(\mathbb{TW}_4) = \{[\mathbb{1}], [\mathrm{Z}_2], [\mathrm{Z}_3], [\mathrm{D}_2], [\mathrm{D}_3], \dots, [\mathrm{Z}_2^c], [\mathrm{Z}_2 \otimes \mathrm{Z}_2^c], [\mathrm{D}_2 \otimes \mathrm{Z}_2^c], [\mathrm{D}_3 \otimes \mathrm{Z}_2^c], [\mathrm{D}_4 \otimes \mathrm{Z}_2^c], [\mathcal{O} \otimes \mathrm{Z}_2^c], [\mathrm{O}(2) \otimes \mathrm{Z}_2^c], [\mathrm{O}(3)], [\mathrm{Z}_2^-], [\mathrm{Z}_4^-], [\mathrm{D}_2^v], [\mathrm{D}_3^v], [\mathrm{D}_4^v], [\mathrm{D}_2^h], [\mathrm{D}_4^h], [\mathrm{D}_6^h], [\mathcal{O}^-], [\mathrm{O}(2)^-]\}$$







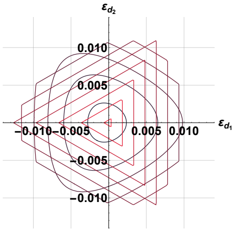
Approximation of the threshold function



Points in stress space

→ "NonLinearModelFit" → Find harmonic parameters

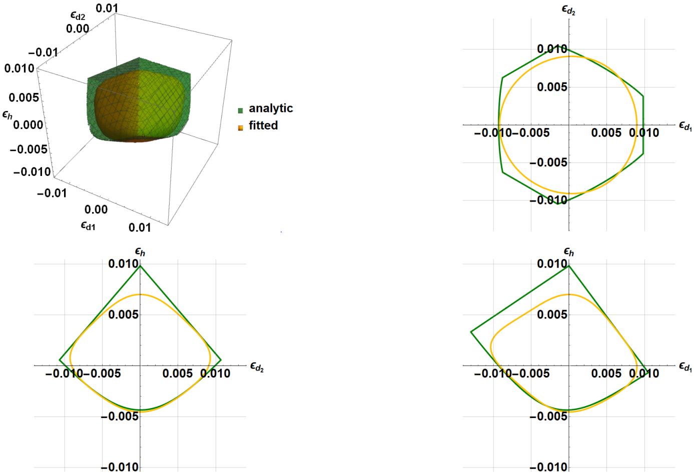
Approximation example: triangular 2D lattice.



Terms degree	$\mathbb{K}^8$	$\mathbb{K}^6$	$\mathbb{K}^4$	$\mathbb{K}^2$	$\mathbb{K}^0$
4	$\underset{\approx}{\mathrm{E}}^{8,4}$	$\underset{\approx}{\mathrm{S}}^{8,3}$	$\underset{\approx}{\mathrm{H}}^{8,4}, \underset{\approx}{\mathrm{H}}^{8,2}$	$\underset{\sim}{\mathrm{h}}^{8,3}, \underset{\sim}{\mathrm{h}}^{8,2}$	$\alpha^{8,4}, \alpha^{8,2}, \alpha^{8,0}$
3		$\underset{\approx}{\mathrm{S}}^{6,3}$	$\underset{\approx}{\mathrm{H}}^{6,2}$	$\underset{\sim}{\mathrm{h}}^{6,3}, \underset{\sim}{\mathrm{h}}^{6,2}$	$\alpha^{6,0}, \alpha^{6,2}$
2			$\underset{\approx}{\mathrm{H}}^{4,2}$	$\underset{\sim}{\mathrm{h}}^{4,1}$	$\alpha^{4,2}, \alpha^{4,0}$
1				$\underset{\sim}{\mathrm{h}}^{2,1}$	$\alpha^{2,0}$

7 parameters in total.

Approximation example: triangular 2D lattice (harmonic basis)



SYNTHESIS

- A quartic polynomial criterion function was proposed, material tensors of order $(2, 4, 6, 8)$ were considered.
- The harmonic decomposition allowed the set ofto get better insight of symmetries that TW4 can describe from the two points of view: the 2D anisotropy ($\mathcal{G}_{\mathcal{S}}^{2D}$) and the 3D symmetry of criterion surface ($\mathcal{G}_{\mathcal{S}}^{3D}$).
- Approximation of some selected threshold criteria from the literature were performed.

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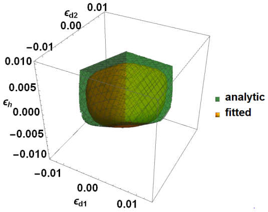

Conclusion

Versitle thresohold function for *lattice* materials

1. A theoretical framework has been establish for the anisotropy in the stress space.
2. Generalised quartic polynomial criterion was proposed.
3. Using of harmonic decomposition allowed to get insights of all possible symmetries with respect to 2D anisotropy and 3D surface symmetry.

Thank you for your attention.

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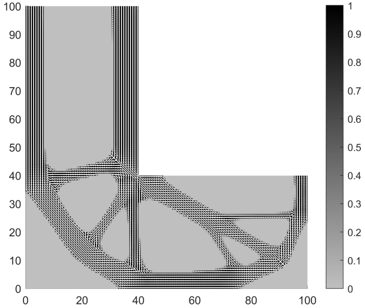
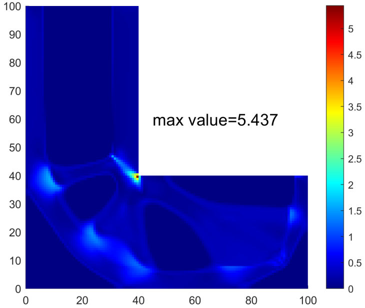
Stiffness constraint: anisotropic case ($\theta_0(\underline{x}) = -45$)



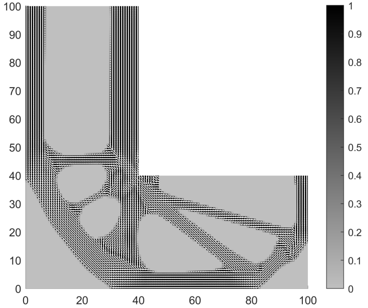
optimal orientation + density



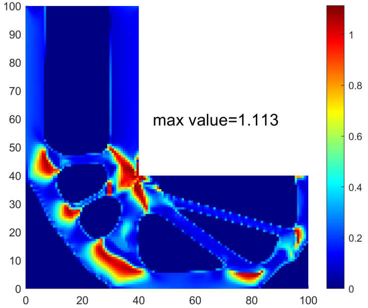
The inverse of the safety factor ($\frac{1}{\kappa}$)

$$
\begin{cases}
\min_{\rho,\theta} \int_{\Omega} \rho \, \mathrm{dv}, \\
\frac{1}{C_0}\left(\int_{\Gamma_N} \underline{u} \cdot \underline{t} \, \mathrm{ds} - 1\right) \mathrm{ds} \leq 0, \\
a_\rho(\underline{u}, \underline{v}) = L(\underline{v}) \quad \forall \, \underline{v} \in U_{ad}, \\
0 \leq \rho \leq 1, \quad -3\frac{\pi}{2} < \theta < 3\frac{\pi}{2},
\end{cases}
$$

Initial orientation: $\theta(\underline{x}) = -45 \; \forall \underline{x} \in \Omega$



optimal orientation + density



The inverse of the safety factor ($\frac{1}{\kappa}$)

$$\begin{cases} \min\limits_{\rho,\theta} \int_\Omega \rho \, \mathrm{dv}, \\ \Psi_{KS}^{L}(\bar{g}) \leq 0, \\ \frac{1}{\mathcal{C}_0}\left(\int_{\Gamma_N} \underline{u}\cdot\underline{t}\,\mathrm{ds} - 1\right)\mathrm{ds} \leq 0, \\ a_\rho(\underline{u},\underline{v}) = L(\underline{v}) \quad \forall\, \underline{v} \in U_{ad}, \\ 0 \leq \rho \leq 1, \quad -3\frac{\pi}{2} < \theta < 3\frac{\pi}{2}, \end{cases}$$

Initial orientation: $\theta(\underline{x}) = -45 \; \forall \underline{x} \in \Omega$

Example 1: Cazacu and Barlat 2004

Cazacu and Barlat criterion (3D) [12]:

$$F(\underset{\sim 3D}{\sigma}) = (J_2)^{3/2} - cJ_3 = \sigma_{lim}^3,$$

where c is a material parameter; $J_2 = \mathrm{tr}(\underset{\sim 3D}{\sigma}^{(2)} \cdot \underset{\sim 3D}{\sigma}^{(2)})/3$ and $J_3 = \mathrm{tr}(\underset{\sim 3D}{\sigma}^{(2)} \cdot \underset{\sim 3D}{\sigma}^{(2)} \cdot \underset{\sim 3D}{\sigma}^{(2)})/3$.

Properties:

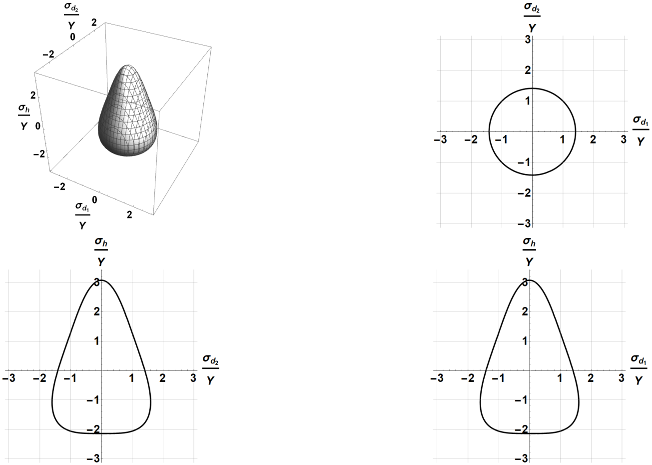
- Isotropic
- Dissymmetric in traction and compression

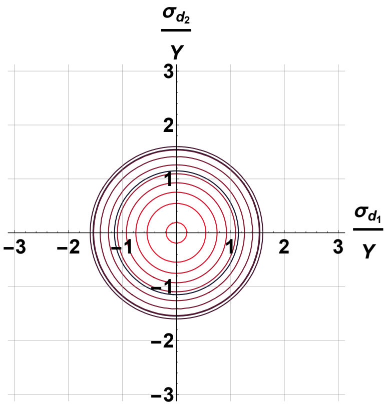
In planar stress:

$$F_{2D}(\underset{\sim}{\sigma}) = \left[\frac{1}{3}\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right)\right]^{3/2} - \frac{c}{27}\left[2\sigma_1^3 + 2\sigma_2^3 - 3\left(\sigma_1 + \sigma_2\right)\sigma_1\sigma_2\right]$$

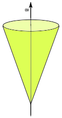
[12] O. Cazacu et al. In: *International Journal of Plasticity* (2004)

Example 1: Cazacu and Barlat 2004 (harmonic basis)





- $\mathcal{G}_{\mathcal{S}}^{2D} = \mathrm{O}(2)$
- $\mathcal{G}_{\mathcal{S}}^{3D} = \mathrm{O}(2)^-$



Example 1: Cazacu and Barlat 2004 (harmonic basis)

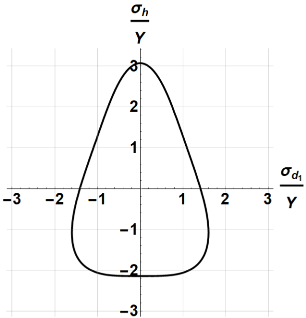


Figure: Threshold surface of Cazacu's criterion for $c = 1.28$.

EXAMPLES: "POLY4" (SOARE ET AL. 2010)

- The "Poly4" criterion (2D) S. Soare et al. “On Using Homogeneous Polynomials To Design Anisotropic Yield Functions With Tension/Compression Symmetry/Assymetry”. In: *AIP Conference Proceedings* (2007) is given by:

$$F(\underset{\sim}{\sigma}) = a_1\sigma_{11}^4 + a_2\sigma_{11}^3\sigma_{22} + a_3\sigma_{11}^2\sigma_{22}^2 + a_4\sigma_{11}\sigma_{22}^3 + a_5\sigma_{22}^4 +$$
$$+ \left(a_6\sigma_{11}^2 + a_7\sigma_{11}\sigma_{22} + a_8\sigma_{22}^2\right)\sigma_{12}^2 + a_9\sigma_{12}^4,$$

where $(a_i,\ i = 1...9)$ are material parameters.

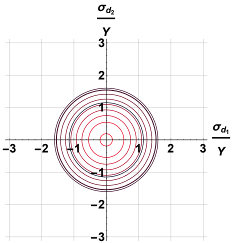
- The criterion is anisotropic and it is used for orthotropic materials.

About harmonic tensors: Pressure independent

The threshold function independent of the hydrostatic stress.

Terms degree	$\mathbb{K}^8$	$\mathbb{K}^6$	$\mathbb{K}^4$	$\mathbb{K}^2$	$\mathbb{K}^0$
4	$\underset{\approx\approx}{\mathrm{E}}^{8,4}$	$\underset{\approx\approx}{\mathrm{S}}^{8,3}$	$\underset{\approx}{\mathrm{H}}^{8,4}, \underset{\approx}{\mathrm{H}}^{8,2}$	$\underset{\sim}{\mathrm{h}}^{8,3}, \underset{\sim}{\mathrm{h}}^{8,2}$	$\alpha^{8,4}, \alpha^{8,2}, \alpha^{8,0}$
3		$\underset{\approx\approx}{\mathrm{S}}^{6,3}$	$\underset{\approx}{\mathrm{H}}^{6,2}$	$\underset{\sim}{\mathrm{h}}^{6,3}, \underset{\sim}{\mathrm{h}}^{6,2}$	$\alpha^{6,2}, \alpha^{6,0}$
2			$\underset{\approx}{\mathrm{H}}^{4,2}$	$\underset{\sim}{\mathrm{h}}^{4,1}$	$\alpha^{4,2}, \alpha^{4,0}$
1				$\underset{\sim}{\mathrm{h}}^{2,1}$	$\alpha^{2,0}$

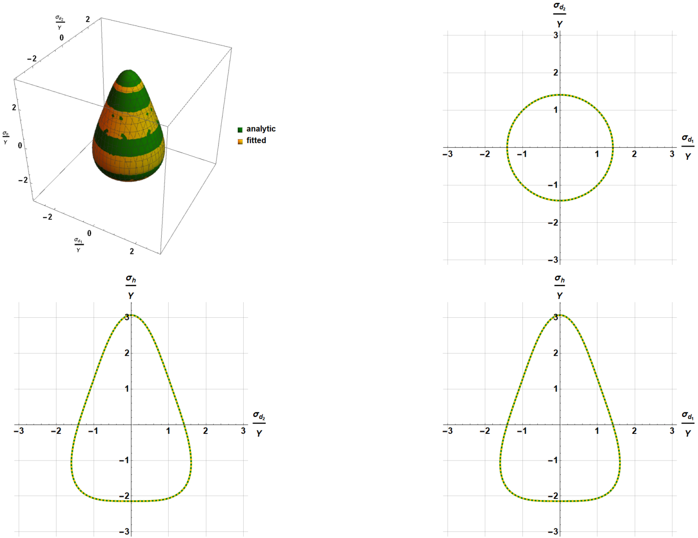
APPROXIMATION EXAMPLE : CAZACU AND BARLAT CRITERION



The criterion is isotropic which means:

Terms degree	$\mathbb{K}^8$	$\mathbb{K}^6$	$\mathbb{K}^4$	$\mathbb{K}^2$	$\mathbb{K}^0$
4	$\underset{\approx}{\underset{\approx}{\mathrm{E}}}^{8,4}$	$\underset{\approx}{\underset{\sim}{\mathrm{S}}}^{8,3}$	$\underset{\approx}{\mathrm{H}}^{8,4}, \underset{\approx}{\mathrm{H}}^{8,2}$	$\underset{\sim}{\mathrm{h}}^{8,3}, \underset{\sim}{\mathrm{h}}^{8,2}$	$\alpha^{8,4}, \alpha^{8,2}, \alpha^{8,0}$
3		$\underset{\approx}{\underset{\sim}{\mathrm{S}}}^{6,3}$	$\underset{\approx}{\mathrm{H}}^{6,2}$	$\underset{\sim}{\mathrm{h}}^{6,3}, \underset{\sim}{\mathrm{h}}^{6,2}$	$\alpha^{6,2}, \alpha^{6,0}$
2			$\underset{\approx}{\mathrm{H}}^{4,2}$	$\underset{\sim}{\mathrm{h}}^{4,1}$	$\alpha^{4,2}, \alpha^{4,0}$
1				$\underset{\sim}{\mathrm{h}}^{2,1}$	$\alpha^{2,0}$

Approximation example : Cazacu and Barlat criterion



THRESHOLD FUNCTION: HOW TO INCLUDE ANISOTROPY?

At least 3 methods:

- Representation theorems [13]

$$F(\underset{\sim}{\sigma}, \mathbf{M}), \qquad \mathrm{G}_M = \mathrm{H} < \mathrm{O}(2).$$

- Linear Transformations [14]

$$F(\underset{\sim}{\sigma}) = h_{iso}(\underset{\sim}{\Sigma}) \qquad \underset{\sim}{\Sigma} = \underset{\approx}{\mathrm{T}} : \underset{\sim}{\sigma}, \quad \mathrm{G}_{\underset{\approx}{\mathrm{T}}} = \mathrm{H} < \mathrm{O}(2).$$

- High degree polynomials[15]

[13] J. P. Boehler. *Applications of Tensor Functions in Solid Mechanics*. Springer, Vienna, 1987

[14] F. Barlat et al. In: *International Journal of Plasticity* (2007)

[15] S. Soare. In: *European Journal of Mechanics - A/Solids* (2022)

EXPLICIT HARMONIC DECOMPOSITION OF $\underset{\approx}{\mathrm{B}}$

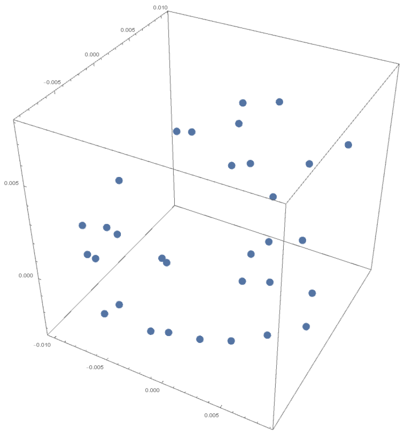
$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{d} + \underset{\sim}{\sigma}^{s}$$

$$\begin{aligned}
\underset{\approx}{\mathrm{B}}\overset{6}{\cdot}(\underset{\sim}{\sigma}\otimes\underset{\sim}{\sigma}\otimes\underset{\sim}{\sigma}) &= (\underset{\sim}{\sigma}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}):\underset{\sim}{\sigma} = \left[(\underset{\sim}{\sigma}^{d}+\underset{\sim}{\sigma}^{s}):\underset{\approx}{\mathrm{B}}:(\underset{\sim}{\sigma}^{d}+\underset{\sim}{\sigma}^{s})\right]:(\underset{\sim}{\sigma}^{d}+\underset{\sim}{\sigma}^{s}),\\
&= (\underset{\sim}{\sigma}^{d}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}^{d}):\underset{\sim}{\sigma}^{d} + (\underset{\sim}{\sigma}^{d}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}^{d}):\underset{\sim}{\sigma}^{s} + (\underset{\sim}{\sigma}^{d}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}^{s}):\underset{\sim}{\sigma}^{d} + (\underset{\sim}{\sigma}^{s}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}^{d}):\underset{\sim}{\sigma}^{d}\\
&+ (\underset{\sim}{\sigma}^{s}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}^{s}):\underset{\sim}{\sigma}^{d} + (\underset{\sim}{\sigma}^{s}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}^{d}):\underset{\sim}{\sigma}^{s} + (\underset{\sim}{\sigma}^{d}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}^{s}):\underset{\sim}{\sigma}^{s} + (\underset{\sim}{\sigma}^{s}:\underset{\approx}{\mathrm{B}}:\underset{\sim}{\sigma}^{s}):\underset{\sim}{\sigma}^{s}.
\end{aligned}$$

$$\begin{aligned}
\underset{\approx}{\mathrm{B}} &= \underset{\approx}{\mathrm{S}}^{6,3} + \underset{\approx}{\Phi}^{6,3}:\underset{\sim}{\mathrm{h}}^{6,3} + \frac{1}{2}\left[\underset{\approx}{\mathrm{Q}}^{6,2}\otimes\underset{\sim}{\mathrm{I}} + \varsigma_{(35)(46)}\star\left(\underset{\approx}{\mathrm{Q}}^{6,2}\otimes\underset{\sim}{\mathrm{I}}\right) + \underset{\sim}{\mathrm{I}}\otimes\underset{\approx}{\mathrm{Q}}^{6,2}\right]\\
&+ \frac{1}{4}\left[\underset{\sim}{\mathrm{h}}^{6,1}\otimes\underset{\sim}{\mathrm{I}}\otimes\underset{\sim}{\mathrm{I}} + \underset{\sim}{\mathrm{I}}\otimes\underset{\sim}{\mathrm{h}}^{6,1}\otimes\underset{\sim}{\mathrm{I}} + \underset{\sim}{\mathrm{I}}\otimes\underset{\sim}{\mathrm{I}}\otimes\underset{\sim}{\mathrm{h}}^{6,1}\right] + \frac{\alpha^{6,0}}{8}(\underset{\sim}{\mathrm{I}}\otimes\underset{\sim}{\mathrm{I}}\otimes\underset{\sim}{\mathrm{I}}),
\end{aligned}$$

Explicit harmonic decomposition of $\underset{\approx}{\mathrm{A}}$

$$
\begin{aligned}
\underset{\approx}{\mathrm{A}} = & \underset{\approx}{\mathrm{E}}^{8,4} + \underset{\approx}{\Phi}^{8,4} : \underset{\approx}{\mathrm{H}}^{8,4} + \alpha^{8,4} \underset{\approx}{\Theta} \\
& + \frac{1}{2} \left[\underset{\approx}{\mathrm{Q}}^{8,3} \otimes \underset{\sim}{\mathrm{I}} + \varsigma_{(57)(68)} \star \left(\underset{\approx}{\mathrm{Q}}^{8,3} \otimes \underset{\sim}{\mathrm{I}} \right) + \varsigma_{(37)(48)} \star \left(\underset{\approx}{\mathrm{Q}}^{8,3} \otimes \underset{\sim}{\mathrm{I}} \right) + \underset{\sim}{\mathrm{I}} \otimes \underset{\approx}{\mathrm{Q}}^{8,3} \right] \\
& + \frac{1}{4} \left[\underset{\approx}{\mathrm{Q}}^{8,2} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} + \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\approx}{\mathrm{Q}}^{8,2} + \underset{\sim}{\mathrm{I}} \otimes \underset{\approx}{\mathrm{Q}}^{8,2} \otimes \underset{\sim}{\mathrm{I}} + \varsigma_{(37)(48)} \star \left(\underset{\approx}{\mathrm{Q}}^{8,2} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \right) \right. \\
& \quad \left. + \varsigma_{(35)(46)} \star \left(\underset{\approx}{\mathrm{Q}}^{8,2} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \right) + \varsigma_{(17)(28)} \star \left(\underset{\approx}{\mathrm{Q}}^{8,2} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \right) \right] \\
& + \frac{1}{8} \left[\underset{\sim}{\mathrm{h}}^{8,1} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} + \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{h}}^{8,1} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} + \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{h}}^{8,1} \otimes \underset{\sim}{\mathrm{I}} + \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{h}}^{8,1} \right] \\
& + \frac{\alpha^{8,0}}{16} \left(\underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \otimes \underset{\sim}{\mathrm{I}} \right),
\end{aligned}
$$



Figure

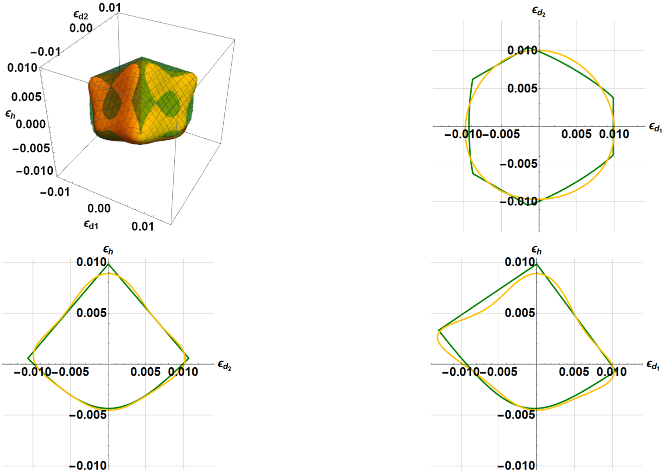


Figure: The threshold surface a triangular lattice 33 points.

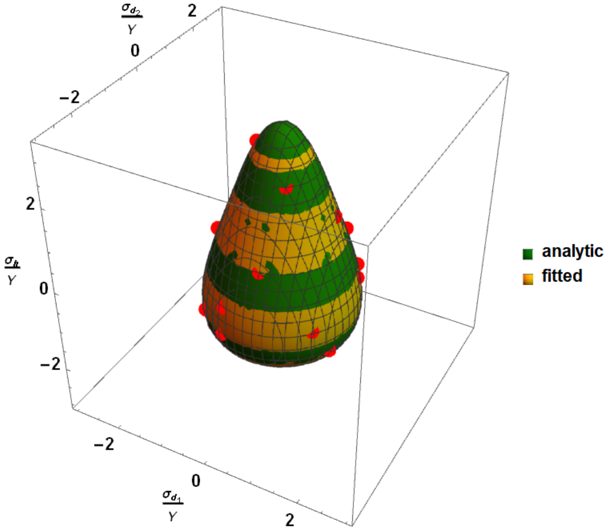


Figure: 20 points

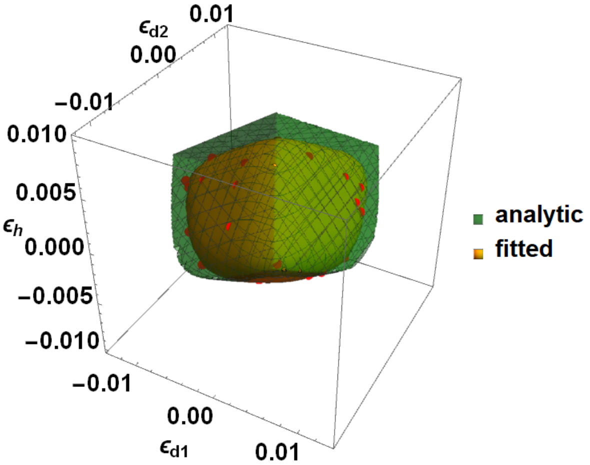
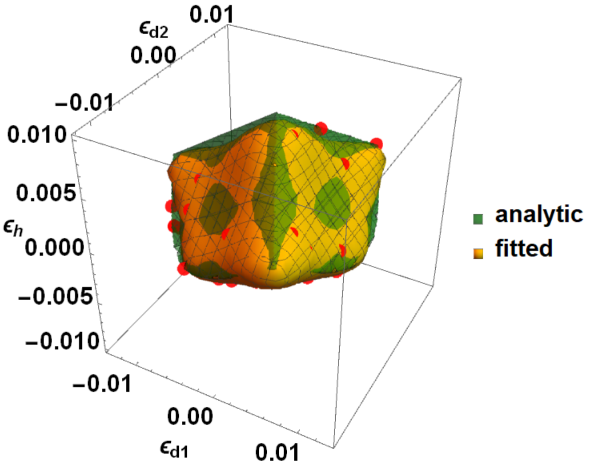


Figure: 39 points



ϵ_{d2}
0.01
0.00
−0.01
0.010
0.005
ϵ_h
0.000
−0.005
−0.010
−0.01
0.00
ϵ_{d1}
0.01
analytic
fitted

Figure