Monge–Ampère Geometry and the Navier–Stokes Equations

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Terminology and Notation

- Configuration Space / Background $\mathbb{R}^m$
  with coordinates (x^i), $i = 1, 2, \ldots m$.
  In 2D, we may set $x^1 = x$, $x^2 = y$.

- Phase Space / Cotangent Bundle $T^*\mathbb{R}^m$
  with coordinates (x^i, q_i), $i = 1, 2, \ldots, m$
  (q's are fibre coordinates).

- Use Einstein summation convention throughout.

(Contact) Monge–Ampère Equations

- Monge–Ampère (MA) Equations are non-linear second-order PDEs which are quasi-linear w.r.t. second order partial derivatives, up to determinants of the Hessian or its minors.

- In two dimensions, they take the form

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D(\psi_{xx}\psi_{yy} - \psi_{xy}^2) + E = 0$$

  where $A, B, \ldots E$ can depend on $x, y, \psi, \psi_x, \psi_y$ non-linearly.

- If $A, B, \ldots E$ do not depend on ψ, we have a symplectic Monge–Ampère equation.

Monge–Ampère Structures and Solutions

- A Monge–Ampère Structure is a triple $(T^*\mathbb{R}^m, \omega, \alpha)$ with
  - $\omega \in \Omega^2(T^*\mathbb{R}^m)$ symplectic, e.g. $\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i$,
  - $\alpha \in \Omega^m(T^*\mathbb{R}^m)$ is ω-effective, i.e. $\alpha \wedge \omega = 0$,

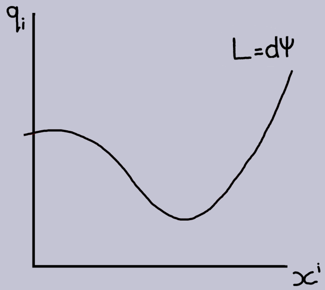
  We call α the Monge–Ampère Form. [Banos 2002]

- A Generalised Solution to a MA equation, w.r.t. a MA structure, is a submanifold $L \hookrightarrow T^*\mathbb{R}^m$ s.t.
  - L is Lagrangian, i.e. $\dim(L) = m$ and $\omega|_L = 0$.
  - α vanishes on L, i.e. $\alpha|_L = 0$.

  [Kushner et al. 2007]

Recovering PDEs and Classical Solutions

- Consider $L = \mathrm{d}\psi$ with coordinates $(x^i, \partial_i \psi)$ for some $\psi \in \mathscr{C}^\infty(\mathbb{R}^m)$.
- Trivially Lagrangian for canonical ω as $\omega|_{\mathrm{d}\psi} = 0$.
- The condition $\alpha|_{\mathrm{d}\psi} = 0$ corresponds to a MA equation, with classical solution ψ. [Lychagin 1979]
- The projection $\pi : L \to \mathbb{R}^m$ is a diffeomorphism.



Monge–Ampère Equations in Two Dimensions

The ω-effective MA forms for 2D background (4D phase space) are

$$\alpha = A\,\mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\left(\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2\right) + C\,\mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

.

Imposing that $\alpha|_{\mathrm{d}\psi} = 0$ yields ($x^1 = x$, $x^2 = y$, and $q_i = \partial_i \psi$)

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0$$

This correspondence is a bijection – unique MA form in ω-effective class.

The Pfaffian of a Monge–Ampère Form

- The Pfaffian of α is defined by $\alpha \wedge \alpha =: f\omega \wedge \omega$ and in 2D, is given by $f = AC - B^2 - DE$.
- Hence, the MA equation $\alpha|_{d\psi} = 0$ is

$$\textit{elliptic} \Leftrightarrow f > 0.$$
$$\textit{hyperbolic} \Leftrightarrow f < 0.$$
$$\textit{parabolic} \Leftrightarrow f = 0.$$

- Two MA forms (hence equations) α_1, α_2 are locally equivalent if there exists a local symplectomorphism $F : (T^*\mathbb{R}^2, \omega, \alpha_1) \to (T^*\mathbb{R}^2, \omega, \alpha_2)$ such that $F^*\alpha_2 = \alpha_1$.

The Lychagin–Rubtsov Theorem and Equivalence

- .[Lychagin et al. 1993] define the endomorphism of vector fields $J : \mathfrak{X}(T^*\mathbb{R}^2) \to \mathfrak{X}(T^*\mathbb{R}^2)$ by

$$\frac{1}{\sqrt{|f|}}\alpha(\cdot\,,\cdot) =: \omega(J\,\cdot\,,\cdot)$$

  $f > 0 \Leftrightarrow J$ is almost complex $(J^2 = -1)$
  $f < 0 \Leftrightarrow J$ is almost para-complex $(J^2 = 1)$

- The Lychagin–Rubtsov theorem states t.f.a.e:
  - $\mathrm{d}(\frac{1}{\sqrt{|f|}}\alpha) = 0$.
  - $\alpha|_{\mathrm{d}\psi} = 0$ is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$.
  - J is integrable.

The Navier–Stokes Equations, Vorticity, and Strain

- Homogeneous, Incompressible Navier–Stokes on $\mathbb{R}^m$ background:

$$\partial_t v = -(v \cdot \nabla)v - \nabla p + \nu \Delta v \ (-c)\,. \qquad (*)$$

- Continuity equation is then $\nabla \cdot v = 0$ and applying ∇ to $(*)$ yields:

$$\Delta p \ (+\nabla \cdot c) = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij}\,.$$

where $\zeta_{ij} = \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)$ and $S_{ij} = \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)$.

- Vorticity term dominates $\Leftrightarrow$ $\Delta p > 0$.
  Strain term dominates $\Leftrightarrow$ $\Delta p < 0$.

The Weiss–Okubo Criterion for 2D Flows

- In 2D, solving $\nabla \cdot v = 0$ yields a stream function ψ with $v_1 = -\psi_y \text{ and } v_2 = \psi_x$.
- Pressure equation is then a Monge–Ampère equation for ψ:

$$\Delta p = 2\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) .$$

- *Vorticity dominates* $\Leftrightarrow \Delta p > 0 \Leftrightarrow$ *Elliptic equation.*
  Strain dominates $\Leftrightarrow \Delta p < 0 \Leftrightarrow$ *Hyperbolic equation.*
  No dominance $\Leftrightarrow \Delta p = 0 \Leftrightarrow$ *Parabolic equation.*
  [Weiss 1991, Larchevêque 1993]

Monge–Ampère Geometry for the Poisson Equationn

- The pressure equation $\Delta p = 2(\psi_{xx}\psi_{yy} - \psi_{xy}^2)$ is recovered from

$$\alpha = \mathrm{d}q_1 \wedge \mathrm{d}q_2 - \frac{1}{2}\Delta p \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \,,$$

- Pfaffian is $f = \frac{1}{2}\Delta p$, hence:

  Vorticity dominates $\Leftrightarrow \Delta p > 0 \Leftrightarrow f > 0 \Leftrightarrow$ *Elliptic.*
  Strain dominates $\Leftrightarrow \Delta p < 0 \Leftrightarrow f < 0 \Leftrightarrow$ *Hyperbolic.*
  No dominance $\Leftrightarrow \Delta p = 0 \Leftrightarrow f = 0 \Leftrightarrow$ *Parabolic*

- The Lychagin–Rubtsov theorem says $\Delta p = 2(\psi_{xx}\psi_{yy} - \psi_{xy}^2)$ is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$ iff Δp is constant.

The Lychagin–Rubtsov Metric

- For choice of $K \in \Omega^2(T^*\mathbb{R}^2)$, we define the Lychagin–Rubtsov metric $\hat{g}(\cdot\,,\cdot) := -K(J\,\cdot\,,\cdot)$ [Roulstone et al. 2001]:

$$\hat{g} = \begin{pmatrix} f I_2 & 0 \\ 0 & I_2 \end{pmatrix}$$

- The pull-back of this metric to classical solution $L = \mathrm{d}\psi$ is

$$\hat{g}\big|_{\mathrm{d}\psi} = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

where $\zeta = \Delta\psi$.

Summary Table

Dominance	Vorticity	Strain	None
Δp	> 0	< 0	$= 0$
f	> 0	< 0	$= 0$
$\alpha\vert_{\mathsf{d}\psi} = 0$	Elliptic	Hyperbolic	Parabolic
J^2	-1	1	Singular
$\hat{g}$	Riemannian $(4, 0)$	Kleinian $(2, 2)$	Degenerate**
$\hat{g}\vert_{\mathsf{d}\psi}$	Riemannian $(2, 0)$	Kleinian $(1, 1)$*	Degenerate**

*Except when $\zeta = 0$, in which case it is degenerate.

**Degeneracies when $\Delta p = 0$ correspond to singularities of scalar curvature – they persist under coordinate changes.

Topology of 2D Vortices

- For simply connected regions Σ of 2D flows on which $\Delta p > 0$ and with boundary given by a closed stream-line, all streamlines within Σ are also closed (and convex). [Larchevêque 1993]
- Σ is topologically a disc $[\chi(\Sigma) = \chi(\mathrm{d}\psi(\Sigma)) = 1]$ and Gauß–Bonnet theorem on $L = \mathrm{d}\psi(\Sigma)$ is:

$$\int_{\mathrm{d}\psi(\partial\Sigma)} \mathrm{d}s\ \kappa(x(s)) = 2\pi - \int_{\mathrm{d}\psi(\Sigma)} \mathrm{vol}_{\mathrm{d}\psi(\Sigma)} R(\hat{g}|_{\mathrm{d}\psi})$$

- The mean curvature of the boundary of a ‘vortex’ is described by gradients of vorticity and strain.

A Jacobi System Formulation?

- Rather than working with the stream function, use velocity directly. Consider L with coordinates $(x^i, v_i(x))$.

- $\omega|_L = 0$ no longer trivial and implies vorticity vanishes. We need a different symplectic form:

$$\varpi = \mathrm{d}q_i \wedge \star(\mathrm{d}x^i)$$

such that $\varpi|_L = 0$ gives $\nabla \cdot v = 0$.

- Our MA form can be written

$$\alpha = \frac{1}{2}\mathrm{d}q_i \wedge \mathrm{d}q_j \wedge \star(\mathrm{d}x^i \wedge \mathrm{d}x^j) - \frac{1}{2}\Delta p \, \mathrm{vol}_m$$

and $\alpha|_L = 0$ yields $\Delta p = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} = 2\det(J(v,x))$.

Towards Multi-symplectic Monge–Ampère Systems

- A k-Plectic Form is a closed and non-degenerate $\varpi \in \Omega^{k+1}(T^*\mathbb{R}^m)$. [Cantrijn et al. 2009]
- A (Higher) Monge–Ampère Structure will be a triple $(T^*\mathbb{R}^m, \varpi, \alpha)$ where ϖ is $(m-1)$-plectic (no effectiveness condition yet).
- Generalised Solutions are now submanifolds $L \hookrightarrow T^*\mathbb{R}^m$ satisfying $\varpi|_L = 0$ and $\alpha|_L = 0$ (not necessarily Lagrangian).
- We focus on L with coordinates $(x^i, v_i(x))$, diffeomorphic to $\mathbb{R}^m$, in lieu of classical solutions.

The Lychagin–Rubtsov Metric in Higher Dimensions

- Can again define a metric $\hat{g}(\cdot\,,\cdot) = -K(J\cdot\,,\cdot)$ on $T^*\mathbb{R}^m$ of the form

$$\hat{g} = \begin{pmatrix} fI_m & 0 \\ 0 & I_m \end{pmatrix} .$$

- For $A_{ij} = \nabla_j v_i$, the pullback metric is

$$(\hat{g}|_L)_{ij} = A^k{}_i A_{kj} - \tfrac{1}{2}\delta_{ij} A_{kl} A^{lk} .$$

- In general, signature change of $\hat{g}|_L$ does not coincide with sign change in f — more complicated relationship.

Topology of 3D Vortices

- No Gauss–Bonnet Theorem in odd dimensions – how to extract topological information?
- Let $\theta = q_i \mathrm{d}x^i$ be the tautological form. Then the helicity density is

$$(\theta \wedge \omega)|_L = v_i \zeta^i \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

- Under ideal conditions, helicity is an invariant quantity and vorticity is conserved.
- Helicity can be related to topological quantities from knot theory i.e. the Gauss linking number, Călugăreanu invariant, and Jones Polynomial [Liu and Ricca 2012, Ricca and Moffatt 1992].

Extension to Riemannian Manifold

- On a Riemannian manifold (M, g), the approach is broadly the same:

$$\Delta p + R_{ij} v^i v^j \; (+\nabla_i c^i) = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \; .$$

- Schematically take

$$\mathrm{d}q_i \to \mathrm{d}q_i - \mathrm{d}x^j \Gamma_{ij}{}^k q_k .$$
$$I \to g .$$
$$f = \tfrac{1}{2}\Delta p \to f = \tfrac{1}{2}(\Delta p + R^{ij} q_i q_j) .$$

- Geometric justification for Weiss criterion for equation type still applies on a manifold, e.g. $\mathbb{S}^2$ [Napper et al. 2023].

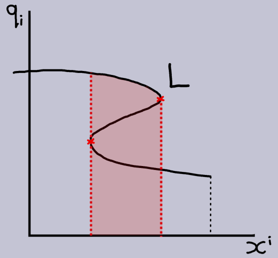
Navier–Stokes equations in spherical geometry describe ocean/atmosphere dynamics (Joshua Stevens - NASA Earth Observatory)

Summary

- We introduced MA geometry as a tool for studying the Poisson equation for the pressure of an incompressible flow.
- We provided a geometric validation for the Weiss–Okubo criterion and showed how the Lychagin–Rubtsov metric could be used to generalise this to flows in higher dimensions/on curved background.
- We highlighted select results concerning solutions, vortices, and their topologies from the wider framework laid out by [N. et al. 2023].

Outlook – Generalised Solutions

- Generalised solutions may have non-immersive projections (Arnold's Singularities) and contain the multivalued solutions. See [Ichikawa et al. 2007, Vinogradov 1973]



- In semi-geostrophic theory, these produce additional degeneracy of $\hat{g}|_L$ and type change, which represent weather fronts. [D'Onofrio et al. 2023]

- The geometry of classical solutions models flows with elliptic vortices, vortex tubes, and lines. Perhaps singular locus of projections could be used to model vortex sheets.

Outlook – Open Questions

- Can one make precise the notion of 'Higher' Monge–Ampère equations? What do we replace effectiveness and Lagrangian with?

- Is it possible to encode dynamics as well as kinematics? Could the vorticity equation

$$\partial_t \zeta + \nabla(\zeta \cdot v) - \nu \Delta \zeta = 0$$

  be used as a (Ricci-like) flow equation for the solutions L?

- .[Lychagin et al. 1993, Banos. 2003] respectively classify 2D and 3D MA equations using integrability of a (para-)complex structure J (and the metric $\hat{g}$). Can we use generalised complex structures to classify 'higher' Monge–Ampère equations?

Thank you!

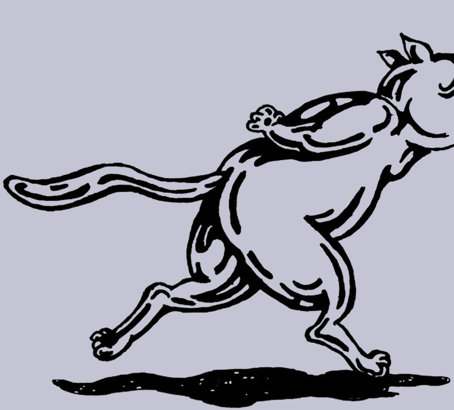


Image Credit [Kushner, Lychagin, Rubtsov. 2007]