Fluide d'Oldroyd B et second principe de la thermodynamique
The Oldroyd B fluid and the second principle of thermodynamics

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SCIENCES SORBONNE UNIVERSITÉ

MATHÉMATIQUES ET INFORMATIQUE Sciences Université Paris Cité

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Overview

`https://arxiv.org/abs/2308.01211` or
`https://hal.sorbonne-universite.fr/hal-04166054`

General notation

Lagrangian description: uppercase. Eulerian description: lowercase. For instance, $(x, t) = (\phi(X, t), t)$, $v(x, t) = V(X, t) = \frac{\partial \phi}{\partial t}(X, t)$ and so on.

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- Dynamics equation $\rho \dot{v} - \operatorname{div}_x \sigma = b$.

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The material derivative is not objective.

A characterization of objective derivatives

Unfortunately, there are infinitely many different objective derivatives. Consider those of the form $\overset{\pentagon}{\sigma} = \dot{\sigma} + \mathrm{Ob}(\sigma, h)$, with $\mathrm{Ob}\colon \mathrm{Sym}_3 \times \mathbb{M}_3 \to \mathrm{Sym}_3$.

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All functions $\mathrm{Ob}_s\colon \mathrm{Sym}_3 \times \mathrm{Sym}_3 \to \mathrm{Sym}_3$ such that $\mathrm{Ob}_s(Q\sigma Q^T, QdQ^T) = Q\mathrm{Ob}_s(\sigma, d)Q^T$ are known (*objective functions*), *cf.* Smith (1971). In particular, all symmetric-valued polynomials in (σ, d).

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Theorem

Such an operator is objective if and only if

$$\mathrm{Ob}(\sigma, h) = \sigma w - w\sigma + \mathrm{Ob}_s(\sigma, d).$$

with $w = \mathrm{Skew}(h)$, $d = \mathrm{Sym}(h)$, *and* $\mathrm{Ob}_s\colon \mathrm{Sym}_3 \times \mathrm{Sym}_3 \to \mathrm{Sym}_3$ *is an objective function.*

(Gurtin-Fried-Anand, 2010).

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Something special about Oldroyd B (incompressible case):

$$\overset{\triangledown}{\sigma} = F\frac{\partial\Sigma}{\partial t}F^T,$$

where Σ is the second Piolà-Kirchhoff stress. $(\sigma = F\Sigma F^T)$.

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Looks like this:

$$\sigma + \lambda_1 \overset{\triangledown}{\sigma} = 2\eta\big(d + \lambda_2 \overset{\triangledown}{d}\big),$$

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$\Leftrightarrow \sigma = \sigma_s + \sigma_p$, $\sigma_s = 2\eta_s d$ and $\sigma_p + \lambda_1 \overset{\triangledown}{\sigma}_p = 2\eta_p d$,
$\eta_s = \frac{\lambda_2}{\lambda_1}\eta$ and $\eta_p = \big(1 - \frac{\lambda_2}{\lambda_1}\big)\eta$.
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Many different ways of deriving the Oldroyd B model. Here a phenomenological Lagrangian approach for testing its compatibility with the second principle of thermodynamics.

Viscoelastic materials with internal variables 1

A pretty general thermo-visco-elastic framework of HLD-AR,[1] based on the Coleman-Noll procedure exploiting the second principle of thermodynamics or Clausius-Duhem inequality.

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$$\frac{\partial B_i}{\partial t} = \widehat{K}(F, H, B_i).$$

What about initial conditions?

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Second principle, C-N version: $\widehat{D}_{\mathrm{int}}(F, H, B_i) \geq 0$ for all possible arguments. A constitutive restriction, equivalent to the second principle in this case.

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HLD-AR 2022 framework: Helmholtz free energy specific density $\widehat{A}_m$ based on neo-Hookean:

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What about the flow rule?

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Decomposition of the 1st PK stress $\leftrightarrow$ decomposition of the Cauchy stress $\sigma = \sigma_s + \sigma_p$ with polymer part $\sigma_p = \frac{\partial \widehat{A}_m}{\partial F}(F, B_i)F^T$, here,

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when written with the 2nd PK polymer stress.

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Some reverse engineering $\rightarrow$ $\left(\frac{\partial B_i}{\partial t} = \widehat{K}(F, H, B_i) \text{ and } \overset{\triangledown}{\sigma} = F \frac{\partial \Sigma}{\partial t} F^T \right)$

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i.e., Oldroyd B!

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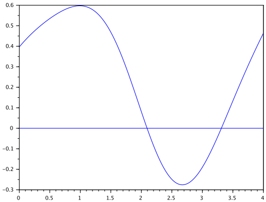

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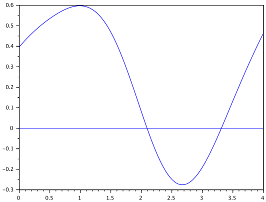

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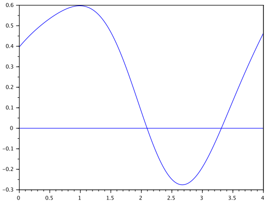

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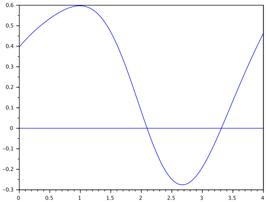

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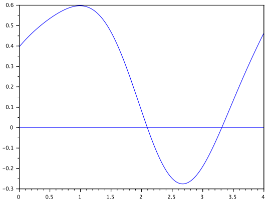

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Anyway $\widehat{d}_{\text{int}}(h, \xi) = 2\eta_s \|d\|^2 + \frac{1}{2\lambda_1}\operatorname{tr}\xi$ does take strictly negative values...

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Role of indeterminate pressure?

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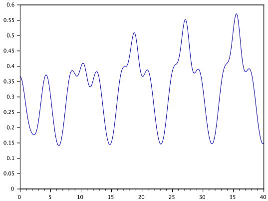
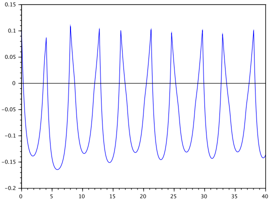


Oldroyd B dissipation with positive definite initial σ_p



Smallest eigenvalue of σ_p

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$\to k = 0$, $\mu_0 = 1$: traditional, linear Oldroyd B,
$\to k = 1$: quadratic Oldroyd B,

$$\sigma^2 - 2\eta_s(\sigma d + d\sigma) + \mu_1 \lambda_1 \overset{\triangledown}{\sigma} = 2(\mu_1 \eta_p d - 2\eta_s^2 d^2 + \mu_1 \lambda_1 \eta_s \overset{\triangledown}{d}).$$

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