From Yang-Mills theory to Maxwell equations

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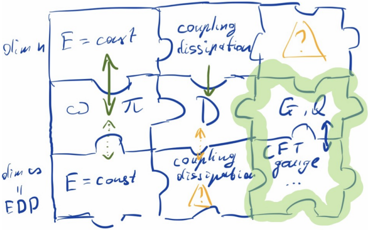
Work in (beginning of) progress with Vladimir Salnikov

GdR–GDM,
Paris, 23 November 2023

Previous episodes

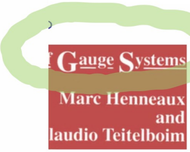
Instead of conclusion – big puzzle and questions

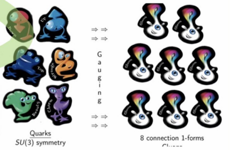


And what precisely about mechanics? What phenomena?

La Rochelle, France, 08/07/2021, 11:40

skip recap





Important "tool": graded geometry, differential graded manifolds.

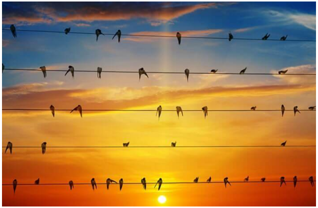
Equivariant cohomology and gauging in a nutshell

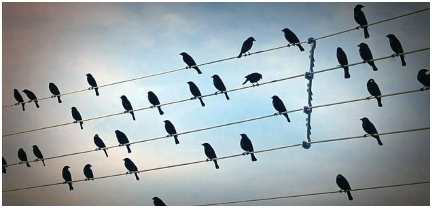
"equations of motion"	↔	Q-morphisms
"symmetries"	↔	Q-homotopies
"gauge invariant"	↔	"equivariantly Q-closed"

Flashback to Vladimir's childhood

Saclay 25/11/2022

Gauged birds

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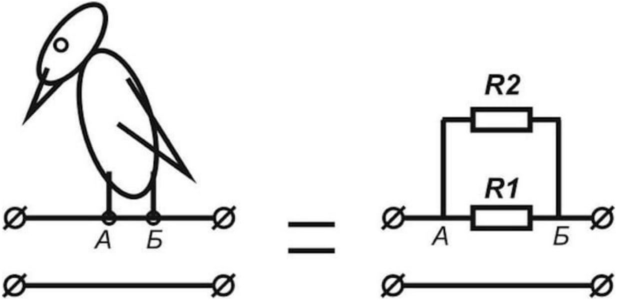


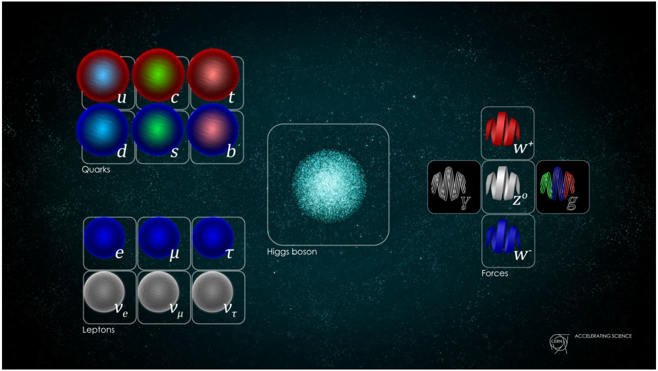
[1]Most pictures credits: internet :-)

No birds were harmed

in the making

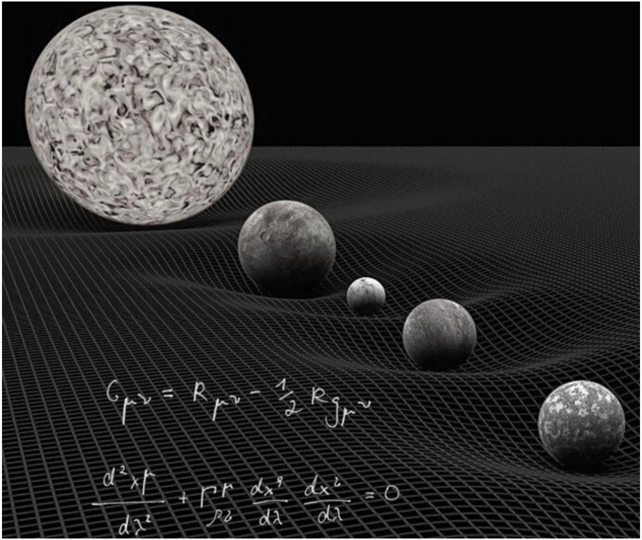
of this presentation

The Standard Model



$$G = SU(3) \times SU(2) \times U(1)$$

General relativity



$$G = Diff(M)$$

Pure Yang-Mills theory.

M – differentiable manifold (space-time)
G – compact connected Lie group
$\mathfrak{g} = Lie(G)$ – its Lie algebra
T_a – generators (basis) of $\mathfrak{g} \leftrightarrow g = exp(\theta_a T_a) \in G$

$$[T_a, T_b] = f_{ab}^c T_c$$

f_{ab}^c – structure constants of $\mathfrak{g}$

Vector field (gauge potential): $A_\mu(x) = A_\mu^a(x) T_a$
Gauge transformation: $A_\mu \to A_\mu^g = g A_\mu g^{-1} + \partial_\mu g g^{-1}$

Field strength tensor:
$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c$
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$
Gauge transformation: $F_{\mu\nu} \to F_{\mu\nu}^g = g F_{\mu\nu} g^{-1}$

Pure Yang-Mills theory. Lagrangian description.

$$L = -\frac{1}{2g^2} Tr F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu}_a$$

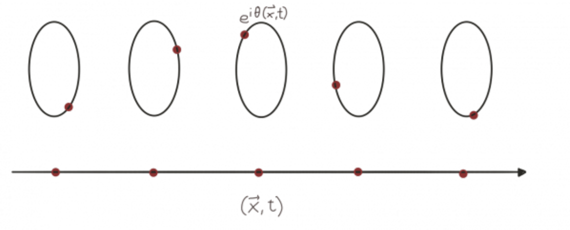
$$S_{YM} = \int_M L = -\frac{1}{4g^2} \int_M d^4x \, F^a_{\mu\nu} F^{\mu\nu}_a$$

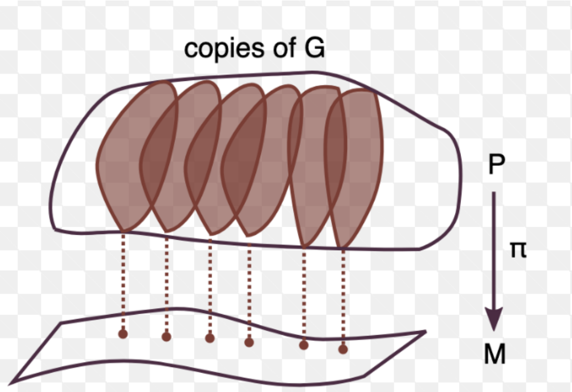
Euler–Lagrange equations:

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0$$

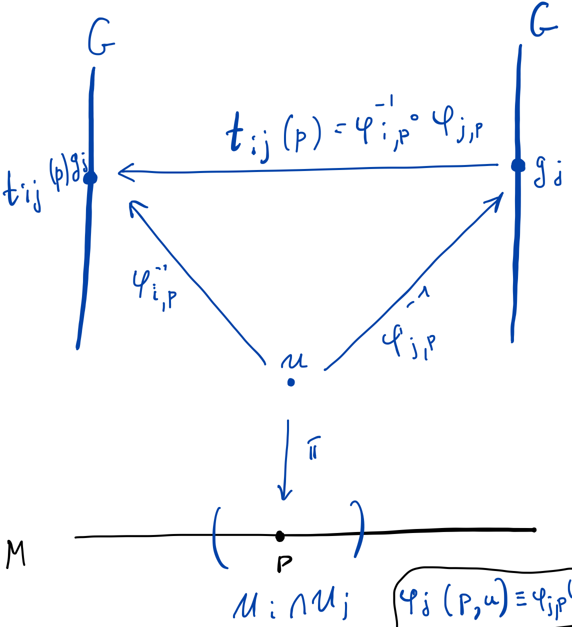
$$D_\mu F^{\mu\nu} = 0$$

Geometry of gauge theories. Principal bundles





Geometry of gauge theories. Principal bundles



$\boxed{\varphi_j(p,u) \equiv \varphi_{j,p}(u)}$

- $G \circlearrowright P$
- $\pi^{-1}(p) = F_p = F \simeq G$
- $\{\mathcal{U}_i\}$ of M
- diffeo $\varphi_i : \mathcal{U}_i \times G \to \pi^{-1}(\mathcal{U}_i)$ s.t. $\pi \circ \varphi_i(p,u) = p$

  $\uparrow \quad \varphi_i^{-1}(u) = (p, g_i)$

  local trivialization
- On $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$:

  $t_{ij}(p) \equiv \varphi^{-1}_{i,p} \circ \varphi_{j,p} : F \to F$

  transition functions

  $t_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \to G$

  $\varphi_j(p,u) = \varphi_i(p, t_{ij}(p) u)$

Connection. Gauge field

Definition 1. A connection on the principal G-bundle is the choice of a horizontal subbundle $HP \hookrightarrow TP$, which is an invariant complement to the vertical subbundle $VP \subset TP$, such that
(i) $TP = VP \oplus HP$
(ii) $H_{gu}P = g_*(H_uP)$,$u \in P$, $g \in G$,
g_* is a map induced by the action $G \times P \to P$.

Definition 2. A connection g-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ is a projection of T_uP to the vertical subspace $V_uP \cong \mathfrak{g}$, satisfying the following conditions:
(i) $\iota_{\xi^P} A = \xi$ for all $\xi \in \mathfrak{g}$,
(ii) A is G-equivariant, i. e. $g^*A = gAg^{-1}$, $g \in G$.
Then the horizontal subspace is given by:
$H_uP = \{X \in T_uP | \iota_X A = 0\}$.

Gauge transformation

Let U_i be an open cover of M and $\sigma_i : U_i \to \pi^{-1}(U_i)$ be a local section for each U_i. Then we can define a g-valued 1-form A_i on each U_i:

$$A_i = \sigma_i^* A \in \Omega^1(M, \mathfrak{g}).$$

For a non-trivial bundle on the intersections $U_i \cap U_j$ the local forms agree in the following way:

$$A_j = t_{ij} A_i t_{ij}^{-1} - dt_{ij} t_{ij}^{-1},$$

where $t_i j : U_i \cap U_j \to G$ are the transition functions.
If we know a local section σ_i and a g-valued 1-form A_i on each U_i we can define a global connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ on the bundle, such that $A_i = \sigma_i^* A$, by the formula

$$A|_{\pi^{-1}(U_i)} = g_i \pi^* A_i g_i^{-1} - dg_i g_i^{-1},$$

where d is the exterior differentiation on P and g_i is the local trivialization defined by $\varphi_i(u) = (p, g_i)$ for $u = g_i \sigma_i(p)$ and $\varphi_i : P \to U_i \times G$.

Covariant derivative and curvature

The covariant derivative is induced by the connection on the principal G-bundle.
Recall: a horizontal n-form with values in the Lie algebra $\varpi \in \Omega^n(P, g)$ is a form satisfying the condition $\iota_{\xi_P}\varpi$ for $\xi \in \mathfrak{g}$ and the fundamental vector field $\xi_P \in VP$.
The connection gives rise to a projection operator $P^h_A : \Omega^n(P, g) \to \Omega^n_{hor}(P, g)$.
The **covariant derivative** is the following composition:

$$d_A = P^h_A \circ d : \Omega^n(P, g) \to \Omega^{n+1}_{hor}(P, g).$$

$$d_A = d + A$$

Definition 3. Let $A \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form on the principal G-bundle $P \to M$. The curvature of the connection is the $\mathfrak{g}$-valued 2-form $F_A \in \Omega^2(P, \mathfrak{g})$ given by the covariant derivative of the connection:

$$F_A = d_A A = dA + \frac{1}{2}[A, A].$$

Curvature. Interpretation.

The curvature F_A measures for how much the covariant differential fails to be a true differential:

$$(d_A)^2\varpi = [F_A, \varpi]$$

for $\varpi \in \Omega^n(P, \mathfrak{g})$.
The curvature is **covariantly constant**.
This important property is called the **Bianchi identity:**

$$d_A F_A = 0.$$

Local curvature on the base space: $F_i = \sigma_1^* F_A \in \Omega^2(U_i, \mathfrak{g})$

Gauge transformation: $F_A \to F_A^g = gF_Ag^{-1}$.

Yang–Mills theory

$$S_{YM}(A) = -\frac{1}{2g^2}\int_M Tr\, F_A \wedge *F_A$$

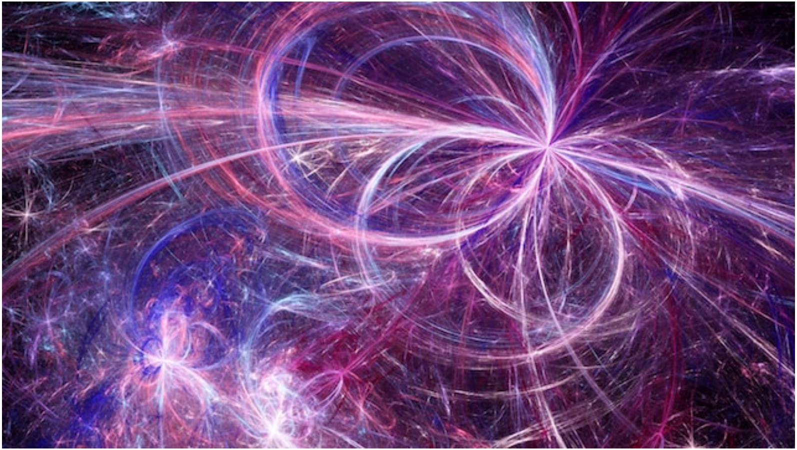
Euler–Lagrange equations:

$$d_A * F_A = d * F_A + [A, *F_A] = 0$$

Bianchi identity:

$$d_A F_A + [A, F_A] = 0.$$

Electromagnetism



$$G = U(1)$$

Electromagnetism

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$

$$A_\mu(x) = A^a_\mu(x)\, T_a$$

$$F^{0i} = -E^i$$

$$A_\mu \to A_\mu - \partial_\mu \theta$$

$$F^{ij} = -\epsilon^{ijk} B^k$$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(E^2 - B^2\right)$$

$$\partial_\mu F^{\mu\nu} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \partial_0 E$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\partial_0 B$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Beginning of work in progress

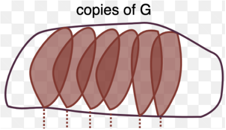
- First discussion on interactions.
  e.g. minimal coupling to a scalar field

$$S_{YM}(A) = -\frac{1}{2g^2}\int_M Tr\Big(F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^2 - m^2\varphi^2\Big)$$

→ Bosonic and fermionic variables

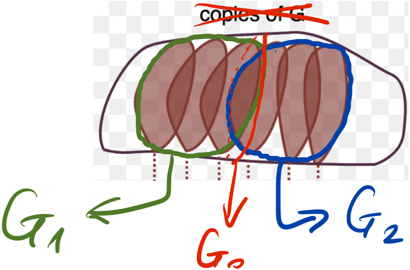
Actual work in progress

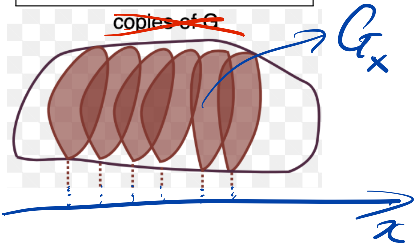
- Homogeneous Maxwell $\leftarrow$ Pure Yang–Mills with $f_{ab}^{c} = 0$
- Real-life Maxwell: i.e. inhomogeneous, eventually with charges, in anisotropic continuous media with varying electromagnetic parameters $\leftarrow$ several options:
  - non-abelian gauge group ($f_{ab}^{c} \neq 0$)



  - Yang-Mills with interaction terms

Lie algebroid Yang–Mills





Merci pour l'attention!