New PDEs governing both fluids and solids

Symmetric-hyperbolic balance laws to model viscoelastic flows of Maxwell fluids

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Viscoelastic flows, Maxwell fluids & hyperbolic PDEs

To compute unequivocal solutions to Cauchy problems
we propose a *symmetric-hyperbolic system of balance laws*

that contains $\lambda \overset{\diamond}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\mu \boldsymbol{D}(\boldsymbol{u})$, and that models

denoting $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T\right) \equiv \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right)$, $\overset{\diamond}{\boldsymbol{\tau}} \equiv \overset{\triangledown}{\boldsymbol{\tau}} := \partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\tau} - (\boldsymbol{\nabla u})\boldsymbol{\tau} - \boldsymbol{\tau}(\boldsymbol{\nabla u})^T$

- *Hookean solids* when $\lambda, \mu \equiv G\lambda \to \infty$: $\overset{\diamond}{\boldsymbol{\tau}} = 2G\boldsymbol{D}(\boldsymbol{u})$

where $\boldsymbol{\tau} = G\left(\boldsymbol{F}\boldsymbol{F}^T - \boldsymbol{I}\right)$, $(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{F} = \boldsymbol{L}\,\boldsymbol{F} \Rightarrow \overset{\diamond}{\boldsymbol{\tau}} \equiv \overset{\triangledown}{\boldsymbol{\tau}} := \partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\tau} - \boldsymbol{L}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{L}^T$

- *Newtonian fluids* when $\lambda, \lambda/\mu \equiv 1/G \to 0$: $\boldsymbol{\tau} = 2\mu \boldsymbol{D}(\boldsymbol{u})$

Outline

Continuum Mechanics

We look for $\mathbb{R}^d = \{\boldsymbol{x} = \phi_t^i(\boldsymbol{a})\boldsymbol{e}_i\,,\ \boldsymbol{a} = a^\alpha \boldsymbol{e}_\alpha\} \quad \forall t \in [0, T)$, i.e.

velocity $\boldsymbol{u} \equiv \partial_t \phi_t$ and *deformation gradient* $\boldsymbol{F} \equiv \partial_\alpha \phi_t^i \boldsymbol{e}_i \otimes \boldsymbol{e}^\alpha$

$$\partial_t F_\alpha^i - \partial_\alpha u^i = 0 \tag{1}$$

$$\partial_t |\boldsymbol{F}| - \partial_\alpha \left(\hat{F}_\alpha^i\, u^i \right) = 0 \tag{2}$$

$$\partial_t \hat{F}_\alpha^i + \sigma_{ijk} \sigma_{\alpha\beta\gamma} \partial_\beta \left(F_\gamma^j\, u^k \right) = 0 \tag{3}$$

where $|\boldsymbol{F}|$ and $\hat{\boldsymbol{F}}$ denote *determinant* and *cofactor matrix* of $\boldsymbol{F}$

while Piola's identities hold ($\sigma_{\alpha\beta\gamma}$ is Levi-Civita's symbol)

$$\sigma_{\alpha\beta\gamma} \partial_\beta F_\gamma^i = 0 = \partial_\alpha \hat{F}_\alpha^i \quad \forall i \tag{4}$$

$(\partial_b F_a^i = \partial_a F_b^i)$

Newtonian physics

We require balance of energy using *material coordinates* i.e.

$$\hat{\rho}\partial_t\left(\frac{|\boldsymbol{u}|^2}{2}+e\right)=\partial_\alpha\left(S_i^\alpha u^i\right)+\hat{\rho}f_i u^i \tag{5}$$

where *stored energy* $e(\boldsymbol{F})$ defines *first Piola-Kirchoff stress* $\boldsymbol{S}$

$$S_i^\alpha=\hat{\rho}\partial_{F^i_\alpha}e\,. \tag{6}$$

We require momentum balance in fact, i.e. in Cartesian system

$$\hat{\rho}\partial_t u_i=\partial_\alpha S_i^\alpha+\hat{\rho}f_i \tag{7}$$

by Galilean invariance. *Here, we also assume* $\hat{\rho}$ *constant.*

Lagrangian description

Then, computable motions $\phi_t(\boldsymbol{a})$ are defined on specifying
i) *constitutive relations* $e(\boldsymbol{F})$, strictly convex in $\boldsymbol{F}$ e.g. like

$$e(\boldsymbol{F}) = \frac{c_1^2}{2}(F_\alpha^k F_\alpha^k - d) \tag{8}$$

($c_1^2 \equiv G > 0$ is Lamé's second coefficient or shear modulus)
hence $\boldsymbol{S}(\boldsymbol{F}) = c_1^2 \boldsymbol{F}^T$ in the (symmetric-hyperbolic) system

$$\partial_t \boldsymbol{F}^T = \boldsymbol{\nabla}_{\boldsymbol{a}} \boldsymbol{u} \tag{9}$$

$$\hat{\rho}\partial_t \boldsymbol{u} = \mathrm{div}_{\boldsymbol{a}} \boldsymbol{S} + \hat{\rho}\boldsymbol{f} \tag{10}$$

plus ii) *initial conditions* for (9–10), e.g. in $[H^s(\mathbb{R}^d)]^{3d}$
(whatever $s \in \mathbb{R}$, $\forall T > 0$ here: (9–10) is linear !)

Neo-Hookean materials

A more realistic constitutive relation (for rubber, resine. . .) is

$$e(\boldsymbol{F}) = \frac{c_1^2}{2}(\boldsymbol{F} : \boldsymbol{F} - d) - \frac{d_1^2}{1-\gamma}|\boldsymbol{F}|^{1-\gamma} \quad (11)$$

(where d_1^2 is Lamé's first coefficient). Properties of (11):

- $e(\boldsymbol{F})$ is *polyconvex* in $\boldsymbol{F}$ as soon as $\gamma > 1$

  $$e(\boldsymbol{F}) \equiv \tilde{e}(|\boldsymbol{F}|, \boldsymbol{F}) \text{ convex in } |\boldsymbol{F}|, \boldsymbol{F}$$

  well defining solutions with $S_i^\alpha(\boldsymbol{F}) = \hat{\rho} c_1^2 F_\alpha^i - \hat{\rho} d_1^2 |\boldsymbol{F}|^{-\gamma} \hat{F}_\alpha^i$ to

  $$\partial_t \boldsymbol{F}^T = \nabla_{\boldsymbol{a}} \boldsymbol{u} \quad (12)$$

  $$\partial_t |\boldsymbol{F}| = \text{div}_{\boldsymbol{a}}(\boldsymbol{u} \cdot \hat{\boldsymbol{F}}) \quad (13)$$

  $$\partial_t \boldsymbol{u} = \text{div}_{\boldsymbol{a}}(\boldsymbol{S}/\hat{\rho}) + \boldsymbol{f} \quad (14)$$

- $e(\boldsymbol{F})$ is *material-frame indifferent*

  $$e(\boldsymbol{F}) \equiv \bar{e}(\boldsymbol{C}, |\boldsymbol{C}|) \text{ where } \boldsymbol{C} = C_{\alpha\beta} \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta, \; C_{\alpha\beta} = F_\alpha^i F_\beta^i,$$

  $\bar{e}$ is monotone convex in each argument for polyconvexity

Eulerian description

Smooth solutions to (11–14) preserving $\boldsymbol{\nabla}_{\boldsymbol{a}} \times \boldsymbol{F}^T = 0$
are equivalently (smooth) solutions preserving $\operatorname{div}(\rho \boldsymbol{F}^T) = 0$ to

$$\partial_t (\rho \boldsymbol{u}) + \operatorname{div} (\rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}) = \rho \boldsymbol{f} \tag{15}$$

$$\partial_t (\rho \boldsymbol{F}) - \boldsymbol{\nabla} \times \left(\rho \boldsymbol{F}^T \times \boldsymbol{u} \right) = \boldsymbol{0} \tag{16}$$

$$\partial_t \rho + \operatorname{div} (\rho \boldsymbol{u}) = 0 \tag{17}$$

using *mass density* $\rho = \hat{\rho}/|\boldsymbol{F}|$ and Cauchy stress

$$\sigma^{ij} := |\boldsymbol{F}|^{-1} S^{i\alpha} F^j_\alpha \equiv \rho c_1^2 F^i_\alpha F^j_\alpha - \rho d_1^2 \left(\frac{\rho}{\hat{\rho}} \right)^\gamma \delta^{ij} \,.$$

It allows one to define *isentropic*, time-reversible motions of "solids", isotropic (motions depend only on c_1^2, d_1^2, not direction)

Fluid motions

Within liquids, stress are mostly spheric i.e. $\boldsymbol{\sigma} = -p\boldsymbol{I}$

like in the famous barotropic case $e(\boldsymbol{F}) = \frac{C_0}{\gamma-1}\rho^{\gamma-1}$, $p = C_0\rho^\gamma$

$$\partial_t (\rho \boldsymbol{u}) + \operatorname{div} (\rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}) = \rho \boldsymbol{f} \tag{18}$$

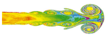
$$\partial_t \rho + \operatorname{div} (\rho \boldsymbol{u}) = 0 \tag{19}$$

well posed – though not in Lagrangian description.

Anyway, real liquids are also viscous, and flow non-reversibly.

Newtonian fluids $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}$, $\boldsymbol{\tau} = 2\mu \boldsymbol{D}(\boldsymbol{u})$ produce entropy,

but lack shear elasticity as in e.g. gels, letting alone that

shear then propagates at infinite speed and fails at 

Fluids (micro-)structure

Rheology of solids & liquids depends on (micro-)structure

Use Maxwell constitutive relation with structural variable $\boldsymbol{\tau}$?

Objective suspension flow models $\lambda \overset{\diamond}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\mu \boldsymbol{D}(\boldsymbol{u})$ where e.g. $\overset{\diamond}{\boldsymbol{\tau}} \equiv \overset{\triangledown}{\boldsymbol{\tau}} := \partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\tau} - (\nabla \boldsymbol{u})\boldsymbol{\tau} - \boldsymbol{\tau}(\nabla \boldsymbol{u})^T$ are not well-posed.

The *linearized* system is hyperbolic if $\boldsymbol{c} := \boldsymbol{I} + \frac{\lambda}{\mu}\boldsymbol{\tau} > 0$,
but the *nonlinear* system has no conservative formulation.

Let's use a *structural tensor* $\boldsymbol{A} = \boldsymbol{A}^T > 0$ (like in plastic solids !)
modelling anisotropy in stored energy through $\mathrm{tr}(\boldsymbol{AC})$.

Outline

Anisotropic elastodynamics

Defects inducing anisotropy in solids can be modelled on modifying elastodynamics system preserving $\operatorname{div}(\rho \boldsymbol{F}) = 0$

$$\begin{aligned} &\partial_t (\rho \boldsymbol{u}) + \operatorname{div} (\rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}) = \rho \boldsymbol{f} \\ &\partial_t (\rho \boldsymbol{F}) - \boldsymbol{\nabla} \times \left(\rho \boldsymbol{F}^T \times \boldsymbol{u} \right) = \boldsymbol{0} \\ &\partial_t \rho + \operatorname{div} (\rho \boldsymbol{u}) = 0 \end{aligned} \tag{20}$$

where $\boldsymbol{\sigma} := \rho \left(\partial_{\boldsymbol{F}} e \right) \cdot \boldsymbol{F}^T$ is given by
$e(\boldsymbol{F}) = \bar{e}(\boldsymbol{C}, |\boldsymbol{C}|)$ material-frame indifferent, polyconvex in $\boldsymbol{F}$.

Introducing a structure parameter $\boldsymbol{A} = \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_p^{-T} > 0$ in e.g.

$$e(\boldsymbol{F}) = \frac{c_1^2}{2} (\operatorname{tr}(\boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T) - d) - \frac{d_1^2}{1-\gamma} |\boldsymbol{F}|^{1-\gamma} \tag{21}$$

(still polyconvex in $\boldsymbol{F}$!) yields $\boldsymbol{\sigma} = \rho c_1^2 \boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T - p\boldsymbol{I}$

with *strain* $\boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T$ like in [Green, Naghdi 1965] [Lee, Liu 1967]

Maxwell fluids with hyperbolic PDEs

Assuming (20) and a modified neo-Hookean stored energy with structure parameter $\boldsymbol{A}(t,\boldsymbol{x})$ as in (21),
Maxwell fluids $\lambda \overset{\diamond}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\mu \boldsymbol{D}(\boldsymbol{u})$ result from requiring $\mu = \lambda c_1^2$

$$\boldsymbol{\tau} = \rho c_1^2 (\boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T - \boldsymbol{I})$$

$$\lambda(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} = \boldsymbol{F}^{-1}\boldsymbol{F}^{-T} \tag{22}$$

$$\overset{\diamond}{\boldsymbol{\tau}} = \partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\tau} - \nabla \boldsymbol{u} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \nabla \boldsymbol{u}^T + (\operatorname{div} \boldsymbol{u})\, \boldsymbol{\tau} \,. \tag{23}$$

Theorem (Lieb, 1973)

$(\boldsymbol{F}, \boldsymbol{Y}) \in \mathbb{R}^{d\times d} \times SDP^{d\times d} \to \operatorname{tr}\left(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^T\right)$ *is convex*

By Godunov-Mock theorem, the system of conservation laws (20–22) is symmetric hyperbolic when $\operatorname{div}(\rho \boldsymbol{F}) = 0$ [Boyaval M2AN 2021]

Thermodynamics consistency

The solutions preserving $\mathrm{div}(\rho \boldsymbol{F}) = 0$ to

$$\begin{aligned}
&\partial_t (\rho \boldsymbol{u}) + \mathrm{div}\,(\rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}) = \rho \boldsymbol{f} \\
&\partial_t (\rho \boldsymbol{F}) - \nabla \times \left(\rho \boldsymbol{F}^T \times \boldsymbol{u}\right) = \boldsymbol{0} \\
&\partial_t \rho + \mathrm{div}\,(\rho \boldsymbol{u}) = 0 \\
&\partial_t \boldsymbol{A} + \boldsymbol{u} \cdot \nabla \boldsymbol{A} = \tfrac{1}{\lambda}(\boldsymbol{F}^{-1}\boldsymbol{F}^{-T} - \boldsymbol{A})
\end{aligned} \tag{24}$$

$\boldsymbol{\sigma} = \rho c_1^2 \boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T - p\boldsymbol{I} = \boldsymbol{\tau} - \tilde{p}\boldsymbol{I}$ satisfy the energy balance

$$\partial_t E + \mathrm{div}\,(E\boldsymbol{u} - \boldsymbol{\sigma} \cdot \boldsymbol{u}) = \rho \boldsymbol{f} \cdot \boldsymbol{u} + \tfrac{\rho c_1^2}{2\lambda}(\boldsymbol{I} - \boldsymbol{c}^{-1}) : (\boldsymbol{c} - \boldsymbol{I}) \tag{25}$$

using $\boldsymbol{c} = \boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^T \in \boldsymbol{S}^{+,*}$, $E = \rho \left(\tfrac{1}{2}|\boldsymbol{u}|^2 + \tilde{e}\right)$, $\tilde{p} = p + c_1^2 \rho$

$$\tilde{e}(\boldsymbol{F}) = \tfrac{c_1^2}{2}(\mathrm{tr}(\boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T) - d - \log|\boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T|) - \tfrac{d_1^2}{1-\gamma}|\boldsymbol{F}|^{1-\gamma}$$

Linking solids with fluids

Standard comparison tools for systems of balance laws *rigorously* link the fluid model (24) with (neo-Hookean) elastic *solid* bodies when $\frac{1}{\lambda} \to 0$ i.e. when no energy is dissipated [Boyaval 2023]

Whenever $0 < \lambda < \infty$, flows dissipate and one is considering non-ideal *fluids* with extra-stress

$$\lambda \left(\partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{\tau} - \boldsymbol{\nabla} \boldsymbol{u} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\nabla} \boldsymbol{u}^T + (\operatorname{div} \boldsymbol{u})\, \boldsymbol{\tau} \right) = 2\mu \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{\tau}$$

When $\lambda \to 0$, fluids memory is fading infinitely fast and fluids become formally *Newtonian*, with non-zero viscosity $\mu = \lambda c_1^2$ if $c_1^2 \to \infty$ at the same time

Entropy and temperature

Thermal influences on mechanics were neglected so far

$$-\frac{\rho c_1^2}{2\lambda}(\boldsymbol{I} - \boldsymbol{c}^{-1}) : (\boldsymbol{c} - \boldsymbol{I}) \equiv \rho\theta(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\eta =: \rho(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}) e_s(\eta)$$

$\Rightarrow$ let e depend on η and preserve entropy production ?

If we assume $K(\theta)$ *affine* in a θ-convex Helmholtz free energy

$$e^\star(\boldsymbol{F}, \theta) = \frac{K(\theta)}{2} \bar{e}_A(\boldsymbol{F}) + \psi_0(|\boldsymbol{F}|, \theta) \tag{26}$$

where $\bar{e}_A(\boldsymbol{F}) = \mathrm{tr}(\boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T) - d - \log|\boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T|$, then

$$e(\boldsymbol{F}, \eta) = \frac{K - \theta\partial_\theta K}{2} \bar{e}_A(\boldsymbol{F}) + e_0\left(|\boldsymbol{F}|, \eta + \partial_\theta K \; \bar{e}_A(\boldsymbol{F})\right)$$

after Legendre transform of (26) is jointly convex in $(|\boldsymbol{F}|, \boldsymbol{F}, \eta)$,
while $\boldsymbol{\sigma} = \rho K(\theta) \boldsymbol{F} \cdot \boldsymbol{A} \cdot \boldsymbol{F}^T - \tilde{p}(\rho, \theta)\boldsymbol{I}$ like [Dressler-Edwards-Öttinger 1999]

Adding heat transfer by conduction

Heat conduction at finite-speed can be added using

$$e(\boldsymbol{F},\eta,\boldsymbol{p}) = \frac{K-\theta\partial_\theta K}{2}\bar{e}_A(\boldsymbol{F}) + e_0\left(|\boldsymbol{F}|,\eta+\partial_\theta K\ \bar{e}_A(\boldsymbol{F})\right) + \frac{\tau}{2}|\boldsymbol{p}|^2$$

$$\tau\rho(\partial_t + u^i\partial_i)\boldsymbol{p} + \mathrm{div}(\zeta(\theta)\rho\boldsymbol{F}) = \rho\theta|\zeta'(\theta)|^2\hat{\kappa}^{-1}\boldsymbol{p}$$

as in pioneering works of Cattaneo,

with an additional heat flux in energy balance (25)

$$\partial_t\tilde{E} + \mathrm{div}\left(\tilde{E}\boldsymbol{u} - \boldsymbol{\sigma}\cdot\boldsymbol{u} + \theta\zeta'(\theta)\boldsymbol{p}\right) = \rho\boldsymbol{f}\cdot\boldsymbol{u}$$

where $\tilde{E} = \rho\left(\frac{1}{2}|\boldsymbol{u}|^2 + e(\boldsymbol{F},\eta,\boldsymbol{p})\right)$

Compatibility with Fourier's law

First, balance of energy $\hat{\rho}\partial_t e + \partial_\alpha Q^\alpha = \hat{\rho} r$ for $e(\eta, \boldsymbol{p})$
is compatible with second law $\hat{\rho}\theta\partial_t\eta + \theta\partial_\alpha q^\alpha - \hat{\rho} r = \hat{\rho}\mathcal{D} \geq 0$
when $Q^\alpha = \theta q^\alpha$, $\hat{\rho}(\partial_{p^\alpha} e)\partial_t p^\alpha + q^\alpha\partial_\alpha\theta = -\hat{\rho}\mathcal{D} < 0$ and

$$\rho C_1(\partial_t + u^i\partial_i)\hat{\rho}\theta + \partial_i\left(\theta\rho F^i_\alpha q^\alpha\right) = \rho\left(\hat{\rho}\mathcal{D} + F^i_\alpha q^\alpha \partial_i\theta\right) \qquad (27)$$

where $\theta := \partial_\eta e$, $C_1(\theta) := \theta(\partial^2_{\eta\eta} e)^{-1}$, $r = 0$ implies Fourier's law

$$\theta\rho F^i_\alpha q^\alpha \to -\kappa_{ij}\partial_j\theta \quad \text{and} \quad F^i_\alpha q^\alpha\partial_i\theta + \mathcal{D} \equiv q^\alpha\partial_\alpha\theta + \mathcal{D} \to 0\,,$$

i.e. $\theta\rho\boldsymbol{q} \to -\hat{\kappa}^{-1}\nabla_{\boldsymbol{a}}\theta\,, \hat{\rho}\mathcal{D} \to \theta\rho\boldsymbol{q}^T\hat{\kappa}^{-1}\boldsymbol{q} > 0 \quad (\hat{\kappa} := \boldsymbol{F}^{-1}\kappa\boldsymbol{F}^T)$
e.g. if $\tau\hat{\rho}\partial_t p^\alpha + \partial_\alpha\zeta(\theta) = -\rho\theta|\zeta'(\theta)|^2[\hat{\kappa}^{-1}]_{\alpha\beta}p^\beta\,, \boldsymbol{p} = \boldsymbol{Q}/(\theta\zeta'(\theta))$

Extensions possible

One can change the stored energy
and introduce finite-extensibility:

$$\psi = \psi_0 + K(\theta) b^2 \log\left(1 - \frac{F^i_\alpha F^i_\beta A^{\alpha\beta}}{b^2}\right) - k_B \theta \log|F^i_\alpha F^i_\beta A^{\alpha\beta}| + \frac{\mathcal{T}}{2}|\boldsymbol{p}|^2$$

or add a term function of $\hat{\boldsymbol{F}}$ for 3D flows…
or let λ vary
(as a function of θ, $\boldsymbol{A}$, $\boldsymbol{F}$... or yet another structure parameter)

Outline

Saint-Venant for shallow flows:viscous?

Saint-Venant [1871]: free-surface gravity flows of depth $H(t,x,y) > 0$ are governed by hydrostatic pressure $P = gH/2$

$$\partial_t H + \text{div}(H\boldsymbol{U}) = 0 \tag{28}$$
$$\partial_t(H\boldsymbol{U}) + \text{div}(H\boldsymbol{U}\otimes\boldsymbol{U} + H(P+\Sigma_{zz})\boldsymbol{I} - H\boldsymbol{\Sigma}_h) = -kH\boldsymbol{U} \tag{29}$$

and $\boldsymbol{\Sigma} = 0$, or $\boldsymbol{\Sigma}_h = 2\nu D(\boldsymbol{U})$, $\Sigma_{zz} = -(\Sigma_{xx} + \Sigma_{yy})$

2D shallow elastodynamics

$\lambda \to \infty$: elastodynamics for thin layers $H \equiv F_c^z = |\boldsymbol{F}_h|^{-1} > 0$ of hyperelastic materials with deformation $\boldsymbol{F} = \partial_{a,b,c}(x, y, z)$

$$\partial_t \boldsymbol{F} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{F} = (\boldsymbol{\nabla} \boldsymbol{U})\boldsymbol{F}$$

and with a Hookean stress function of $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$

$$\boldsymbol{\Sigma}_h = \partial_{\boldsymbol{F}_h} \left(\frac{G}{2} \boldsymbol{F}_h : \boldsymbol{F}_h \right) \boldsymbol{F}_h^T , \quad \Sigma_{zz} = \partial_{F_c^z} \left(\frac{G}{2} |F_c^z|^2 \right) F_c^z$$

i.e. $\boldsymbol{\Sigma}_h - \Sigma_{zz} \boldsymbol{I} \equiv G(\boldsymbol{B}_h - B_{zz} \boldsymbol{I}) = (\partial_{\boldsymbol{F}_h} e) \boldsymbol{F}_h^T$;

in fact is as *symmetric hyperbolic* system of conservation laws with *polyconvex* energy $e := \frac{g}{2} |\boldsymbol{F}_h|^{-1} + \frac{G}{2} (\boldsymbol{F}_h : \boldsymbol{F}_h + |\boldsymbol{F}_h|^{-2})$

2D shallow elastodynamics SCL

When $\lambda \to \infty$, SV-UCM should be

$$\partial_t(HF^i_\alpha) + \partial_j(HU^j F^i_\alpha - HF^j_\alpha U^i) = 0$$
$$\partial_t(HU^i) + \partial_j(HU^j U^i + gH^2/2 + GH^3 - GHF^i_\alpha F^j_\alpha) = -KHU^i$$

as long as $\partial_\alpha(\sigma_{\alpha\beta} F^k_\beta) = 0$, $\partial_j(HF^j_\alpha) = 0$ (Piola) so e.g.

$$\partial_t H + \partial_j(HU^j) = 0.$$

It is possible accomodate *viscosity* using “memory” variables.

2D viscoelastic Saint-Venant

Adding $A_{\alpha\beta}$ to the usual dependent variables yields

$$\partial_t H + \partial_j(HU^j) = 0$$
$$\partial_t(HF^i_\alpha) + \partial_j(HU^j F^i_\alpha - HF^j_\alpha U^i) = 0$$
$$\partial_t(HU^i) + \partial_j(HU^j U^i + gH^2/2 + GH^3 A_{cc} - GHF^i_\alpha A_{\alpha\beta} F^j_\beta F^j_\alpha) = -KHU^i$$
$$\partial_t(HA_{\alpha\beta}) + \partial_j(HU^j A_{\alpha\beta}) = H(|\boldsymbol{F}_h|^{-2}\sigma_{\alpha\alpha'}\sigma_{\beta\beta'}F^k_{\alpha'}F^k_{\beta'} - A_{\alpha\beta})/\lambda$$
$$\partial_t(HA_{cc}) + \partial_j(HU^j A_{cc}) = H(H^{-2} - A_{cc})/\lambda$$

i.e. a system of *conservation laws*, with companion law

$$\partial_t(HE) + \partial_x\left(HEU + H(P + \Sigma_{zz} - \Sigma_{xx})U - H\Sigma_{xy}V\right)$$
$$+\partial_y\left(HEV - H\Sigma_{yx}U + H(P + \Sigma_{zz} - \Sigma_{yy})V\right) \leq -KH|\boldsymbol{U}|^2 - HD$$

Computing solutions

$\boldsymbol{\tau} := \boldsymbol{\sigma} + p\,\delta$ satisfies a *compressible UCM* eq.

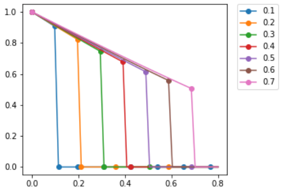
$$\lambda \overset{\triangledown}{\boldsymbol{\tau}} + \boldsymbol{\tau}\ (\operatorname{div} \boldsymbol{u}) + \boldsymbol{\tau} = 2\mu \boldsymbol{D}(\boldsymbol{u})$$

using $\overset{\triangledown}{\boldsymbol{\tau}} := \partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\tau} - \boldsymbol{\nabla} \boldsymbol{u}\, \boldsymbol{\tau} - \boldsymbol{\tau}\, \boldsymbol{\nabla} \boldsymbol{u}^T$

Assuming 1D flow, one retrieves the damped-wave equation

$$\lambda \partial^2_{tt} \tau(t, y) + \partial_t \tau(t, y) = \mu \partial^2_{yy} \tau(t, y)$$

with shear-wave solution to Stokes first-problem in $\{y > 0\}$



But beyond ?

Perspectives

- Vorticity generated locally in an initially-quiescent fluid
- Fluid-Solid contact “seamlessly” modelled (discontinuity)
- Rheology: local re-structuration under shear

References

S. Boyaval.
Viscoelastic flows of Maxwell fluids with conservation laws.
M2AN, 55:807–831, 2021.

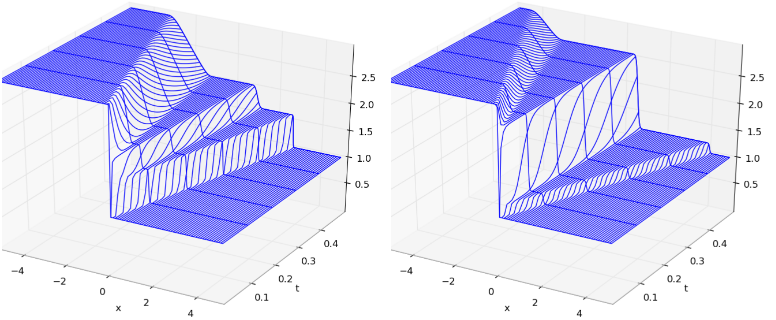
R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager.
Dynamics of Polymeric Liquids, volume 1&2.
John Wiley & Sons, 1987.

C. M. Dafermos.
Hyperbolic conservation laws in continuum physics.
Springer, 2000.

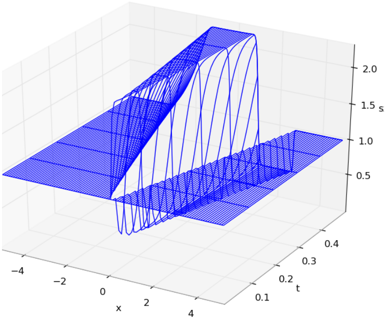
E. H. Lieb.
Convex trace functions and the Wigner-Yanase-Dyson conjecture.
Advances in Math., 11:267–288, 1973.

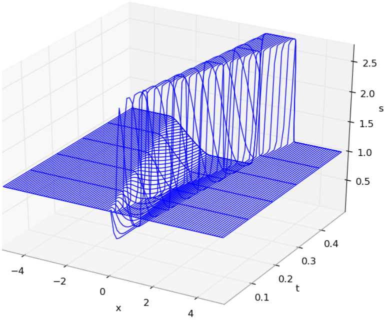
M. Renardy.
A local existence and uniqueness theorem for a K-BKZ-fluid.
ARMA , 88(1):83–94, 1985.

1D case (Stoker): h ; $\mu = 1$ and 10, $\lambda = \infty$

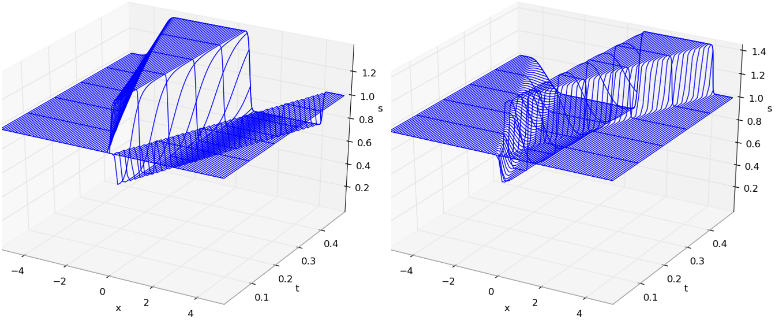


1D case (Stoker): τ_{xx}, τ_{zz} ; $\mu = 1, \lambda = \infty$





1D case (Stoker): τ_{xx}, τ_{zz} when $\mu = 10$, $\lambda = \infty$

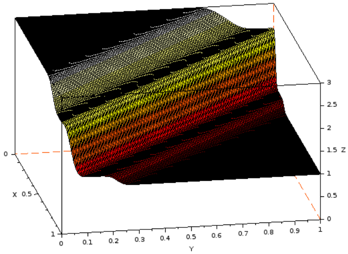


Stoker "dam-break" benchmark test case

Compute a solution for $t \in (0, .2)$ in $(x, y) \in [0, 1]^2$ starting from

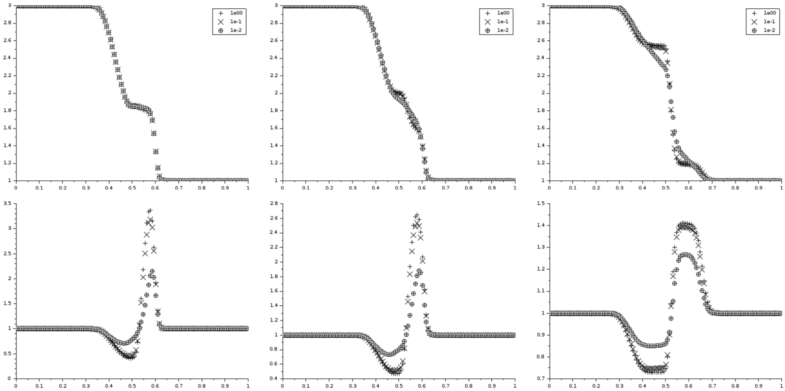
$$(H, U, V, B_{xx}, B_{yy}, B_{xy}, B_{zz}) = \begin{cases} (3, 0, 0, 1, 1, 0, 1) & x + y < 1 \\ (1, 0, 0, 1, 1, 0, 1) & x + y > 1 \end{cases}$$

SV-UCM Depth H
$T = .2$
Froude $g^{-1/2} = .3$
Elasticity $G = 10 \approx g$
Weissenberg $\lambda = 1 \gg T$



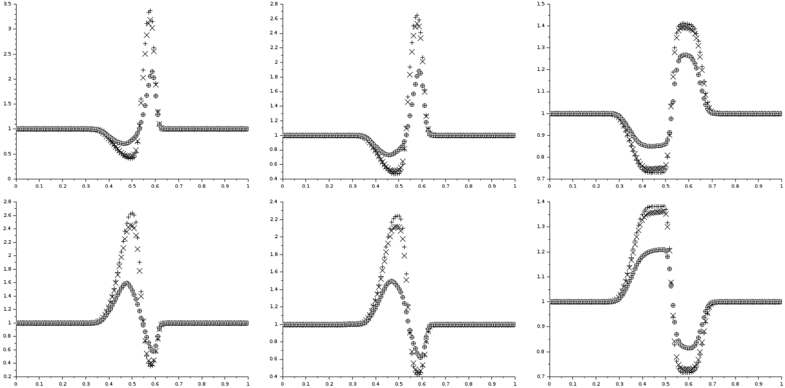
Varying elasticity $G = .1, 1, 10$ at $g = 10$

Depth H (top) and strain B_{xx} (bottom) at $T = .2$ for $\lambda = .01, .1, 1$



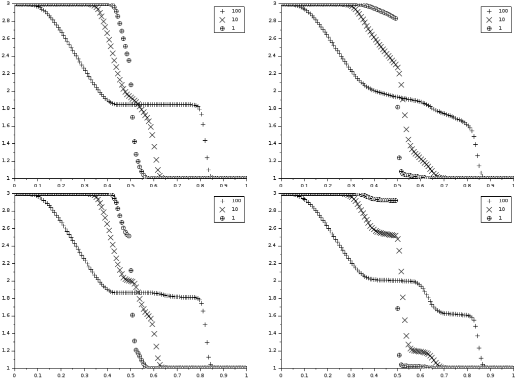
Varying elasticity $G = .1, 1, 10$ at $g = 10$

Strain B_{xx} (top) and B_{zz} (bottom) at $T = .2$ for $\lambda = .01, .1, 1$



Varying Froude $g = 1, 10, 100$

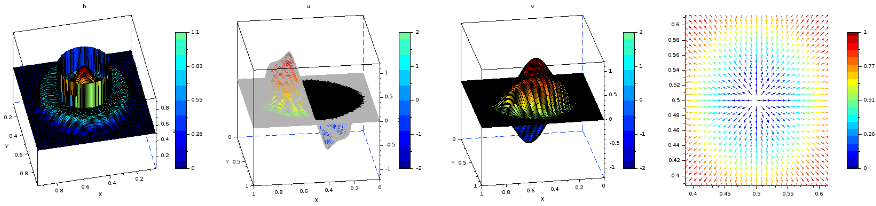
Depth H at $T = .2$ for $\lambda = .01$ (top) and .1 (bottom)
with elasticity $G = 1$ (left) and 10 (right)



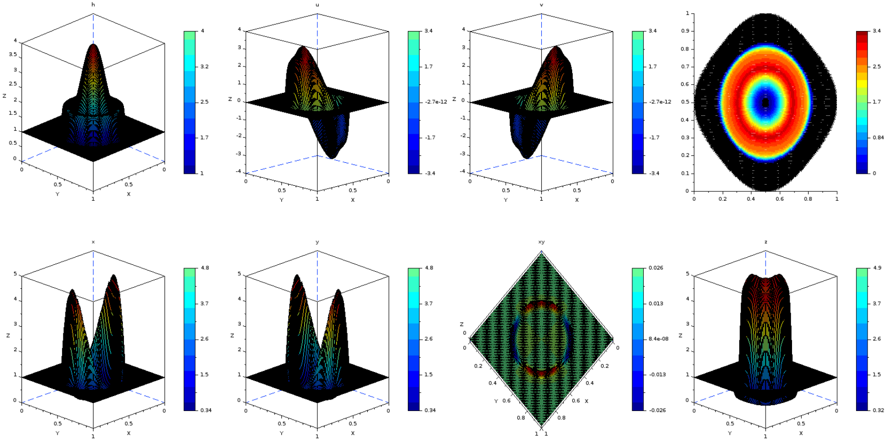
2D column "circular dam-break" benchmark

Solution at $T = .2$ in $(x, y) \in [0, 1]^2$ starting from

$$(H, U, V, B^{xx}, B^{yy}, B^{xy}, B^{zz}) = \begin{cases} (3, 0, 0, 1, 1, 0, 1) & (x - .5)^2 + (y - .5)^2 < .2 \\ (1, 0, 0, 1, 1, 0, 1) & (x - .5)^2 + (y - .5)^2 > .2 \end{cases}$$



$g = 10$, $G = 0.01$, $\lambda = 1$ at $T = .2$



$g = 10$, $G = 1$, $\lambda = 1$ at $T = .2$