Explicit semi-algebraic description of the orbit space of Weyl group actions

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Trigonometric polynomials

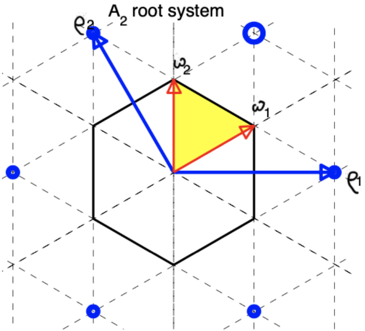
Lattice $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$

Trigonometric polynomial: $f : \mathbb{R}^n \to \mathbb{R}$

where $f(u) = \sum_{\omega\in\Omega} a_\omega \, e^{2\pi i \langle \omega, u\rangle}$

$a_{-\omega} = \overline{a_\omega} \in \mathbb{C}$

f in $\Omega^\perp$-periodic



Compute (numerically)

$$\min_{u\in\mathbb{R}^n} f(u)$$

under the asumption that f is invariant w.r.t a reflection group $\mathfrak{G}$ of rank n

Univariate case

$\mathfrak{G} = \{+1, -1\}$ acts on $\mathbb{R}$ and preserves $\Omega = \mathbb{Z}$

$$\begin{array}{ccc} \mathfrak{G} \times \mathbb{R} & \rightarrow & \mathbb{R} \\ (\epsilon, u) & \mapsto & \epsilon\, u \end{array}$$

Invariant trigonometric polynomials : $a_{-k} = a_k \in \mathbb{R}$

$$f(u) = \sum_{k \in \mathbb{N}} a_k \left(e^{2\pi \mathrm{i} k u} + e^{-2\pi \mathrm{i} k u} \right) = \sum_{k \in \mathbb{N}} 2 a_k \cos(2\pi k u)$$

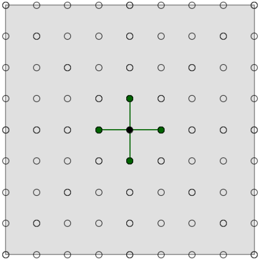
Chebyshev polynomials $\{T_k\}_{k \in \mathbb{N}}$

$$\cos(k\,\theta) = T_k\left(\cos(\theta)\right) \quad \text{where} \quad \cos(\theta) = \frac{1}{2}\left(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}\right)$$

$z = \cos(2\pi u) \in [-1, 1]$

$$\min_{u \in \mathbb{R}} f(u) = \min_{1 - z^2 \geq 0} \sum_{k \in \mathbb{N}} 2 a_k\, T_k(z)$$

2D lattices & symmetry

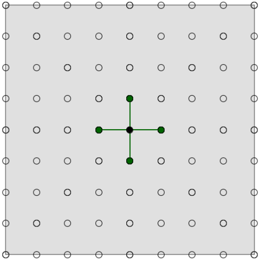


$\mathfrak{G} = \{+1, -1\}^2$

Invariance : $f(-u, v) = f(u, v) = f(u, -v)$

Trigonometric $\rightsquigarrow$ polynomial optimization

$$\min_{u,v\in\mathbb{R}} f(u, v) = \min_{z_1,z_2\in[-1,1]^2} a_{k,l}\, T_k(z_1)\, T_l(z_2)$$



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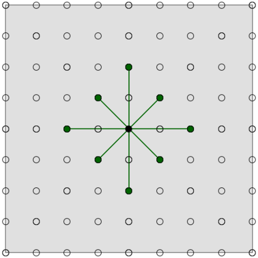
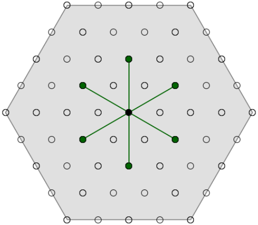
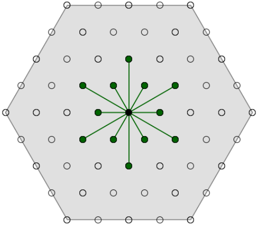


C_2

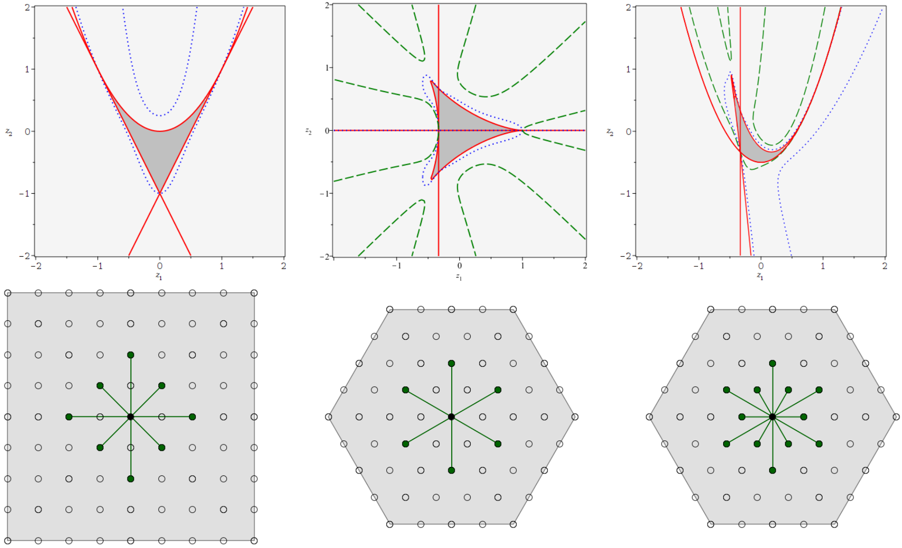


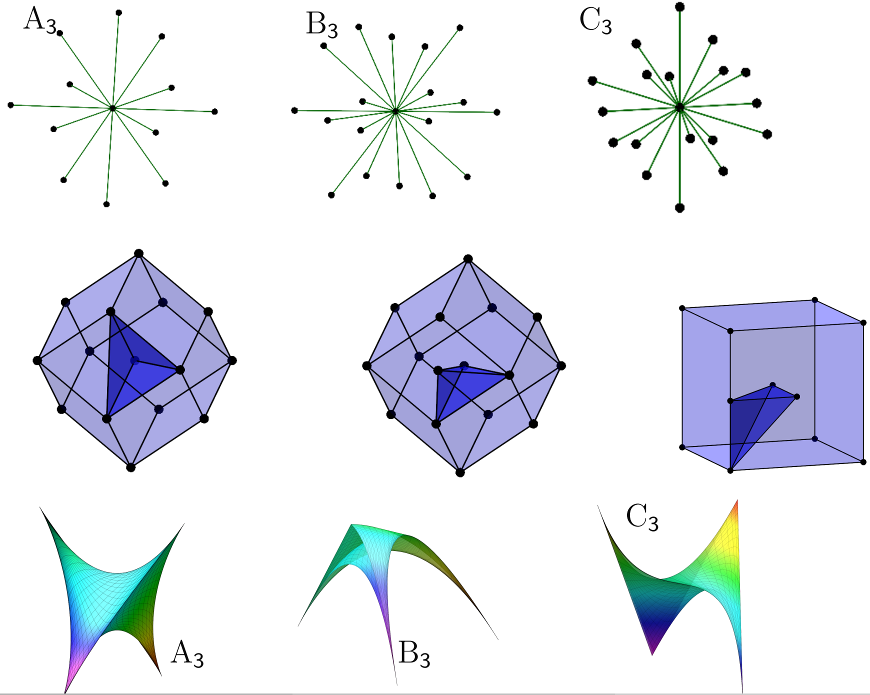
A_2



G_2

2D lattices & symmetry

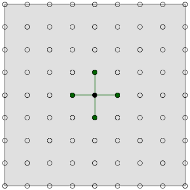



A_3
B_3
C_3
C_3
A_3
B_3

R root system

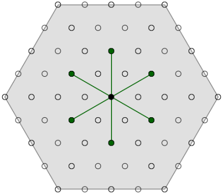
- R is finite and spans $\mathbb{R}^n$
- If $\rho, \tilde{\rho} \in \mathrm{R}$, then $\sigma_\rho(\tilde{\rho}) \in \mathrm{R}$, where $\sigma_\rho(u) := u - \langle u, \rho^\vee \rangle \rho$

♣ If $\rho, \tilde{\rho} \in \mathrm{R}$, then $\langle \tilde{\rho}, \rho^\vee \rangle \in \mathbb{Z}$, where $\rho^\vee := 2\,\frac{\rho}{\langle \rho\,,\rho \rangle}$



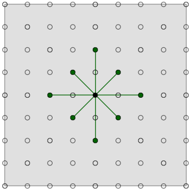
$\mathrm{A}_1 \times \mathrm{A}_1$

$\mathfrak{G} = \{+1, -1\}^2$



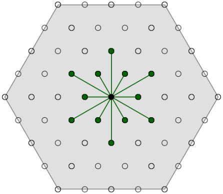
A_2

$\mathfrak{S}_3$



C_2

$\mathfrak{S}_2 \ltimes \{+1, -1\}^2$



G_2

$\mathfrak{S}_3 \ltimes \{+1, -1\}$

- The **Weyl group** $\mathfrak{G}$ is the group generated by the σ_ρ
- The **coroot lattice** Λ is the lattice generated by the $\rho^\vee$
- The **weight lattice** Ω is the dual lattice of Λ:

$$\omega \in \Omega \quad \Leftrightarrow \quad \langle \omega, \rho^\vee \rangle \in \mathbb{Z} \quad \forall \rho \in \mathrm{R}$$

Multivariate Chebysev polynomials

$\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$ the weight lattice of a root system R $\mathfrak{G}$-invariant

For $\omega \in \Omega$

$$\begin{array}{rccc} \mathfrak{e}_\omega : & \mathbb{R}^n & \to & \mathbb{C} \\ & u & \mapsto & e^{2\pi \mathrm{i}\langle \omega, u\rangle} \end{array}$$

form an orthogonal basis of Λ-periodic functions

Generalized cosines

$$\mathfrak{c}_\omega = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \mathfrak{e}_{g\omega} \quad \text{for } \omega \in \mathbb{N}\,\omega_1 \oplus \ldots \oplus \mathbb{N}\,\omega_n$$

form a linear basis of the $\mathfrak{G}$-invariant functions

$\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}$ are [Bourbaki]

- algebraically independent
- generate the ring of invariant trigonometric polynomials

Generalized Chebyshev polynomial T_α, $\alpha \in \mathbb{N}^n$

is the unique element of $\mathbb{Q}[z_1, \ldots, z_n]$ satisfying

$$\mathfrak{c}_{\alpha_1\omega_1 + \ldots + \alpha_n\omega_n} = T_\alpha\left(\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}\right)$$

Optimization of trigonometric polynomials with symmetry

$\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$ is $\mathfrak{G}$-invariant

$$f(u) = \sum_{\omega \in \Omega^+} a_\omega \, \mathfrak{c}_\omega(u)$$

$$\min_{u \in \mathbb{R}^n} f(u) \;=\; \min_{z \in \mathcal{T}} \sum_{\alpha \in \mathbb{N}^n} \tilde{a}_\alpha \, T_\alpha(z)$$

$$\mathcal{T} = \mathfrak{c}\,(\mathbb{R}^n) \quad \text{where} \quad \mathfrak{c} = (\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}) : \mathbb{R}^n \to \mathbb{R}^n$$

For root systems $\mathcal{A}_{n-1}$, $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{D}_n$, $\mathcal{G}_2$

$(z_1, \ldots, z_n) \in \mathcal{T} \quad \Leftrightarrow \qquad e_1 = (1, 0, \ldots, 0)$

$$\begin{pmatrix} T_0 - T_{2e_1} & T_{e_1} - T_{3e_1} & T_0 - T_{4e_1} & \cdots \\ T_{e_1} - T_{3e_1} & T_0 - T_{4e_1} & 2T_{e_1} - T_{3e_1} - T_{5e_1} & \cdots \\ T_0 - T_{4e_1} & 2T_{e_1} - T_{3e_1} - T_{5e_1} & 2T_0 + T_{2e_1} - T_{6e_1} - 2T_{4e_1} + T_{2e_1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \succeq 0$$

i.e.,

$$\left(-T_{(i+j)e_1} + \sum_{k=1}^{\lceil (i+j)/2 \rceil - 1} \left(4\binom{i+j-2}{k-1} - \binom{i+j}{k} \right) T_{(i+j-2k)\,e_1} + \begin{cases} 2\binom{i+j-2}{(i+j)/2-1} - \frac{1}{2}\binom{i+j}{(i+j)/2}, & i+j \text{ even} \\ 0, & i+j \text{ odd} \end{cases} \right)_{1 \le i,j \le n} \succeq 0$$

Comments

- We obtain an explicit and unified formula

R	$\mathcal{A}_{n-1}$	$\mathcal{B}_n$	$\mathcal{C}_n$	$\mathcal{D}_n$	$\mathcal{G}_2$
$\mathfrak{G}$	$\mathfrak{S}_n$	$\mathfrak{S}_n \ltimes \{\pm 1\}^n$	$\mathfrak{S}_n \ltimes \{\pm 1\}^n$	$\mathfrak{S}_n \ltimes \{\pm 1\}_+^n$	$\mathfrak{S}_3 \ltimes \{\pm 1\}$

  We use the interconnection between $\mathfrak{S}_n$ and the roots of polynomials.
  [Procesi Schwarz 85] would lead to different matrices.

- Missing $\mathcal{F}_4$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$

It makes sense to work the Lasserre hierarchy in the basis $\{T_\alpha\}_{\alpha\in\mathbb{N}^n}$

$$T_\alpha\, T_\beta \;=\; \sum_{A\in\mathfrak{G}} T_{\alpha+A\beta} \;=\; T_{\alpha+\beta} \;+ \sum_{\langle\gamma,\rho_0\rangle<\langle\alpha+\beta,\rho_0\rangle} c_\gamma\, T_\gamma$$

along the weighted degree given by the highest root

Explicit semi-algebraic description of the orbit space of Weyl group actions

1. Trigonometric and Chebysev polynomials
2. $\mathcal{T}$ as the orbit space of a multiplicative action
3. Case $\mathcal{C}_n$, $\mathcal{B}_n$, $\mathcal{D}_n$,
4. Case $\mathcal{A}_{n-1}$
5. Spectral bounds on the chromatic number of some infinite graphs

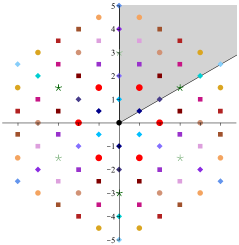
Trigonometric polynomials $\rightsquigarrow$ Laurent polynomials

$$\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$$

$$\mathfrak{G} \to \mathrm{O}_n(\mathbb{R})$$

$$e^{2\pi \mathrm{i} \langle\, \alpha_1\omega_1 + \ldots + \alpha_n\omega_n\,,\, u\,\rangle}$$

$$u \in \mathbb{R}^n, \text{ or } \mathbb{C}^n$$

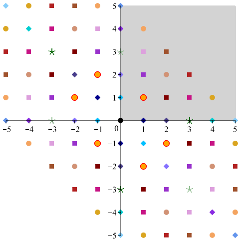


$$\mathbb{Z}^n$$

$$\mathfrak{G} \to \mathrm{GL}_n(\mathbb{Z})$$

$$x^\alpha \in \mathbb{Q}[x, x^{-1}] = \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

$$x \in \mathbb{T}^n, \text{ or } (\mathbb{C}^*)^n$$



A representation $\mathfrak{G} \to \mathrm{GL}_n(\mathbb{Z})$ of a finite group $\mathfrak{G}$

Multiplicative action on $(\mathbb{C}^*)^n$

$$\begin{array}{ccc} \mathfrak{G} \times (\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^n \\ (A,\, x) & \mapsto & x^A = \left(x^{A_{\cdot 1}}, \ldots, x^{A_{\cdot n}}\right) \end{array}$$

$\mathbb{T}^n = \{(x_1, \ldots, x_n) \in (\mathbb{C}^*)^n \mid |x_1| = \cdots = |x_n| = 1\}$ is left invariant

Induced (linear) action on $\mathbb{Q}[x, x^{-1}] = \mathbb{Q}[x_1,\, \ldots,\, x_n, x_1^{-1},\, \ldots,\, x_n^{-1}]$

$$\begin{array}{ccc} \mathfrak{G} \times \mathbb{Q}[x, x^{-1}] & \to & \mathbb{Q}[x, x^{-1}] \\ \left(A,\, x^{\beta}\right) & \mapsto & x^{A\,\beta} \end{array} \qquad \text{i.e. } (A \cdot f)(x) = f\left(x^A\right)$$

Orbit polynomials : $\Theta_\alpha = \sum_{A \in \mathfrak{G}} x^{A\alpha}$

[Bourbaki]

When $\mathfrak{G}$ is a Weil group $\quad \mathbb{Q}[x, x^{-1}]^{\mathfrak{G}} = \mathbb{Q}[\Theta_{e_1}, \ldots, \Theta_{e_m}]$

$T_\alpha \in \mathbb{Q}[z_1, \ldots z_n]$ defined by $\Theta_\alpha = T_\alpha(\theta_1, \ldots, \theta_n)$ with $\theta_i = \Theta_{e_i}$

$$\begin{array}{rccc} \vartheta: & \mathbb{T}^n & \rightarrow & Z \cong \mathbb{R}^n \\ & (x_1, \ldots, x_n) & \mapsto & (\theta_1(x), \ldots, \theta_m(x)) \end{array}$$

Goal: describe $\mathcal{T} = \vartheta\left(\mathbb{T}^n\right)$

- $\mathcal{T}$ is a compact semi-algebraic set
- $\mathcal{T}$ can be understood as the orbit space $\mathbb{T}^n/\mathfrak{G}$
- $\mathcal{T}$ is the region of orthogonality of $\{T_\alpha\}_{\alpha\in\mathbb{N}}$

$\mathcal{C}_n$

SYSTÈMES DE TYPE C_l ($l \geqslant 2$)

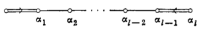
$V = E = \mathbf{R}^l$.
Racines : $\pm 2\varepsilon_i$ $(1 \leqslant i \leqslant l)$, $\pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqslant i < j \leqslant l)$.
Nombre de racines : $n = 2l^2$.

Base : $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\ldots$, $\alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l$, $\alpha_l = 2\varepsilon_l$.

Racines positives
$$\begin{cases} \varepsilon_i - \varepsilon_j = \sum_{i \leqslant k < j} \alpha_k & (1 \leqslant i < j \leqslant l), \\ \varepsilon_i + \varepsilon_j = \sum_{i \leqslant k < j} \alpha_k + 2 \sum_{j \leqslant k < l} \alpha_k + \alpha_l & (1 \leqslant i < j \leqslant l), \\ 2\varepsilon_i = 2 \sum_{i \leqslant k < l} \alpha_k + \alpha_l & (1 \leqslant i \leqslant l). \end{cases}$$

Nombre de Coxeter : $h = 2l$.

Plus grande racine : $\tilde{\alpha} = 2\varepsilon_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$.
Graphe de Dynkin complété :



$R^\vee$ est l'ensemble des vecteurs $\pm \varepsilon_i$, $\pm \varepsilon_i \pm \varepsilon_j$.

$$\Phi_R(x, y) = \frac{(x|y)}{4(l+1)} \qquad \gamma(R) = (l+1)(4l-2).$$

Poids fondamentaux :

$$\varpi_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i \quad (1 \leqslant i \leqslant l)$$
$$= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1} + \tfrac{1}{2}\alpha_l).$$

For $1 \leq i \leq n$

$$\theta_i = 2^{-i} \binom{n}{i}^{-1} \sigma_i(y_1 + y_1^{-1}, \ldots, y_n + y_n^{-1})$$

where

$\sigma_1, \ldots, \sigma_n$ elementary symmetric polynomials

$y_1 = x_1$ and $y_k = x_k x_{k-1}^{-1}$, $1 \leq k \leq n$

Note

- $(x_1, \ldots, x_n) \in \mathbb{T}^n \quad \Leftrightarrow \quad (y_1, \ldots, y_n) \in \mathbb{T}^n$
- $y_i \in \mathbb{T} \quad \Leftrightarrow \quad y_i + y_i^{-1} \in [-2, 2]$

$x \in \mathbb{T}^n \quad \Leftrightarrow \quad \xi^n - \theta_1(x)\xi^{n-1} + \ldots + (-1)^k \theta_k(x)\xi^{n-k} + \ldots + (-1)^n \theta_n(x)$

has all its root in $[-2, 2] \quad \Leftrightarrow \quad$ for any of its root ξ, $4 - \xi^2 \geq 0$

Orbit space of $\mathcal{C}_n$ [HMR]

$\mathcal{T} = \{z \in \mathbb{R}^n \mid H(z) \succ 0\}$ where

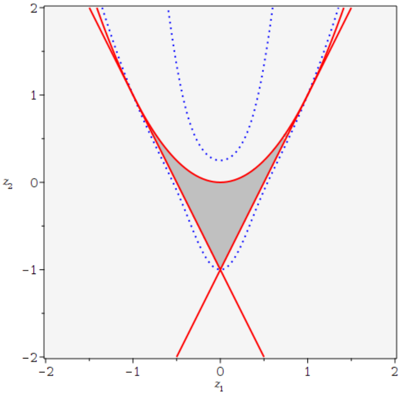
$$H(z) = \left[\mathrm{Trace}\left(C(z)^{i+j-2}(4 - C(z)^2)\right)\right]_{1\le i,j\le n}$$

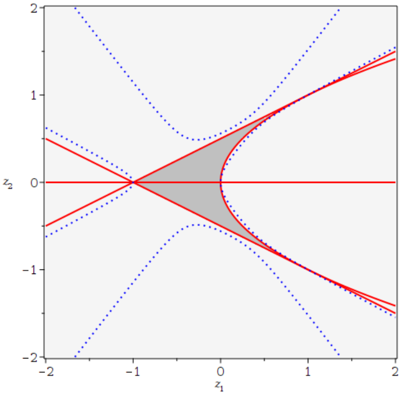
$$C(z) = \begin{bmatrix} 0 & \dots & 0 & z_n \\ 1 & & 0 & z_{n-1} \\ & \ddots & & \vdots \\ 0 & & 1 & z_1 \end{bmatrix}$$

$\mathcal{C}_2$ $\qquad\qquad$ $\mathcal{B}_2$

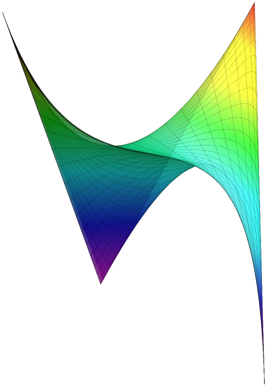
$$\begin{bmatrix} -2z_1^2 + z_2 + 1 & -8z_1^3 + 6z_1z_2 + 2z_1 \\ -8z_1^3 + 6z_1z_2 + 2z_1 & -32z_1^4 + 8z_1^2 + 32z_1^2z_2 - 4z_2^2 - 4z_2 \end{bmatrix} \succeq 0$$

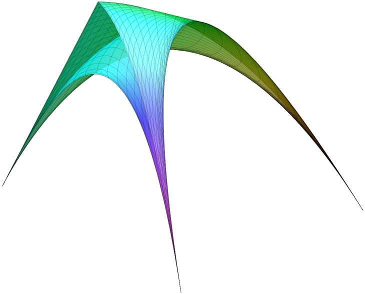
$$\begin{bmatrix} -z_1^2 + 2z_2^2 - z_1 & -4z_1^3 + 12z_1z_2^2 - 6z_1^2 - 2z_1 \\ -4z_1^3 + 12z_1z_2^2 - 6z_1^2 - 2z_1 & -16z_1^4 - \dots - 4z_1 \end{bmatrix} \succeq 0$$





$\mathcal{C}_3$ $\mathcal{B}_3$





Similar approach

$\mathcal{B}_n$

$$y_1 = x_1, \qquad y_n = x_n^2 x_{n-1}^{-1}$$
$$y_k = x_k x_k^{-1}, \quad 2 \leq k \leq n-1$$

$$\sigma_i(y + y^{-1}) = \theta_i, \ 1 \leq i \leq n-1$$
$$\sigma_n(y + y^{-1}) = \Theta_{2e_n}$$
$$= {\theta_n}^2 + \sum_{j=1}^{n-1} \binom{n}{j} \theta_{j-1}$$

$\mathcal{D}_n$

$$y_1 = x_1, \ y_k = x_k x_{k-1}^{-1}, \ 2 \leq k \leq n-1$$
$$y_{n-1} = x_n x_{n-1} x_{n-2}^{-1}, \ y_n = x_n x_{n-1}^{-1}$$

$$\sigma_i(y + y^{-1}) = \theta_i, \quad 1 \leq i \leq n-2$$
$$\sigma_{n-1}(y + y^{-1}) = \Theta_{e_{n-1}+e_n} = \dots$$
$$\sigma_n(y + y^{-1}) = \Theta_{2e_{n-1}} + \Theta_{2e_n} = \dots$$

$\mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n$

$$z \in \mathcal{T} \quad \Leftrightarrow \quad H(z) \succeq 0$$

$$H(z) = \left[\mathrm{Trace}\left(C(z)^{i+j-2}(4 - C(z)^2)\right)\right]_{1 \le i,j \le n}$$

where $C(z)$ is a companion matrix whose entries belong to $\mathbb{Q}[z_1, \ldots, z_n]$

$$C(\theta) = \begin{bmatrix} 0 & 0 & \ldots & 0 & \sigma_n(y + y^{-1}) \\ 1 & 0 & \ldots & 0 & \sigma_{n-1}(y + y^{-1}) \\ 0 & 1 & \ldots & 0 & \sigma_{n-2}(y + y^{-1}) \\ & & \ddots & & \vdots \\ 0 & & & 1 & \sigma_1(y + y^{-1}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 & ** \\ 1 & 0 & \ldots & 0 & ** \\ 0 & 1 & \ldots & 0 & \theta_{n-2} \\ & & \ddots & & \vdots \\ 0 & & & 1 & \theta_1 \end{bmatrix}$$

$\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{D}_n$

The eigenvalues of $C(z)$ are $y_1 + y_1^{-1}, \ldots, y_n + y_n^{-1}$ so that

$$\mathrm{Trace}\left(C(z)^k\right) = (y_1 + y_1^{-1})^k + \ldots + (y_n + y_n^{-1})^k$$

while $\quad \Theta_{ke_1} = y_1^k + y_1^{-k} + \ldots + y_n^k + y_n^{-k} = T_{ke_1}(\theta_1, \ldots, \theta_n)$

$(z_1, \ldots, z_n) \in \mathcal{T} \quad \Leftrightarrow$

$$\begin{pmatrix} T_0 - T_{2e_1} & T_{e_1} - T_{3e_1} & T_0 - T_{4e_1} & \iddots \\ T_{e_1} - T_{3e_1} & T_0 - T_{4e_1} & 2T_{e_1} - T_{3e_1} - T_{5e_1} & \iddots \\ T_0 - T_{4e_1} & 2T_{e_1} - T_{3e_1} - T_{5e_1} & 2T_0 + T_{2e_1} - T_{6e_1} - 2T_{4e_1} + T_{2e_1} & \iddots \\ \iddots & \iddots & \iddots & \iddots \end{pmatrix} \succeq 0$$

i.e.,

$$\left(-T_{(i+j)e_1} + \sum_{k=1}^{\lceil (i+j)/2 \rceil - 1} \left(4\binom{i+j-2}{k-1} - \binom{i+j}{k} \right) T_{(i+j-2k)\,e_1} + \begin{cases} 2\binom{i+j-2}{(i+j)/2-1} - \frac{1}{2}\binom{i+j}{(i+j)/2}, & i+j \text{ is even} \\ 0, & i+j \text{ is odd} \end{cases} \right) \succeq 0$$

Explicit semi-algebraic description of the orbit space of Weyl group actions

For $1 \leq i \leq n-1$, $\theta_i = \binom{n}{i}\sigma_i(y_1, \ldots, y_n)$ and $\sigma_n(y_1, \ldots, y_n) = 1$

where $y_1 = x_1,\ y_k = x_k x_{k-1}^{-1}$, $2 \leq k \leq n-1$, and $y_n = x_n^{-1}$

As

$$\theta_{n-1-i}(x) = \theta_i(x^{-I})$$

take

$$\check{p} = \xi^n - \theta_1\,\xi^{n-1} + \ldots + (-1)^{n-1}\theta_{n-1}\,\xi + (-1)^n$$

$$\hat{p} = \xi^n - \theta_{n-1}\,\xi^{n-1} + \ldots + (-1)^{n-1}\theta_1\,\xi + (-1)^n$$

and $$\check{p}\,\hat{p} = \xi^n \sum_{k=0}^{n} d_{n-k}\left(\xi^k + \xi^{-k}\right) = 2\xi^n \sum_{k=0}^{n} d_{n-k}\,T_k\left(\frac{\xi+\xi^{-1}}{2}\right)$$

where $d_k = (-1)^k \sum_{\ell=1}^{k} \binom{n}{\ell}\binom{n}{k-\ell}\,\theta_\ell\,\theta_{n-k+\ell}$

Orbit space of $\mathcal{A}_n$ [HMR]

$$x \in \mathbb{T}^{n-1} \quad \Leftrightarrow \quad \check{p}\,\hat{p} \text{ has its roots in } \mathbb{T}_1^n = \{z \in \mathbb{T}^n \mid z_1 \ldots z_n = 1\}$$

$$\Leftrightarrow \quad T_n(\zeta) - d_1\, T_{n-1}(\zeta) - \ldots - d_n \text{ has its roots in } [-1, 1]$$

$$\mathcal{T} = \{z \in Z \mid H(z) \succ 0\} \text{ where } H(z) = \left[\mathrm{Tr}\left(C(z)^{i+j-2}\left(1 - C(z)^2\right) \right)\right]_{1\leq i,j\leq n}$$

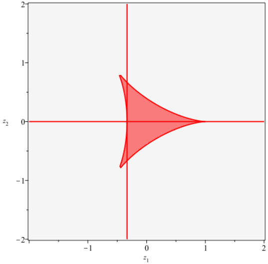
$$C(z) = \begin{bmatrix} 0 & \frac{1}{2} & & & 0 & \frac{d_n}{2} \\ 1 & 0 & \ddots & & & d_{n-1} \\ & \frac{1}{2} & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \frac{1}{2} & d_3 \\ & & & \ddots & 0 & d_2 \\ 0 & & & & \frac{1}{2} & d_1 \end{bmatrix}$$

$$Z = \{z \in \mathbb{C}^n \mid z_{n-i-1} = \bar{z}_i\}$$

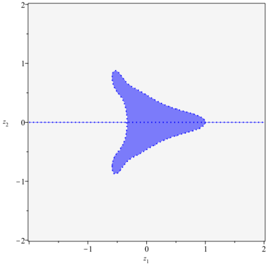
since $\theta_{n-1-i}(x) = \overline{\theta_i(x)}$ for $x \in \mathbb{T}^{n-1}$

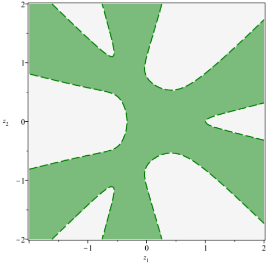
$$d_\ell = (-1)^{\ell+1} \sum_{i=0}^{\ell} \binom{n}{i} \binom{n}{\ell-i} z_i\, z_{n-\ell+i}$$

$\mathcal{A}_2$

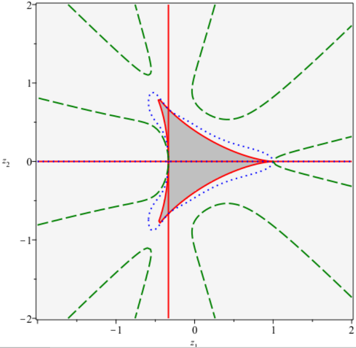


Det $H \geq 0$





Trace $H \geq 0$



Explicit semi-algebraic description of the orbit space of Weyl group actions

Spectral bound on the chromatic number of infinite graphs

Consider the infinite graph (V, E) where $V = \mathbb{R}^n$ or $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$
$S \subset V$ centrally symmetric and $(v_1, v_2) \in E$ if $v_2 - v_1 \in S$

[Bachoc, DeCorte, de Oliveira Filho, Vallentin 14]

The chromatic number χ of the graph is bounded by

$$2^n \geq \chi \geq 1 - \frac{\sup_{u \in \mathbb{R}^n} \hat{\nu}(u)}{\inf_{u \in \mathbb{R}^n} \hat{\nu}(u)}$$

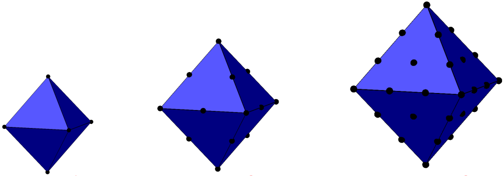
where ν is a measure on S and $\hat{\nu}(u) = \int e^{-2\pi \mathrm{i} \langle x, u \rangle} d\nu(x)$

Known results :

- $n = 2$ for S the Euclidean sphere and $V = \mathbb{R}^n$ [Hardwinger, Nelson 50]
- S Voronoi cell in lattice Ω [Dutour Sikiric, Madore, Moustrou, Pecher]
- S polytope, $V = \mathbb{R}^n$ or Ω : partial results.

$V = \mathbb{Z}^n$ and $S = \mathbb{S}^1_r$ the cross-polytopes

$\mathbb{S}^1_r := \{u \in \mathbb{Z}^n \,|\, |u_1| + \ldots + |u_n| = r\}$ [Füredi, Kang'04]



$r = 1$ $r = 2$ $r = 3$

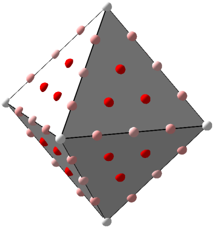
Symmetry: $\mathcal{B}_n$ or $\mathcal{C}_n$

$$2^n \geq \chi_m(\mathbb{Z}^n, \mathbb{S}^1_r) \geq 1 - \frac{1}{F(r)}$$

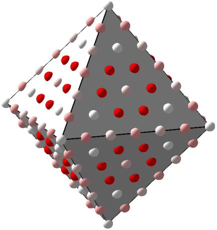
$$F(r) := \max \left\{ \min_{z \in \mathcal{T}} \sum_{\alpha \in S_r^+} f_\alpha \, T_\alpha(z) \;\middle|\; \sum_{\alpha \in S_r^+} f_\alpha = 1, f_\alpha \geq 0 \right\}$$

Analytical bounds (with Chebyshev polynomials) [HMMR 23]

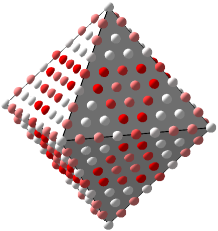
$\chi_m(\mathbb{Z}^2, \mathbb{B}^1_{2\,r}) = 4$ $\chi_m(\mathbb{Z}^n, \mathbb{B}^1_{2\,r+1}) = 2$ $\chi_m(\mathbb{Z}^n, \mathbb{B}^1_{2}) = 2n$



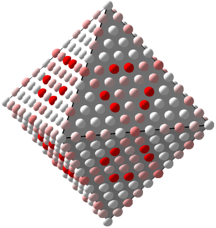
$1 - \frac{1}{F(4)} \geq 6.28148$



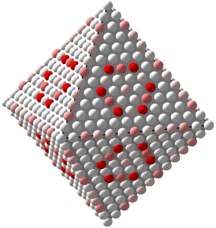
$1 - \frac{1}{F(6)} \geq 6.30269$



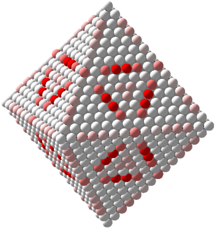
$1 - \frac{1}{F(8)} \geq 6.30229$



$1 - \frac{1}{F(10)} \geq 6.30502$

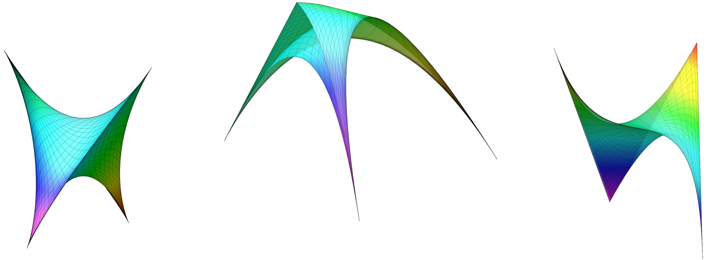


$1 - \frac{1}{F(12)} \geq 6.30229$



$1 - \frac{1}{F(14)} \geq 6.30156$

Thanks



E. Hubert, T. Metzlaff, C. Riener; Orbit spaces of Weyl groups acting on compact tori: a unified and explicit polynomial description. https://hal.science/hal-03590007

E. Hubert, T. Metzlaff, P. Moustrou, C. Riener; Optimization of trigonometric polynomials with symmetry and spectral bounds for set avoiding graphs. https://hal.science/hal-03768067

Explicit semi-algebraic description of the orbit space of Weyl group actions

6. A probable question

$\mathcal{T} = \vartheta\left(\mathbb{T}^n\right)$ where $\vartheta = (\theta_1, \ldots, \theta_m)$ and $\mathbb{Q}[x, x^{-1}]^{\mathfrak{G}} = \mathbb{Q}[\theta_1, \ldots, \theta_m]$

A reasonable conjecture?

$$z \in \vartheta((\mathbb{C}^*)^n) \qquad (z_1, \ldots, z_m) \in \mathcal{T} \quad \Leftrightarrow \quad M(z_1, \ldots, z_n) \succeq 0$$

where

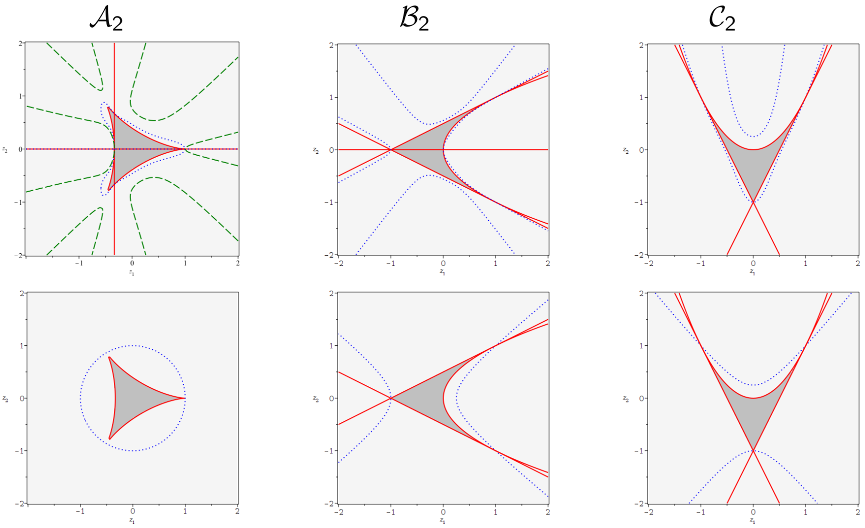
$$M(\theta_1, \ldots, \theta m) = -\left[\langle \tilde{\nabla}\theta_i, \tilde{\nabla}\hat{\theta}_j \rangle\right]_{1 \leq i,j \leq m}$$

$$\tilde{\nabla} = \left(x_1 \frac{\partial}{\partial x_1}, \ldots, x_n \frac{\partial}{\partial x_n}\right) \qquad \hat{\theta}_i(x) = \theta_i(x^{-I})$$

For $\mathcal{A}_n, \mathcal{C}_n, \mathcal{B}_n, \mathcal{D}_n$,

Referee provided a construction of $M(z)$
based on [Section 4, Procesi Schwarz 85]

considering $\mathrm{SU}_n(\mathbb{C})$, Sp_n, $\mathrm{Spin}_{2n+1}(\mathbb{R})$, $\mathrm{Spin}_{2n}(\mathbb{R})$ and their maximal tori.


$\mathcal{A}_2$
$\mathcal{B}_2$
$\mathcal{C}_2$