An Intrinsic Geometric Formulation of Hyper-Elasticity, Pressure Potential and Non-Holonomic Constraints

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OUTLINE

Outline

CONTEXT

- Hyperelasticity is naturally formulated using an elastic energy. To treat a full hyperelasticity problem as the minimization of a functional $\mathcal{L}$ (Ball, 1977), each boundary term must also be recast using a potential energy.
- For dead loads, more generally conservative loads, such potentials exist.
- For prescribed pressure on the boundary, different expressions for such a potential have been proposed (Pearson,1956, Ball, 1977).
  But additional constraints are required for such a potential to exist (Sewell,1965, Beatty, 1970, Podio-Giudugli, 1988).
- Expressions given on a reference configuration Ω_0, with lost information (impossible to pull them back on the body $\mathcal{B}$, Truesdell and Noll, 1965, Noll, 1972, 1978).

A general scheme for potential formulations of boundary terms

We shall introduce a method to build a potential energy for surface forces.

- when certain compatibility conditions are satisfied, concerning the surface forces;
- when these compatibility conditions are not satisfied, some non-holonomic constraints are formulated to bypass these restrictions.

THE CONTRIBUTION OF DIFFERENTIAL GEOMETRY

Classical differential geometry furnishes tools, like the Poincaré lemma

- to decide if the problem admits a potential, and in that case, to calculate such a potential;
- otherwise, thanks to Poincaré integrator, it allows to formulate explicitly the non-holonomic constraints under which a potential is defined.

These tools can be extrapolated to differential geometry in infinite dimension.

In infinite dimension

The approach is the same as the one adopted by Arnold (1965):

- use classical results from finite dimensional differential geometry,
- extrapolate them in this extended infinite dimensional setting,
- and then check that they are still true.

When is a vector field X the gradient of a function f ?

- A classical question is when a vector field

$$X = P\boldsymbol{e}_1 + Q\boldsymbol{e}_2 + R\boldsymbol{e}_3,$$

defined on $\mathbb{R}^3$, is the gradient of a function f ?

- A necessary condition is $\operatorname{rot} X = 0$, or in other words

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) = \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

A REFORMULATION OF THE PROBLEM USING EXTERIOR CALCULUS

- A necessary condition for a differential one-form

$$\alpha = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

to be the differential of a function f, *i.e.* $\alpha = df$ is

$$d\alpha := \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = 0.$$

- Here $dx \wedge dy := dx \otimes dy - dy \otimes dx$ and
$d : \Omega^k(\mathbb{R}^3) \to \Omega^{k+1}(\mathbb{R}^3)$ is the exterior derivative which satisfies

$$d \circ d = 0.$$

Volume forms

Definitions

- A differential form ω of degree k on $\mathbb{R}^d$ (or more generally on a manifold) is a tensor field of order k which is alternate,

$$\omega_{r_1 \cdots r_j \cdots r_i \cdots r_k} = -\omega_{r_1 \cdots r_i \cdots r_j \cdots r_k}.$$

- A volume form on $\mathbb{R}^d$ (or more generally on a manifold of dimension d) is a d-form (maximal degree) which vanishes nowhere.

- **Example**: a volume form on $\mathbb{R}^2$ is written

$$f\, dx \wedge dy, \quad \text{where } f(x,y) \neq 0.$$

- **Example**: a volume form on $\mathbb{R}^3$ is written

$$f\, dx \wedge dy \wedge dz, \quad \text{where } f(x,y,z) \neq 0.$$

$$dx \wedge dy = dx \otimes dy - dy \otimes dx, \qquad dx \wedge dy \wedge dz = (dx \otimes dy \otimes dz)^a .$$

The Riemannian volume form

On every (orientable) Riemannian manifold (M, g) there exists a unique volume form, noted vol_g which is characterized that its value is 1 when evaluated on every direct orthonormal basis.

- **Example**: on $\mathbb{R}^3$, equipped with its natural Euclidean structure $\mathbf{q}$, this volume form is written

$$\mathrm{vol}_{\mathbf{q}} = dx \wedge dy \wedge dz = (dx \otimes dy \otimes dz)^a ,$$

in any system of (direct) orthogonal coordinates (x, y, z).

- **Example**: if Ω is a bounded domain in $\mathbb{R}^3$, the Riemannian volume form on its boundary (area element), $\partial\Omega$, is written

$$da = i_{\boldsymbol{n}} \mathrm{vol}_{\mathbf{q}} = \boldsymbol{n} \cdot \mathrm{vol}_{\mathbf{q}}, \qquad \boldsymbol{n} : \text{outward normal}$$

- **Example**: if Σ is a bounded surface in $\mathbb{R}^3$, with area element da, then the Riemannian volume form (length element) on $\partial\Sigma$ is written

$$d\ell = i_{\boldsymbol{n}} da = \boldsymbol{n} \cdot da, \qquad \boldsymbol{n} : \text{outward normal.}$$

Outline

The configuration space in finite strains

Truesdell and Noll 1965

- The material medium is parameterized by a three-dimensional compact and orientable manifold with boundary, $\mathcal{B}$, the body.
- A configuration is represented by a smooth orientation-preserving embedding (particles cannot occupy the same point in space)

$$p : \mathcal{B} \to \mathcal{E},$$

where $\mathcal{E}$ is the three dimensional Euclidean space.

The configuration space

The configuration space is thus the (infinite dimensional) manifold of smooth embeddings $\mathrm{Emb}(\mathcal{B}, \mathcal{E})$.

Remark

Some authors consider embeddings of class C^k with $k \geq 1$ rather smooth embeddings (Segev 1986, Segev-Epstein 2020).

THE TOPOLOGICAL STRUCTURE OF $\mathrm{Emb}(\mathcal{B},\mathcal{E})$

- $\mathrm{Emb}(\mathcal{B},\mathcal{E})$ is a subset of $\mathrm{C}^{\infty}(\mathcal{B},\mathcal{E})$, the space of smooth mappings from $\mathcal{B}$ to $\mathcal{E}$, which is an infinite dimensional affine space.
- $\mathrm{C}^{\infty}(\mathcal{B},\mathcal{E})$ is not a Banach space, its topology is not defined by a norm but by a countable family of semi-norms (the C^k semi-norms). For this topology, $\mathrm{Emb}(\mathcal{B},\mathcal{E})$ is an open set.

Lemma

Let $p_0 \in \mathrm{Emb}(\mathcal{B},\mathcal{E})$. The neighborhood of p_0, defined by

$$\mathcal{U}_{p_0} := \left\{ p \in \mathrm{C}^{\infty}(\mathcal{B},\mathcal{E});\ \sup_{\mathbf{X}\in\mathcal{B}} \|\mathbf{F}(\mathbf{X}) - \mathbf{F}_0(\mathbf{X})\| < 1 \right\},$$

where $\mathbf{F} = Tp$, $\mathbf{F}_0 = Tp_0$ are the corresponding tangent linear maps, is an open convex set of $\mathrm{C}^{\infty}(\mathcal{B},\mathcal{E})$, which is contained in $\mathrm{Emb}(\mathcal{B},\mathcal{E})$.

The phase space in finite strains

The tangent bundle to the manifold of embeddings

The tangent space at a configuration p to $\mathrm{Emb}(\mathcal{B},\mathcal{E})$ is described as follows.

- To each path of embedding $p(s)$ with $p(0)=p$, we consider the variation

$$\mathbf{W} := \delta p = \partial_s p(0).$$

- The tangent space $T_p\mathrm{Emb}(\mathcal{B},\mathcal{E})$ is the space of all variations at p

$$T_p\mathrm{Emb}(\mathcal{B},\mathcal{E}) := \{\delta p := \partial_s p(0,\mathbf{X});\ p(0)=p\}.$$

- It is useful to introduce the vector field on $\Omega = p(\mathcal{B})$,

$$\boldsymbol{w} := \delta p \circ p^{-1}.$$

Remark

- $\mathbf{W} := \delta p$: a virtual Lagrangian velocity.
- $T_p\mathrm{Emb}(\mathcal{B},\mathcal{E})$: the space of virtual Lagrangian velocities.
- $\boldsymbol{w} := \mathbf{W} \circ p^{-1}$: a virtual displacement on the deformed configuration.

Pull-back and push-forward

From Lagrangian variables to Eulerian variables and *vice versa*

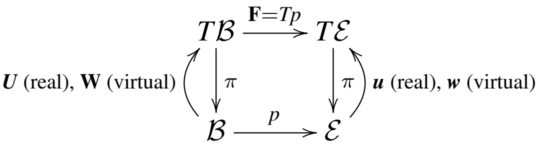
- These operations extend the following operations on functions $f \in C^{\infty}(\Omega, \mathbb{R})$ and $\mathcal{F} \in C^{\infty}(\mathcal{B}, \mathbb{R})$, where $p : \mathcal{B} \to \mathcal{E}$,

$$p^{*}f = f \circ p \quad \text{(pull-back)}, \qquad p_{*}\mathcal{F} = \mathcal{F} \circ p^{-1} \quad \text{(push-forward)}$$

  to any tensor fields.

- **Example**: for velocity vector fields, we get

$$p^{*}\boldsymbol{u} = \mathbf{F}^{-1}.\boldsymbol{u} \circ p \quad \text{(pull-back)}, \qquad p_{*}\boldsymbol{U} = \mathbf{F}.\boldsymbol{U} \circ p^{-1} \quad \text{(push-forward)}$$



Mass measure and mass density

- The 3-dimensional body is equipped with a volume form,

  the mass measure μ,

  which encodes the distribution of mass in the material (Truesdell and Noll, 1965).
- Given a configuration $p : \mathcal{B} \to \mathcal{E}$, the push-forward of μ by p defines a mass measure $p_*\mu$ on $\Omega = p(\mathcal{B})$.
- $p_*\mu$ is necessarily proportional to the volume form $\mathrm{vol}_{\mathbf{q}}$ on $\mathcal{E}$

  $$p_*\mu = \rho\,\mathrm{vol}_{\mathbf{q}},$$

  which defines the mass density ρ.

The metric γ and Cauchy–Green tensors

- By pull-back of $\mathbf{q}$ on $\mathcal{E}$, we get a Riemannian metric γ on the Body $\mathcal{B}$

$$\gamma = p^*\mathbf{q} = \mathbf{F}^\star \mathbf{q}\, \mathbf{F}, \qquad \mathbf{F} = Tp = \left(\frac{\partial p^i}{\partial X^J} = \frac{\partial x^i}{\partial X^J} \right).$$

- Choose a reference configuration p_0 and introduce the deformation φ:

$$p_0 : \mathcal{B} \to \Omega_0, \qquad \varphi = p \circ p_0^{-1}.$$

  - The right Cauchy–Green tensor is defined on $\Omega_0 = p_0(\mathcal{B})$ as

$$\mathbf{C} := \varphi^*\mathbf{q} = \mathbf{F}_\varphi^\star \mathbf{q}\, \mathbf{F}_\varphi = \mathbf{q}\mathbf{F}_\varphi^t \mathbf{F}_\varphi. \qquad \mathbf{F}_\varphi = T\varphi = \left(\frac{\partial x^i}{\partial x_0^J} \right).$$

  - The metrics $\mathbf{C}$ (on $\Omega_0 = p_0(\mathcal{B})$) and γ (on $\mathcal{B}$) are related as

$$p_0^*\mathbf{C} = \gamma.$$

  - $\gamma \equiv \mathbf{C}$ when the the body $\mathcal{B}$ is identified with a reference configuration Ω_0.

STRAIN RATE

- The strain rate is often defined as

$$\widehat{\mathbf{d}} := \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right), \qquad \widehat{\mathbf{d}} = ({d^i}_j).$$

- Its covariant version $\mathbf{d} = \mathbf{q}\widehat{\mathbf{d}}$ writes as

$$\mathbf{d} = \frac{1}{2}\,\mathcal{L}_{\boldsymbol{u}}\,\mathbf{q}, \qquad \mathbf{d} = (d_{ij})$$

where $\mathcal{L}_{\boldsymbol{u}}$ is the Lie derivative with respect to $\boldsymbol{u}$ (Eulerian velocity).

The covariant version $\mathbf{d}$ of the strain rate seems to have more geometric meaning than its mixed form $\widehat{\mathbf{d}}$.

The metric as the primitive of the strain rate when pulled back on the Body

Theorem (Rougée, 1991, generalizing $\partial_t \mathbf{C} = 2\varphi^* \mathbf{d}$)

Along a path of embeddings $p(t)$ (a loading), the metric $\boldsymbol{\gamma}(t) = p(t)^* \mathbf{q}$ on the body, satisfies the evolution equation

$$\partial_t \boldsymbol{\gamma} = 2 p^* \mathbf{d},$$

where $\mathbf{d}$ is the covariant form of the strain rate.

This result is a direct consequence of the more general formula

$$\partial_t (p^* \mathbf{t}) = p^* \left(\partial_t \mathbf{t} + \mathcal{L}_{\boldsymbol{u}} \, \mathbf{t} \right),$$

for any tensor field $\mathbf{t}$ defined on Ω.

Stresses – Dual concept of strains

Mathematically, they are distribution-tensors (virtual works).

- Cauchy stress tensor : it is the special case when this tensor-distribution has a density $\boldsymbol{\sigma}$ (on the deformed configuration)

$$\mathcal{P}^{int}(\boldsymbol{\epsilon}) = -\int_{\Omega} (\boldsymbol{\sigma} : \boldsymbol{\epsilon})\, \mathrm{vol}_{\mathbf{q}}.$$

- We can rewrite this expression on the body, using the change of variables formula

$$\mathcal{P}^{int}(\boldsymbol{\epsilon}) = -\int_{\Omega} (\boldsymbol{\tau} : \boldsymbol{\epsilon})\, \rho \mathrm{vol}_{\mathbf{q}} = -\int_{\mathcal{B}} (\boldsymbol{\theta} : p^{*}\boldsymbol{\epsilon})\, \mu.$$

  where

  - $\boldsymbol{\tau} = \boldsymbol{\sigma}/\rho$ is the Kirchhoff stress tensor (on Ω)
  - and $\boldsymbol{\theta} = p^{*}\boldsymbol{\tau}$ is the Rougée stress tensor (on $\mathcal{B}$).

OUTLINE

THE SET OF STRAIN VARIABLES

AFTER THE WORK OF PAUL ROUGÉE

Besides the configuration space $\mathrm{Emb}(\mathcal{B}, \mathcal{E})$ it is important to describe the space $\mathrm{Met}(\mathcal{B})$ of all Riemannian metrics $\boldsymbol{\gamma}$ on the body (strains variables).

- $\mathrm{Met}(\mathcal{B})$ is an open convex set of the infinite dimensional vector space of second-order covariant vector fields on $\mathcal{B}$.
- The tangent space $T_{\boldsymbol{\gamma}}\mathrm{Met}(\mathcal{B})$ (of metrics variations $\delta\boldsymbol{\gamma}$) corresponds to virtual linearized strains $\boldsymbol{\epsilon}$ at p

$$\delta\boldsymbol{\gamma} = 2p^{*}\boldsymbol{\epsilon}.$$

- The cotangent space $T^{\star}_{\boldsymbol{\gamma}}\mathrm{Met}(\mathcal{B})$ corresponds to the space of virtual powers of stresses (with or without densities).

Rougée metric on strain variables

A Riemannian metric on the space of Riemannian metrics

- Rougée has introduced the following Riemannian metric on $\mathrm{Met}(\mathcal{B})$ (slightly different from Ebin's one):

$$G^{\mu}_{\gamma}(\delta_1\gamma, \delta_2\gamma) := \int_{\mathcal{B}} \mathrm{tr}\left(\gamma^{-1}(\delta_1\gamma)\gamma^{-1}(\delta_2\gamma)\right)\,\mu, \quad \delta_k\gamma \in T_{\gamma}\mathrm{Met}(\mathcal{B})$$

- This metric induces an injective (but not surjective) linear mapping

$$T_{\gamma}\mathrm{Met}(\mathcal{B}) \to T^{\star}_{\gamma}\mathrm{Met}(\mathcal{B}), \qquad \boldsymbol{\eta} \mapsto G^{\mu}_{\gamma}(\boldsymbol{\eta}, \cdot);$$

- The range of this mapping in $T^{\star}_{\gamma}\mathrm{Met}(\mathcal{B})$ corresponds exactly to virtual works with densities

$$\mathcal{W}_{\gamma}(\delta\gamma) = \int_{\mathcal{B}} (\boldsymbol{\theta} : \delta\gamma)\mu, \qquad \boldsymbol{\theta} = \gamma^{-1}\boldsymbol{\eta}\gamma^{-1}.$$

Elastic constitutive laws

A geometric point of view

- An elastic constitutive law (in the sense of Cauchy) can be interpreted (Rougée) as a vector field on the manifolfd of the Riemannian metrics

$$S : \gamma \mapsto S(\gamma) \quad \textit{on}\ \mathrm{Met}(\mathcal{B})$$

$$\begin{array}{ccc} T\mathrm{Met}(\mathcal{B}) & \xrightarrow{G^{\mu}} & T^{\star}\mathrm{Met}(\mathcal{B}) \\ S \uparrow \ \downarrow \pi & \nearrow_{\theta=\gamma^{-1}S(\gamma)\gamma^{-1}} & \\ \mathrm{Met}(\mathcal{B}) & & \end{array}$$

- In this framework, an hyperelastic law (elasticity in the sense of Green) corresponds to a vector field S which is the gradient (for the metric G^{μ}) of a functional $H \in \mathrm{C}^{\infty}(\mathrm{Met}(\mathcal{B}))$

$$(d_{\gamma}H).\delta\gamma = G^{\mu}_{\gamma}(S(\gamma), \delta\gamma) = \int_{\mathcal{B}} \mathrm{tr}(\gamma^{-1} S(\gamma)\, \gamma^{-1} \delta\gamma)\, \mu.$$

$H/2$ is the called the elastic energy.

Theorem

Given a reference configuration p_0 and $\boldsymbol{\gamma}_0 = p_0^* \mathbf{q}$, local isotropic hyperelasticity recasts on the body using the functional

$$H_{\gamma_0}(\boldsymbol{\gamma}) = \int_{\mathcal{B}} 2\psi(\boldsymbol{\gamma})\mu,$$

where $\psi(\boldsymbol{\gamma}) = \widetilde{\psi}(I_1, I_2, I_3)$ *and* $I_k = \operatorname{tr}(\boldsymbol{\gamma}_0^{-1}\boldsymbol{\gamma})^k$*. We have then*

$$d\mathcal{H}_{\gamma_0}.\delta\boldsymbol{\gamma} = G^{\mu}_{\gamma}(\boldsymbol{\gamma}^{-1}\boldsymbol{\theta}\boldsymbol{\gamma}^{-1}, \delta\boldsymbol{\gamma}) = \int_{\mathcal{B}} (\boldsymbol{\theta} : \delta\boldsymbol{\gamma})\mu,$$

where

$$\boldsymbol{\theta} = 2\frac{\partial\psi}{\partial\boldsymbol{\gamma}}.$$

Recovering hyperelasticity on a configuration

On the reference configuration ($\Omega_0 = p_0(\mathcal{B}) = \varphi^{-1}(\Omega)$)

$$\mathbf{S} = 2\frac{\partial \psi}{\partial \mathbf{C}} \qquad \begin{cases} \mathbf{S} = p_* \boldsymbol{\theta} = \varphi^* \boldsymbol{\tau} = \mathbf{F}_\varphi^{-1}(\boldsymbol{\tau} \circ \varphi)\mathbf{F}_\varphi^{-\star} \\ \mathbf{C} = p_* \gamma = \varphi^* \mathbf{q} = \mathbf{F}_\varphi^\star \mathbf{q} \mathbf{F}_\varphi \end{cases}$$

($\mathbf{S}$: second Piola–Kirchhoff stress tensor, $\mathbf{C}$: right Cauchy–Green tensor)

On the deformed configuration ($\Omega = p(\mathcal{B})$) – Doyle–Ericksen formula

$$\boldsymbol{\tau} = 2\frac{\partial \psi}{\partial \mathbf{q}} \qquad \begin{cases} \boldsymbol{\tau} = p_* \boldsymbol{\theta} \\ \mathbf{q} = p_* \gamma \end{cases}$$

($\boldsymbol{\tau} = \boldsymbol{\sigma}/\rho$: Kirchhoff stress tensor, $\mathbf{q}$: Euclidean metric)

OUTLINE

Differential forms on the configuration space

- A zero-form on $\mathrm{Emb}(\mathcal{B},\mathcal{E})$ is a smooth functional

$$\mathcal{L} : p \mapsto \mathcal{L}(p), \qquad p \in \mathrm{Emb}(\mathcal{B},\mathcal{E}).$$

- A one-form on $\mathrm{Emb}(\mathcal{B},\mathcal{E})$ is a continuous linear functional

$$\mathcal{W}_p(\delta p), \qquad p \in \mathrm{Emb}(\mathcal{B},\mathcal{E}),\ \delta p \in \mathrm{C}^\infty(\mathcal{B},\mathcal{E})$$

depending smoothly on p.

- A two-form on $\mathrm{Emb}(\mathcal{B},\mathcal{E})$ is a skew-symmetric continuous bilinear functional

$$\mathcal{K}_p(\delta_1 p, \delta_2 p), \qquad p \in \mathrm{Emb}(\mathcal{B},\mathcal{E}),\ \delta_k p \in \mathrm{C}^\infty(\mathcal{B},\mathcal{E})$$

depending smoothly on p.

VIRTUAL WORK PRINCIPLE AS THE VANISHING OF A ONE-FORM

- Virtual work principle (no dynamics) states that

$$\mathcal{P}^{int}(\boldsymbol{w}) + \mathcal{P}^{ext}(\boldsymbol{w}) = 0,$$

for all kinematic admissible virtual displacement $\boldsymbol{w}$.

- Both

$$\mathcal{P}^{int} = -\mathcal{W}^{int} \quad \text{and} \quad \mathcal{P}^{ext} = -\mathcal{W}^{ext}$$

may involve various one-forms $\mathcal{W}^k$.

- Once recast on the body, using pull-back, each work $\mathcal{W}^k$ is interpreted as a one-form on $\mathrm{Emb}(\mathcal{B}, \mathcal{E})$ and the Virtual work principle recast as

$$\sum_k \mathcal{W}_p^k(\delta p) = 0, \qquad \mathbf{W} = \delta p,$$

for all $(p, \delta p)$ satisfying certain non-holonomic conditions.

Virtual work principle as the vanishing of a one-form

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$$\sum_{k} \mathcal{W}_{p}^{k} = 0,$$

for all $(p, \delta p)$ satisfying certain non-holonomic conditions.

Potential formulation

When virtual works correspond to exact one-forms

- Virtual work principle, once recast on the body, involves various differential one-forms

$$\mathcal{W}^k \quad \text{defined on } \mathrm{Emb}(\mathcal{B}, \mathcal{E}),$$

- If each involved one-form $\mathcal{W}^k$ is exact, meaning that

$$\mathcal{W}^k = d\mathcal{L}^k,$$

  then, each solution of the mechanical problem is a critical point of the functional

$$\mathcal{L} := \sum_k \mathcal{L}^k.$$

Virtual work of internal forces

For hyperelasticity

$$
\begin{aligned}
\mathcal{P}^{int}(\boldsymbol{w}) = -\mathcal{W}^{int}(\boldsymbol{w}) &= -\int_{\Omega} (\boldsymbol{\sigma} : \boldsymbol{\epsilon}) \, \mathrm{vol}_{\mathbf{q}}, \qquad && \boldsymbol{\epsilon} := \frac{1}{2} \mathcal{L}_{\boldsymbol{w}} \, \mathbf{q} \\
&= -\int_{\mathcal{B}} (\boldsymbol{\theta} : \frac{1}{2}\delta\boldsymbol{\gamma}) \, \mu, \qquad && p^{*}\boldsymbol{\epsilon} = \frac{1}{2}\delta\boldsymbol{\gamma} \\
&= -\frac{1}{2} d\mathcal{H}_{\gamma_0}.\delta\boldsymbol{\gamma}
\end{aligned}
$$

Internal forces virtual work

$\mathcal{W}^{int}$ recast on the body as the exact one-form

$$
\mathcal{W}_{p}^{int}(\delta p) = \frac{1}{2}\delta H = \delta \int_{\mathcal{B}} \psi \, \mu,
$$

i.e., as the variation of the so-called hyperelastic energy.

Virtual work of external forces

It has for general expression

$$\mathcal{P}^{ext} = -\mathcal{W}^{ext} = -\mathcal{W}^{ext,v} - \mathcal{W}^{ext,s}.$$

Here:

- No exterior volume forces (gravitation/electromagnetism) involved.
- We assume $\mathcal{W}^{ext,v} = 0$: we shall consider only boundary terms $\mathcal{W}^{ext,s}$.

Surface forces term: dead load DL + prescribed pressure P

$$\mathcal{W}^{ext,s}(\boldsymbol{w}) = -\int_{\Sigma_0^{(DL)}} (\boldsymbol{t}_0 \cdot \delta\varphi)\, da_0 + \int_{\Sigma^{(P)}} P\,(\boldsymbol{w} \cdot \boldsymbol{n})\, da, \qquad (\delta\varphi = \boldsymbol{w} \circ \varphi)$$

where

- $\boldsymbol{t}_0$ (for dead load): a vector valued function defined on $\Sigma_0^{(DL)} \subset \partial\Omega_0$,
- P : pressure, a scalar function defined on $\Sigma^{(P)} \subset \partial\Omega$.

Pressure virtual work

A prescribed pressure P is defined on the boundary part $\Sigma^{(P)} \subset \partial\Omega$ and induces the boundary condition

$$\widehat{\boldsymbol{\sigma}}\boldsymbol{n}\big|_{\Sigma^{(P)}} = -P\boldsymbol{n}.$$

Prescribed pressure virtual work

The virtual work of the pressure corresponds to the one-form $-\mathcal{W}^{(P)}$, where

$$\mathcal{W}_p^{(P)}(\delta p) = \int_{\Sigma^{(P)}} P\,(\boldsymbol{w}\cdot\boldsymbol{n})\,da = \int_{\Sigma^{(P)}} P\,i_{\boldsymbol{w}}\mathrm{vol}_{\mathbf{q}} = \int_{\Sigma_{\mathcal{B}}^{(P)}} (P\circ p)\,\omega,$$

where ω is the two-form defined by

$$\omega = \mathrm{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot) \quad \text{i.e.} \quad \omega(A,B) = \mathrm{vol}_{\mathbf{q}}(\delta p, \mathbf{F}A, \mathbf{F}B),$$

$\mathbf{F} = Tp$, $\delta p = \boldsymbol{w}\circ p$ and $\Sigma_{\mathcal{B}}^{(P)} := p^{-1}(\Sigma^{(P)}) \subset \partial\mathcal{B}$.

Remark : pressure boundary terms on Ω_0

If we identify the body $\mathcal{B}$ with a reference configuration Ω_0, we have

$$p_0 \equiv \mathrm{Id}, \quad p \equiv \varphi, \quad \Sigma_{\mathcal{B}}^{(P)} \equiv \Sigma_0^{(P)} \subset \partial\Omega_0, \quad \mathbf{F} \equiv \mathbf{F}_\varphi, \quad \text{and} \quad \delta\boldsymbol{\xi} \equiv \delta\varphi,$$

and thus

$$\int_{\Sigma_{\mathcal{B}}^{(P)}} (P \circ p)\, \mathrm{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot) = \int_{\Sigma_0^{(P)}} (P \circ \varphi)\, \mathrm{vol}_{\mathbf{q}}(\delta\boldsymbol{\xi}, \mathbf{F}_\varphi\cdot, \mathbf{F}_\varphi\cdot).$$

Now, since

$$\mathrm{vol}_{\mathbf{q}}(\delta\boldsymbol{\xi}, \mathbf{F}_\varphi\cdot, \mathbf{F}_\varphi\cdot) = J_\varphi {\mathbf{F}_\varphi}^{-1}\delta\boldsymbol{\xi} \cdot \boldsymbol{n}_0\, da_0,$$

we recover the well-known expression of the pressure virtual work on Ω_0,

$$-\int_{\Sigma^{(P)}} P\,(\boldsymbol{w} \cdot \boldsymbol{n})\, da = -\int_{\Sigma_0^{(P)}} (P \circ \varphi)\, J_\varphi \mathbf{F}_\varphi^{-1}\delta\boldsymbol{\xi} \cdot \boldsymbol{n}_0\, da_0, \qquad J_\varphi = \det \mathbf{F}_\varphi.$$

Outline

Poincaré lemma

An exact differential form $\alpha = d\beta$ on U is always closed $d\alpha = 0$ ($d \circ d = 0$), but the converse is false in general.

Poincaré lemma asserts that the converse is true when U is convex.

Lemma (Poincaré)

Let $U \subset \mathbb{R}^3$ be a convex open set. If $\alpha \in \Omega^k(U)$ is closed ($d\alpha = 0$), then α is exact ($\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(U)$).

The proof of Poincaré lemma is constructive. It relies on the explicit definition of a linear operator $K : \Omega^{k+1}(U) \to \Omega^k(U)$, the Poincaré integrator, such that

$$Kd + dK = \mathrm{id}.$$

Explicit solution of the problem

An explicit primitive β of α is then $\beta = K\alpha$ since

$$d\beta = d(K\alpha) = \alpha - K(d\alpha) = \alpha, \qquad \text{if} \quad d\alpha = 0.$$

$K(d\alpha)$ is called the obstruction for β to be a primitive of α.

Poincaré lemma in infinite dimension

- The Poincaré integrator K is still meaningful in infinite dimension, in particular, for variational problems in finite strains.
- In this setting
  - f is replaced by a functional $\mathcal{L}$ and its exterior derivative

    $$(df)_i = \partial_i f,$$

    corresponds to the first variation of $\mathcal{L}$

    $$(d\mathcal{L})_p(\delta p) = \delta\mathcal{L}.$$

  - α is replaced by a virtual work $\mathcal{W}$ and its exterior derivative

    $$(d\alpha)_{ij} = \partial_i\alpha_j - \partial_j\alpha_i$$

    corresponds to the skew symmetric part of the first variation of $\mathcal{W} = \mathcal{W}_p(\delta p)$

    $$(d\mathcal{W})_p(\delta_1 p, \delta_2 p) = \delta_1(\mathcal{W}_p(\delta_2 p)) - \delta_2(\mathcal{W}_p(\delta_1 p)).$$

The Poincaré integrator K

- The Poincaré integrator for a one-form $\mathcal{W}$ (virtual work) is defined as

$$(K\mathcal{W})(p) = \int_{-\infty}^{0} e^t \mathcal{W}_{\phi^t(p)}(\boldsymbol{\xi}(p))\, dt$$

on the convex set $\mathcal{U}_{p_0}$, and where

$$\phi^t(p) = e^t p + (1 - e^t)p_0$$

is the flow of the radial field $\boldsymbol{\xi}(p) = p - p_0$ (the displacement).

- The obstruction for $\mathcal{L}$ to be a primitive of $\mathcal{W}$ is

$$K(d\mathcal{W})(\delta p) = \int_{-\infty}^{0} e^{2t} (d\mathcal{W})_{\phi^t(p)}(\boldsymbol{\xi}(p), \delta p)\, dt \neq 0.$$

The two cases: $d\mathcal{W} = 0$ or $d\mathcal{W} \neq 0$

- If $\mathcal{W}$ is closed ($d\mathcal{W} = 0$), then, a potential for $\mathcal{W}$

$$\mathcal{L}(p) = (K\mathcal{W})(p) = \int_{-\infty}^{0} e^{t}\mathcal{W}_{\phi^t(p)}(\boldsymbol{\xi}(p))\, dt$$

  is obtained locally, using the Poincaré integrator.

- If $\mathcal{W}$ is not closed ($d\mathcal{W} \neq 0$), then,

$$d\mathcal{L} = \mathcal{W} - K(d\mathcal{W}),$$

  and the condition

$$K(d\mathcal{W})(\delta p) = \int_{-\infty}^{0} e^{2t}(d\mathcal{W})_{\phi^t(p)}(\boldsymbol{\xi}(p), \delta p)\, dt = 0,$$

  is a non-holonomic constraint required for $\mathcal{W} = d\mathcal{L}$.

Dead loads virtual work is closed

The corresponding one-form defined on $\mathrm{Emb}(\mathcal{B},\mathcal{E})$ is written

$$\mathcal{W}_p^{(DL)}(\delta p) = -\int_{\Sigma_{\mathcal{B}}^{(DL)}} (\delta p \cdot \boldsymbol{t}_0 \circ p_0)\, da_{\gamma_0}.$$

This form is obviously closed, since it does not depend explicitly on p.

A potential for dead loads

A potential $\mathcal{L}^{(DL)}(p)$ exists always for dead loads and is given by

$$\mathcal{L}^{(DL)}(p) = -\int_{\Sigma_{\mathcal{B}}^{(DL)}} ((p - p_0) \cdot \boldsymbol{t}_0 \circ p_0)\, da_{\gamma_0}.$$

Pressure virtual work is not closed

Assuming $P = 1$, we get

$$\mathcal{W}_p^{(P)}(\delta p) = \int_{\Sigma^{(P)}} (\boldsymbol{w} \cdot \boldsymbol{n})\, da = \int_{\Sigma_{\mathcal{B}}^{(P)}} \mathrm{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot)$$

Lemma

The differential form

$$\mathcal{W}_p^{(P)}(\delta p) = \int_{\Sigma_{\mathcal{B}}^{(P)}} \mathrm{vol}_{\mathbf{q}}(\delta p, \mathbf{F}\cdot, \mathbf{F}\cdot)$$

defined on $\mathrm{Emb}(\mathcal{B}, \mathcal{E})$ *is not closed. Its exterior derivative is written*

$$\left(d\mathcal{W}^{(P)}\right)_p (\delta p_1, \delta p_2) = \int_{\partial \Sigma_{\mathcal{B}}^{(P)}} (\delta p_2 \times \delta p_1) \cdot \mathbf{F}\, d\boldsymbol{\ell}_{\mathcal{B}},$$

where $d\boldsymbol{\ell}_{\mathcal{B}} = p_0^* \, d\boldsymbol{\ell}_0$ *is the oriented length element on* $\partial \Sigma_{\mathcal{B}}^{(P)}$.

Theorem

Let us consider the functional

$$\mathcal{L}^{(P)}(p) = \frac{1}{6}\int_{\Sigma_{\mathcal{B}}^{(P)}} 2\,\mathrm{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}\cdot, \mathbf{F}\cdot) + 2\,\mathrm{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0\cdot, \mathbf{F}_0\cdot) + \left(\mathrm{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}\cdot, \mathbf{F}_0\cdot) + \mathrm{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0\cdot, \mathbf{F}\cdot)\right)$$

where $\boldsymbol{\xi}(p) := p - p_0$ *is the displacement field. Then*

$$\left(d\mathcal{L}^{(P)}\right)_p(\delta p) = \mathcal{W}_p^{(P)}(\delta p) + \frac{1}{6}\oint_{\partial\Sigma_{\mathcal{B}}^{(P)}} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot (2\mathbf{F} + \mathbf{F}_0)\,d\boldsymbol{\ell}_{\mathcal{B}}.$$

In particular, the condition for the functional $\mathcal{L}^{(P)}$ *to be a primitive of* $\mathcal{W}^{(P)}$ *is thus*

$$\oint_{\partial\Sigma_{\mathcal{B}}^{(P)}} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot (2\mathbf{F} + \mathbf{F}_0)\,d\boldsymbol{\ell}_{\mathcal{B}} = 0.$$

First formulation (in components) of a pressure potential: Pearson (1956) (then Sewell (1965, 1967)). Intrinsic expression (with typos): Beatty (1970).

Corrected expression

$$\mathcal{L}^{(P)}(\varphi) = \frac{P}{3}\int_{\Sigma_0}\left(J_\varphi \mathbf{F}_\varphi{}^{-1}\boldsymbol{\xi} + \frac{1}{2}\Big((\operatorname{tr}\mathbf{F}_\varphi)\boldsymbol{\xi} - \mathbf{F}_\varphi\boldsymbol{\xi}\Big) + \boldsymbol{\xi}\right)\cdot \boldsymbol{n}_0\, da_0,$$

where $\boldsymbol{\xi} = \varphi - \mathrm{Id}$.

This expression corresponds to the case of the body $\mathcal{B}$ identified with a reference configuration Ω_0, embedded in Euclidean space $\mathcal{E}$,

$$\Sigma_{\mathcal{B}} = \Sigma_0, \qquad p \equiv \varphi, \qquad \mathbf{F}_0 \equiv \mathbf{Id}, \qquad \mathbf{F} \equiv \mathbf{F}_\varphi, \qquad \boldsymbol{\xi} \equiv \varphi - \mathrm{Id}.$$

Optimal non-holonomic constraints

When formulated on the body $\mathcal{B}$, our constraint recasts on Ω_0 as

$$\oint_{\partial\Sigma_0^{(P)}} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot (2\mathbf{F}_\varphi + \mathbf{Id}) d\boldsymbol{\ell}_0 = 0, \qquad \boldsymbol{\xi} = \varphi - \mathrm{Id}. \tag{1}$$

This is an improvement compared to the two Beatty conditions

$$\oint_{\partial\Sigma_0^{(P)}} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot d\boldsymbol{\ell}_0 = 0 \quad \text{and} \quad \oint_{\partial\Sigma_0^{(P)}} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot \mathbf{F}_\varphi d\boldsymbol{\ell}_0 = 0, \tag{2}$$

which are stronger since (2) implies (1), but the converse does not hold.

Summary

Solutions of the full mechanical problem (hyperelasticity + dead loads + prescribed pressure) have been recast as critical points of the functional

$$\mathcal{L}(p) = \int_{\mathcal{B}} \psi\, \mu - \int_{\Sigma_{\mathcal{B}}^{(DL)}} (\boldsymbol{\xi} \cdot \boldsymbol{t}_0 \circ p_0)\, da_{\gamma_0} + \sum_k \mathcal{L}^{(P_k)}(p),$$

where

$$\mathcal{L}^{(P_k)}(p) = \frac{P_k}{6} \int_{\Sigma_{\mathcal{B}}^{(P_k)}} 2\,\mathrm{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}\cdot, \mathbf{F}\cdot) + 2\,\mathrm{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0\cdot, \mathbf{F}_0\cdot) + \left(\mathrm{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}\cdot, \mathbf{F}_0\cdot) + \mathrm{vol}_{\mathbf{q}}(\boldsymbol{\xi}, \mathbf{F}_0\cdot, \mathbf{F}\cdot)\right),$$

under non-holonomic constraints (kinematic admissible virtual displacements)

$$\oint_{\partial\Sigma_{\mathcal{B}}^{(P_k)}} (\boldsymbol{\xi} \times \delta\boldsymbol{\xi}) \cdot (2\mathbf{F} + \mathbf{F}_0) d\boldsymbol{\ell}_{\mathcal{B}} = 0.$$

Conclusion

- We have formulated directly on the body $\mathcal{B}$, a three-dimensional compact and orientable manifold with boundary (equipped with a mass measure), and not necessarily embedded as a reference configuration in space,
  - hyperelasticity as a variational problem,
  - the dead load and pressure types boundary conditions on $\partial\mathcal{B}$.
- The Poincaré lemma (extended to infinite dimension) has allowed us to obtain in a straightforward manner both the pressure potential and optimal non-holonomic constraints for such a potential to exist.
- The proposed methodology is based on the interpretation of virtual works as one-forms on the configuration space $\mathrm{Emb}(\mathcal{B}, \mathcal{E})$. It is general and can be applied to many others situations.

Intégration du problème d'hypo-élasticité

Interprétation géométrique

Le problème d'hypo-élasticité se reformule (sur le body $\mathcal{B}$) sous la forme géométrique

$$\left(\nabla_{\partial_t \gamma} \mathcal{P}\right)(\boldsymbol{\epsilon}) = \mathcal{H}(\partial_t \boldsymbol{\gamma}, \boldsymbol{\epsilon}), \quad \text{où}$$

$$\mathcal{P}(\boldsymbol{\epsilon}) = \int_{\mathcal{B}} (p^* \boldsymbol{\tau} : \boldsymbol{\epsilon})\, \mu, \quad \text{et} \quad \mathcal{H}(\partial_t \boldsymbol{\gamma}, \boldsymbol{\epsilon}) = \int_{\mathcal{B}} (\boldsymbol{\epsilon} : \mathbf{H} : \partial_t \boldsymbol{\gamma})\, \mu,$$

$\mathcal{P}$ désignant la puissance virtuelle des efforts intérieurs et $\mathbf{H}$, un champ de tenseurs covariants d'ordre 2 sur $\mathrm{Met}(\mathcal{B})$.

La question posée se résume donc à savoir

si le champ de tenseurs $\mathcal{H}$ correspond
à la dérivée covariante d'une 1-forme $\mathcal{P}_\gamma$

(*i.e.* une loi de comportement élastique).
Les conditions d'intégrabilité sont connues.

- On peut étudier numériquement la dépendance de l'intégration au chemin de chargement. Il est clair que si cette intégration dépend du chemin, alors $\mathcal{P}$ n'existe pas !

Lectures complémentaires

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