Déformations en relativité poincaréenne et galiléenne : les tenseurs de conformation et de friction

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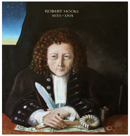


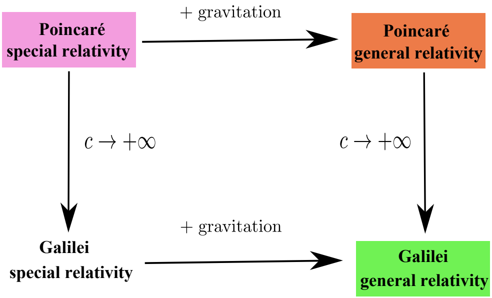
Two deformation tensors in Relativity

- Relativistic reversible media Jean-Marie Souriau
  Géométrie et Relativité (1964)
  Conformation tensor

- Relativistic dissipative media Jean-Marie Souriau
  Lect. Notes in Math. 676 (1976)
  Friction tensor

Galilei and Poincaré relativities

Christian Cardall terminology :
A unified perspective on
Poincaré and Galilei relativity 2023



Galilei group and geometry

Galilei general relativity

- **Chart :** $\mathbb{R}^4 \ni X = \begin{pmatrix} t \\ x \end{pmatrix} \mapsto \boldsymbol{X} \in \mathcal{M}$ space-time
- **4-velocity :** $\vec{\boldsymbol{U}}$ represented by $U = \frac{dX}{dt} = \begin{pmatrix} 1 \\ v \end{pmatrix}$
- **Galilei group** $\mathbb{GAL}$ is the one of the **Galilean transformations** :

$$X = P\,X' + C \qquad \text{with} \qquad P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

  where $u \in \mathbb{R}^3$ is the **Galilean boost** and R is a rotation

- **Clock form** $\boldsymbol{\tau} = dt$ represented by the Galilean invariant $\begin{pmatrix} 1 & 0^T \end{pmatrix}$

Galilei group and geometry

- The closure condition $d\omega = 0$ of the presymplectic form gives

$$curl\, g + 2\, \frac{\partial \Omega}{\partial t} = 0, \qquad\qquad div\, \Omega = 0$$

  then there exist **potentials** ϕ, A generating
  the **gravity** $g = -grad\, \phi - \dfrac{\partial A}{\partial t}$ and **Coriolis' effect** $\Omega = \frac{1}{2}\, curl\, A$

- The corresponding Lagrangian

$$\mathcal{L} = \frac{1}{2}\, m \parallel v \parallel^2 + m\, A \cdot v - m\, \phi$$

  is covariant provided

$$\phi' = \phi - A \cdot u - \frac{1}{2} \parallel u \parallel^2, \quad A' = R^T (A + u)$$

Spacetime metrics

Poincaré general relativity

For weak gravitational fields

- $G = \begin{pmatrix} c^2 + 2\,\phi & -A^T \\ -A & -1_{\mathbb{R}^3} \end{pmatrix} = \epsilon^{-1} \overset{(-1)}{G} + \overset{(0)}{G}$

  where $\epsilon = c^{-2}$, $\overset{(-1)}{G} = dt \otimes dt$ and $\overset{(0)}{G}$ are Galilean tensors
- Truncated expansion of

$$G^{-1} \cong \begin{pmatrix} 0 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{pmatrix} + \epsilon \begin{pmatrix} 1 & -A^T \\ -A & A\,A^T \end{pmatrix}$$

  Each term is a Galilean 2-contravariant tensor
- $\sqrt{-\det\ G} \cong c\,(1 + \epsilon\,(\phi + \frac{\|A\|^2}{2}))$

  Galilean invariant

Hilbert-Einstein functional $\mathcal{A} = \mathcal{A}_M + \mathcal{A}_G = \int (p_M + p_G)\sqrt{-\det G}\, d^4X$

Adjoint map

Let $(\mathcal{M}_0, \boldsymbol{G}_0), (\mathcal{M}, \boldsymbol{G})$ be two Riemaniann spaces
The **adjoint** of a map

$$\boldsymbol{A} : T_{\boldsymbol{x}_0}\mathcal{M}_0 \rightarrow T_{\boldsymbol{x}}\mathcal{M}$$

is the map

$$\boldsymbol{A}^* : T_{\boldsymbol{x}}\mathcal{M} \rightarrow T_{\boldsymbol{x}_0}\mathcal{M}_0$$

such that

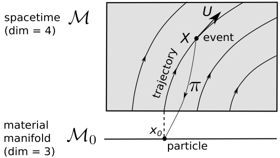
$$\boldsymbol{G}(\boldsymbol{A}(\vec{\boldsymbol{U}}_0), \vec{\boldsymbol{V}}) = \boldsymbol{G}_0(\vec{\boldsymbol{U}}_0, \boldsymbol{A}^*(\vec{\boldsymbol{V}}))$$

In a local chart, if $\boldsymbol{A}$ and $\boldsymbol{A}^*$ are represented by the matrices A and A^*

$$A^* = G^{-1} A^T G_0$$

where G_0 and G are Gram's matrices of $\boldsymbol{G}_0$ and $\boldsymbol{G}$

Reversible material : conformation tensor



line bundle : $\boldsymbol{\pi} : \mathcal{M} \rightarrow \mathcal{M}_0$
represented by : $x_0 = \pi(X)$

- **Conformation tensor** [Souriau 1964 Géométrie & Relativité]

$$\mathcal{D} = -\frac{\partial x_0}{\partial X}\left(\frac{\partial x_0}{\partial X}\right)^* = -\frac{\partial x_0}{\partial X}\, G^{-1}\left(\frac{\partial x_0}{\partial X}\right)^T G_0$$

where $\frac{\partial x_0}{\partial X} = (-F^{-1}\, v, F^{-1})$ with $F = \frac{\partial x}{\partial x_0}$

- For an **isotropic material**, $G_0 = 1_{\mathbb{R}^3}$
- mass density (Galilean invariant) $\rho = \frac{\rho_0(x_0)}{\det F}$
- $\mathcal{D} \cong F^{-1}\left(1_{\mathbb{R}^3} - \epsilon(v + A)\,(v + A)^T\right) F^{-T}$

Conformation tensor

Galilei general relativity

Hypothesis (Souriau) : the functional due to the matter is a function of the conformation tensor.

For the simplest matter (dust), we claim that

$$p_M \sqrt{-\det G} = c^2 \rho_0(x_0) \sqrt{\det \mathcal{D}(-\det G)} \cong c[\rho\, c^2 - \rho\,(\frac{\|v\|^2}{2} + A \cdot v - \phi)]$$

energy at rest — Lagrangian density

In Relativity, the three following phenomena

- the deformation (in the simplest form, contained in the density)
- the kinetic
- and the gravitation

are coupled through a unique tensor, the **conformation**

Conformation tensor

Poincaré special relativity

Comment : if $F = 1_{\mathbb{R}^3}$ and $\rho_0 = C^{te}$,
we recover the proper-time (exactly)

$$\mathcal{A}_M = c.m_0 c^2 \int \sqrt{1 - \parallel v \parallel^2 / c^2}\, dt$$

Galilei general relativity

More general case

$$p_M \sqrt{-\det G} = \left[c^2 \rho_0(x_0) \sqrt{\det \mathcal{D}} + \mathcal{W}(x_0, \mathcal{D}) \right] \sqrt{-\det G}$$

Neo-Hookean hyperelastic material $\mathcal{W}(x_0, \mathcal{D}) = \mu \, Tr(\mathcal{D}^{-1})$

$$p_M \sqrt{-\det G} \cong c[\rho \, c^2 - \rho \left(\frac{\|v\|^2}{2} + A \cdot v - \phi \right)] + \mu \, Tr(C)$$
$$+ \frac{\mu}{c^2} \left\{ \frac{\|F^T v\|^2}{1 - \|v\|^2 / c^2} + Tr(C) \left(\phi + \frac{\|A\|^2}{2} \right) \right\}$$

with $C = F^T F$

Dissipative material : friction tensor

Poincaré general relativity

- Relativistic Thermodynamics Jean-Marie Souriau
  Lect. Notes in Math. 676 (1976)

- Dissipative constitutive laws in Special Relativity
  Claude Vallée, IJES (1981)

Galilei general relativity

- **Book** with **Claude Vallée :**
  Galilean Mechanics and Thermodynamics of Continua
  (ISTE-Wiley, 2016)

Bargmannian transformations

Galilei general relativity

- The space-time $\mathcal{M}$ is embedded into a space $\hat{\mathcal{M}}$ of dimension 5 : $\mathcal{M} \to \hat{\mathcal{M}} \,:\, \boldsymbol{X} \mapsto \hat{\boldsymbol{X}} = \hat{f}\,(\boldsymbol{X})$
- We built a group of affine transformations $\hat{X}' \mapsto \hat{X} = \hat{P}\,\hat{X}' + \hat{C}$ of $\mathbb{R}^5$ which are Galilean when acting onto the space-time of which the linear part is :

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 \\ u & R & 0 \\ \frac{1}{2} \parallel u \parallel^2 & u^T R & 1 \end{pmatrix}$$

Their set is the **Bargmann's group**,

Temperature 5-vector

- The reciprocal temperature $\beta = 1/\theta$ is generalized as a Bargmannian 5-vector :

$$\hat{W} = \begin{pmatrix} W \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix} ,$$

where W is **Planck's temperature vector**
and ζ is **Planck's potential**

- The transformation law $\hat{W}' = \hat{P}^{-1} \hat{W}$ leads to the **normal form**

$$\hat{W} = \begin{pmatrix} \beta \\ \beta \, v \\ \zeta_{int} + \frac{\beta}{2} \parallel v \parallel^2 \end{pmatrix} .$$

Friction tensor

Friction tensor

The **friction tensor** is a linear map from $T_{\mathbf{X}}\mathcal{M}$ into itself, hence a **mixed tensor** $\boldsymbol{f}$ of rank 2

$$\boldsymbol{f} = \nabla \vec{\boldsymbol{W}}$$

represented by the 4×4 matrix $f = \nabla W$

$$= \begin{pmatrix} \frac{\partial \beta}{\partial t} & \frac{\partial \beta}{\partial x} \\ \frac{\partial}{\partial t}(\beta v) - \beta g + \Omega \times \beta v & \frac{\partial}{\partial x}(\beta v) + \beta j(\Omega) \end{pmatrix}$$

- This object introduced by Souriau merges the **temperature gradient** and the **strain velocity**
- In dimension 5, we can also introduce $\hat{\boldsymbol{f}} = \nabla \hat{\vec{\boldsymbol{W}}}$ represented by a 5×4 matrix $\hat{f} = \nabla \hat{W} = \begin{pmatrix} f \\ \nabla \zeta \end{pmatrix}$

Dual of the friction : momentum tensor

Momentum tensor

Linear map from $T_{\hat{f}(\boldsymbol{X})}\hat{\mathcal{M}}$ into $T_{\boldsymbol{X}}\mathcal{M}$, hence a **mixed tensor** $\hat{\boldsymbol{T}}$ of rank 2

it is represented by a 4×5 matrix of the form $\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ k & \sigma_\star & p \end{pmatrix}$

- ρ as the **density**
- $p = \rho\, v$ as the **linear momentum**
- $\sigma_\star = \sigma - \rho v v^T$ as the **dynamical stresses**
- $\mathcal{H} = \rho\, \left(e_{int} + \frac{1}{2} \parallel v \parallel^2 \right)$ as the **total energy**
- $k = \quad h \quad + \quad \mathcal{H} v \quad - \quad \sigma\, v$ as the **energy flux** by
  conduction convection stress

with :

- the **heat flux** $h = R\, h'$
- the **statical stresses** $\sigma = R\, \sigma'\, R^T$

Friction and momentum tensor

Theorem : reversible medium

if ζ is a function of

- the right Cauchy strain $C = F^T F$
- the temperature vector W
- and the Lagrangean coordinates x_0

then the 4×5 matrix $\hat{T}_R = (T_R, \rho U)$ with $T_R = U \otimes \Pi_R + \begin{pmatrix} 0 & 0 \\ -\sigma_R v & \sigma_R \end{pmatrix}$

where $\Pi_R = -\rho \, \frac{\partial \zeta}{\partial W}$ $\qquad \sigma_R = -\frac{2\rho}{\beta} \, F \, \frac{\partial \zeta}{\partial C} \, F^T$

is a momentum tensor such that :

$\diamondsuit$ $\boxed{Tr\left(\hat{T}_R \nabla \hat{W}\right) = 0}$ (justifies the terminology of friction tensor)

Second principle

- **Additive decomposition of the momentum tensor** $\hat{\mathrm{T}} = \hat{\mathrm{T}}_R + \hat{\mathrm{T}}_I$
- **Thermodynamics of irreversible processes (TIP)** : Planck's potential depends on extra internal variables (for instance the plastic strain F^p)

Covariant form of the second principle

The **local production of entropy** of a medium caracterized by a temperature vector $\hat{\vec{\boldsymbol{W}}}$ and a momentum tensor $\hat{\boldsymbol{T}}$ is non negative

$$\Phi = \boldsymbol{Div}\ \left(\hat{\boldsymbol{T}}\ \hat{\vec{\boldsymbol{W}}}\right) - \left(\boldsymbol{\tau}(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\ \left(\boldsymbol{\tau}(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right) \geq 0$$

and vanishes if and only if the process is reversible

[de Saxcé & Vallée IJES 2012]

- In the classical form, we recover **Clausius-Duhem inequality**
  $\Phi = \rho\ \frac{ds}{dt} - \frac{\rho}{\theta}\ \frac{dq_I}{dt} + div\ \left(\frac{h}{\theta}\right) \geq 0$

Dissipative constitutive laws

Theorem

If $\mathbf{Div}\,\hat{\mathbf{T}} = \mathbf{0}$ (First principle), the local production of entropy reads

$$\Phi = h \cdot grad\,\beta + \beta\, Tr(\sigma_I D)$$

that reveals the duality between
thermodynamic forces (or **affinities**) $a = grad\ \beta$, $A = \beta\, grad_s v = \beta\, D$
and corresponding **thermodynamic fluxes** h, σ_I

Assuming a linear isotropic law $(h, \sigma_I) = F(a, A)$,
we recover the heat conduction and Navier-Stokes equations

Epistemological reversal : coming back to the relativistic model

Poincaré general relativity

Relativistic form of the 2^{nd} principle

The **local production of entropy** of a medium characterized by a temperature vector $\vec{W}$ and a momentum tensor T is non negative :

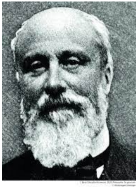
$$\Phi = Div\,(T\,\vec{W} + \zeta\,\vec{N}) - \frac{1}{c^2}\left(U^*(f(\vec{U}))\right)\frac{1}{c^2}\left(U^*(T_I(\vec{U}))\right) \geq 0\ ,$$

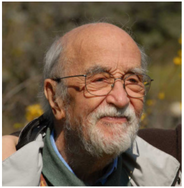
and vanishes if and only if the process is reversible

- **Dissipative constitutive laws :** see Claude Vallée, IJES (1981)
  In contrast to the Galilean case, there is a metric
  The momentum tensor T is self-adjoint and the friction too :
  $f = \frac{1}{2}\left[\nabla\,\vec{W} + (\nabla\,\vec{W})^*\right]$

Thank you !

This work is dedicated to Pierre Duhem





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