Construction of reduced-order models for solving numerically linear and nonlinear problems coming from transfer phenomenon.

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November 20, 2025

Geometry-dependent periodic homogenization

Motivation

Early degradation of the buildings

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- Chloride Cl^- penetration in the pores
- Decrease of PH in the interstitial solution, ignition of oxydo-reduction reactions
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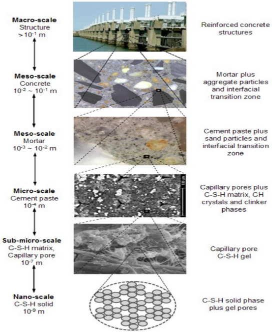
Multiscale heterogeneities



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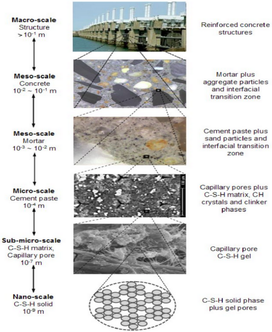

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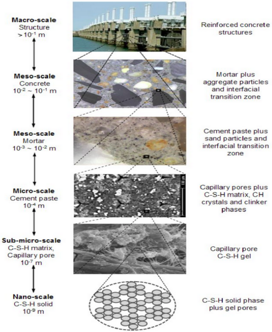

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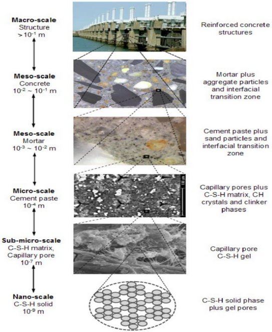
- Aim: affording this variability



Modelling ionic diffusion in porous media

Ionic diffusion : Nernst-Planck equations

$$\begin{cases} \dfrac{\partial c_\pm}{\partial t} - D_\pm \,\mathrm{div}\left(\nabla c_\pm + B\, c_\pm \nabla \Psi\right) & = & 0 & \text{in } \mathrm{Y_f} \\ -D_\pm \left(\nabla c_\pm + B\, c_\pm \nabla \Psi\right) \cdot \boldsymbol{n} & = & 0 & \text{on } \Gamma_{\mathrm{sf}} \end{cases}$$

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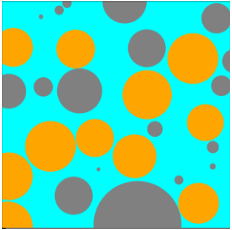
- Charge density $\rho = F(c_+ - c_-)$ for a pair (c_+, c_-)
- Electrochemical equilibrium : $\Psi = 0$ and $\nabla \Psi = \mathbf{0}$, we have the Fick law

Influence of the electrical double layer (EDL)

Microstructure



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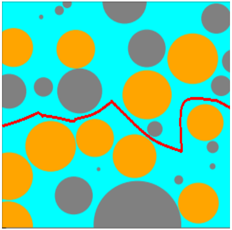

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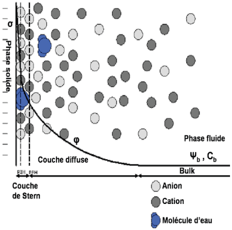
$\ell_{\mathrm{pores}} \leq \ell_{\mathrm{D}} \propto \frac{1}{\sqrt{C_{\mathrm{b}}}}$:



- $\Psi = \psi_{\mathrm{b}} + \varphi$
- $c_{\pm} = C_{\mathrm{b}}\, e^{\mp B\, \varphi}$
- Poisson-Boltzmann equation :

$$\begin{aligned} \epsilon_v \Delta \varphi - 2F\, C_{\mathrm{b}} \sinh\left(B\, \varphi\right) &= 0 \\ \epsilon_v \nabla \varphi \cdot \boldsymbol{n} &= \sigma. \end{aligned}$$

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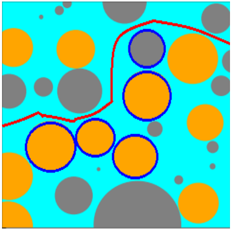

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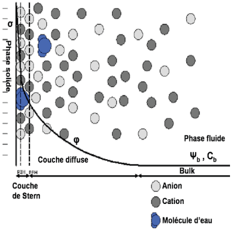

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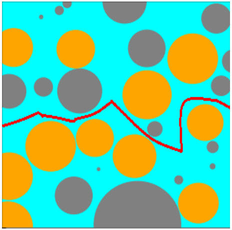
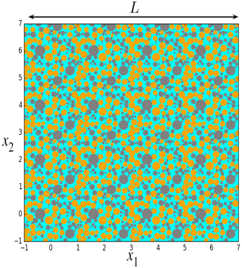


$\ell_{\rm pores} \gg \ell_{\rm D}$:

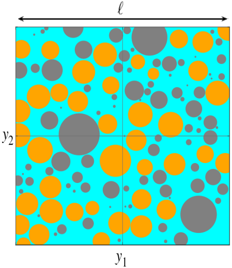
- $\varphi \simeq 0$
- $c_{\pm} = C_{\rm b}$
- Purely geometric effects

Periodic homogenization

Periodical microstructure



Periodical Elementary Cell



1. Scale separation: $\epsilon = \dfrac{\ell}{L} \ll 1$.

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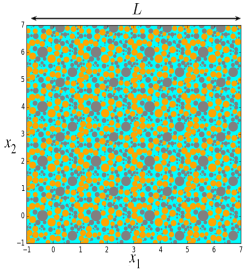

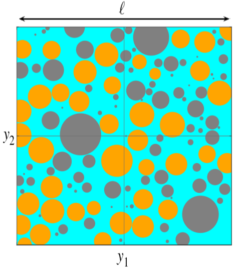

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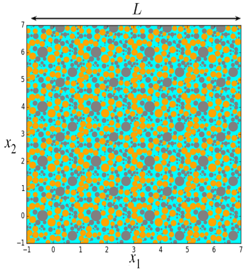

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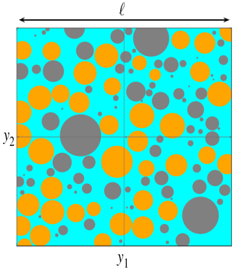

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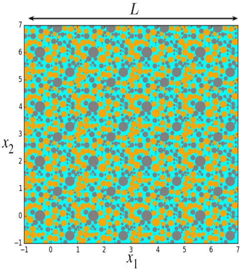

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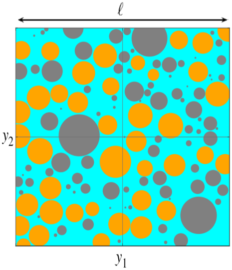

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4. Nested problems to solve: homogenized model

A multiscale problem (Bourbatache et al., 2012)

Homogenized equation

$$\varepsilon_{\mathbf{p}} \frac{\partial C_{\mathrm{b}} \left\langle e^{-B\varphi} \right\rangle_{\boldsymbol{y}}}{\partial t} - \mathrm{div}_{\boldsymbol{x}} \left(\boldsymbol{D}^{\mathrm{hom}} \left(\nabla_{\boldsymbol{x}} C_{\mathrm{b}} + B \; C_{\mathrm{b}} \nabla_{\boldsymbol{x}} \psi_{\mathrm{b}} \right) \right) = 0$$

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Homogenized diffusion tensor

$$\boldsymbol{D}^{\mathrm{hom}} = \frac{1}{|\mathrm{Y}|} \int_{\mathrm{Y}_f(\rho)} D e^{-\beta \varphi} \left(\boldsymbol{I} + \nabla_{\boldsymbol{y}} \boldsymbol{\chi}^{\mathsf{T}} \right) \mathrm{d}\boldsymbol{y} \quad \text{where} \quad \beta = \frac{F}{RT}$$

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Cell problem

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Local variable $\boldsymbol{\chi}$:

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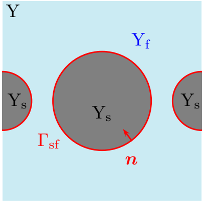

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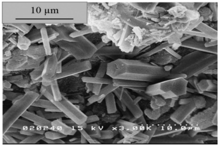

Jewell et al., 2015

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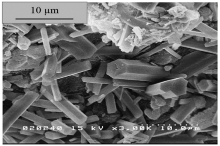

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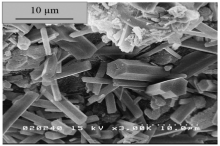

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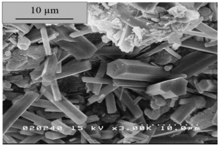

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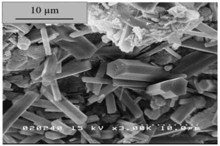

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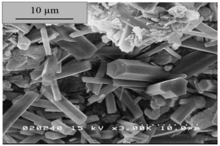

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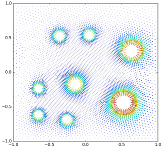
Principle of model order reduction



$\boldsymbol{\chi}(\rho_1^1, \rho_2^1)$

1. Computation of $\boldsymbol{\chi}$ and then $\boldsymbol{D}^{\mathrm{hom}}$ for (ρ_1^1, ρ_2^1), ..., (ρ_1^k, ρ_2^k)

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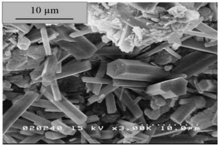

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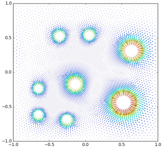

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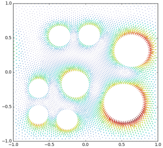

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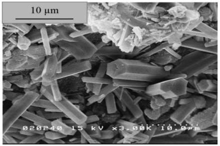

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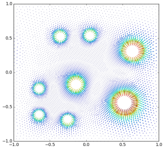

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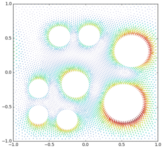

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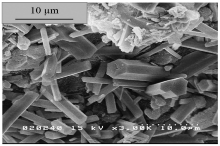

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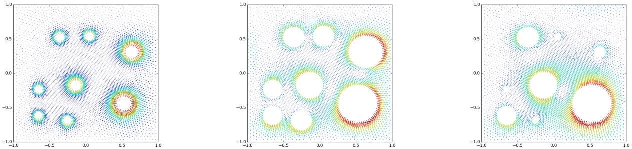

$\boldsymbol{\chi}(\rho_1^1, \rho_2^1)$ $\boldsymbol{\chi}(\rho_1^k, \rho_2^k)$ $\boldsymbol{\chi}(\rho_1^{new}, \rho_2^{new})$

1. Computation of χ and then $\boldsymbol{D}^{\mathrm{hom}}$ for $(\rho_1^1, \rho_2^1), \ldots, (\rho_1^k, \rho_2^k)$
2. Fast computation of $\boldsymbol{D}^{\mathrm{hom}}(\rho_1^{new}, \rho_2^{new})$

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Snapshots method (POD) :

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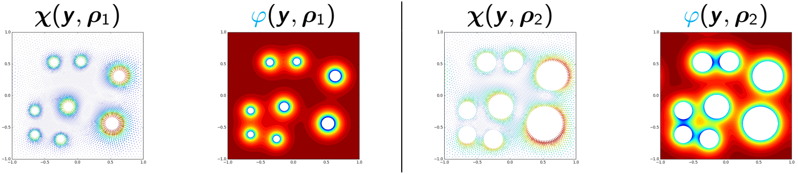
Snapshots method (POD) :

$$\left.\begin{array}{ll} \text{Particular solutions} & \mathbf{u}(\mathbf{x}, \boldsymbol{\mu}_1),\ \ldots,\ \mathbf{u}(\mathbf{x}, \boldsymbol{\mu}_{N_\mu}) \\ \text{Correlations tensor} & \left[\boldsymbol{C}\right]_{kj} = \dfrac{1}{N_\mu} \displaystyle\int_{\mathrm{Y}} \mathbf{u}(\boldsymbol{\mu}_k, \mathbf{x}')\mathbf{u}(\boldsymbol{\mu}_j, \mathbf{x}')\,\mathrm{d}\mathbf{x}' \end{array}\right\} \xrightarrow[\text{of } \boldsymbol{C}]{\text{Reduction}} (\phi_i)_i$$

Issue : ROM construction for the parametrized cell problem depending on ρ

Snapshots for two different cells : $\rho_1 \neq \rho_2$



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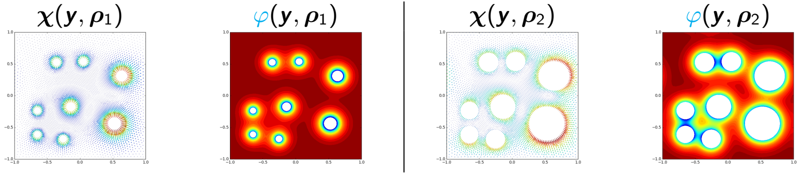
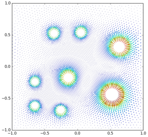


Correlations tensor puisque $\boldsymbol{\rho}_1 \neq \boldsymbol{\rho}_2$:

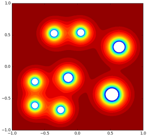
Issue : ROM construction for the parametrized cell problem depending on ρ

Snapshots for two different cells : $\boldsymbol{\rho}_1 \neq \boldsymbol{\rho}_2$

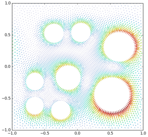
$\boldsymbol{\chi}(\boldsymbol{y}, \boldsymbol{\rho}_1)$



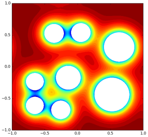
$\varphi(\boldsymbol{y}, \boldsymbol{\rho}_1)$



$\boldsymbol{\chi}(\boldsymbol{y}, \boldsymbol{\rho}_2)$



$\varphi(\boldsymbol{y}, \boldsymbol{\rho}_2)$



Correlations tensor puisque $\boldsymbol{\rho}_1 \neq \boldsymbol{\rho}_2$:

$$[\boldsymbol{C}^{\varphi}]_{12} = \int_{\Omega} \varphi(\boldsymbol{y}; \boldsymbol{\rho}_1)\varphi(\boldsymbol{y}; \boldsymbol{\rho}_2)\,\mathrm{d}\Omega \quad \text{et} \quad [\boldsymbol{C}^{\boldsymbol{\chi}}]_{12} = \int_{\Omega} \boldsymbol{\chi}(\boldsymbol{y}; \boldsymbol{\rho}_1) \cdot \boldsymbol{\chi}(\boldsymbol{y}; \boldsymbol{\rho}_2)\,\mathrm{d}\Omega$$

are not defined

Issue : ROM construction for the parametrized cell problem depending on ρ

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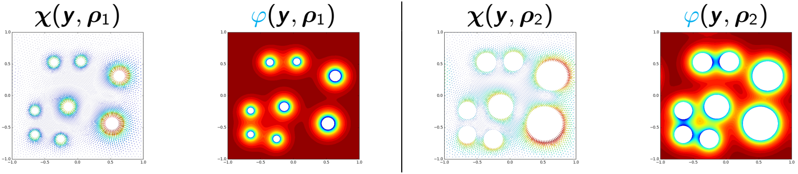

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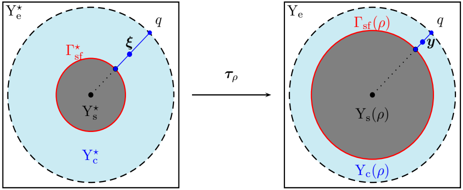
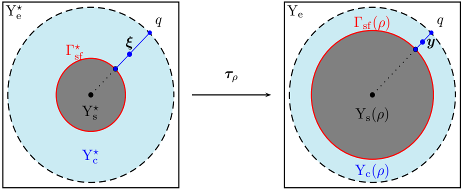
are not defined

POD basis $(\phi_i^{\varphi})_i$ et $(\phi_i^{\chi})_i$: cannot be built

Parametrized transformation $\boldsymbol{\tau}_\rho : \mathrm{Y}^\star \to \mathrm{Y}$



Parametrized transformation $\boldsymbol{\tau}_\rho \colon \mathrm{Y}^\star \to \mathrm{Y}$



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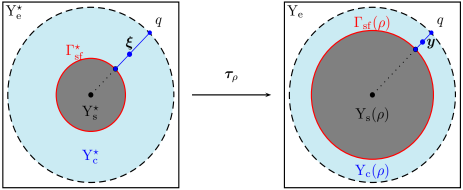

Direct transformation : $\boldsymbol{\tau}_{\rho_\star,\rho}(\boldsymbol{\xi}) = \beta_\rho \boldsymbol{\xi} + \alpha_\rho \dfrac{\boldsymbol{\xi}}{\| \boldsymbol{\xi} \|}$

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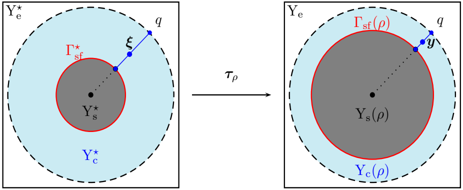

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Dependency in ρ : $\alpha_\rho = q\dfrac{\rho - \rho_\star}{q - \rho_\star}$ et $\beta_\rho = \dfrac{q - \rho}{q - \rho_\star}$

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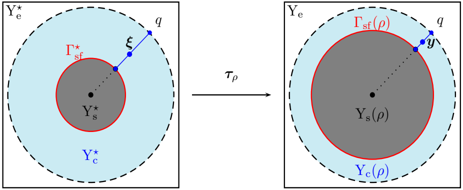

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Inverse transformation : $\boldsymbol{\tau}^{-1}_{\rho_\star,\rho}(\boldsymbol{y}) = \boldsymbol{\tau}_{\rho,\rho_\star}(\boldsymbol{y}) = \frac{1}{\beta_\rho}\boldsymbol{y} - \dfrac{\alpha_\rho}{\beta_\rho}\dfrac{\boldsymbol{y}}{\|\boldsymbol{y}\|}$

Cell problem (EDL neglected)

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- $$\int_{\mathrm{Y_f}(\rho)} \nabla_{\boldsymbol{y}} \boldsymbol{\chi} : \nabla_{\boldsymbol{y}} \boldsymbol{v} \mathrm{d}\boldsymbol{y} = -\int_{\Gamma_{\mathrm{sf}}} \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{y}$$

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- $\displaystyle\boldsymbol{D}^{\mathrm{hom}}(\rho) = D\varepsilon_{\mathbf{p}}\boldsymbol{I} + \frac{1}{|\mathrm{Y}|}\int_{\mathrm{Y}_f(\rho)} D\nabla_{\boldsymbol{y}}\boldsymbol{\chi}^{\intercal}\,\mathrm{d}\boldsymbol{y}$

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Reformulation on the reference domain $\mathrm{Y}^{\star}$

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Reformulation on the reference domain $\mathrm{Y}^\star$

- $$\int_{\mathrm{Y}_{\mathrm{c}}^\star} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \, \boldsymbol{J}_\rho^{\,-2} j_\rho \right) : \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_\star \, \mathrm{d}\boldsymbol{\xi} + \int_{\mathrm{Y}_{\mathrm{e}}^\star} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \right) : \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_\star \, \mathrm{d}\boldsymbol{\xi} = - \int_{\Gamma_{\mathrm{sf}}^\star} \boldsymbol{n} \cdot \boldsymbol{v}_\star \, g_\rho \, \mathrm{d}\boldsymbol{\xi}$$

Cell problem (EDL neglected)

- $$\int_{Y_f(\rho)} \nabla_{\boldsymbol{y}} \boldsymbol{\chi} : \nabla_{\boldsymbol{y}} \boldsymbol{v} \mathrm{d}\boldsymbol{y} = -\int_{\Gamma_{sf}} \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{y}$$
- $$\boldsymbol{D}^{\mathrm{hom}}(\rho) = D\varepsilon_{\mathbf{p}} \boldsymbol{I} + \frac{1}{|Y|} \int_{Y_f(\rho)} D \nabla_{\boldsymbol{y}} \boldsymbol{\chi}^{\intercal} \, \mathrm{d}\boldsymbol{y}$$

Reformulation on the reference domain $Y^\star$

- $$\int_{Y_c^\star} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \, \boldsymbol{J}_\rho^{-2} j_\rho \right) : \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_\star \, \mathrm{d}\boldsymbol{\xi} + \int_{Y_e^\star} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \right) : \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_\star \, \mathrm{d}\boldsymbol{\xi} = -\int_{\Gamma_{sf}^\star} \boldsymbol{n} \cdot \boldsymbol{v}_\star \, g_\rho \, \mathrm{d}\boldsymbol{\xi}$$
- $$\boldsymbol{D}^{\mathrm{hom}}(\rho) = D\varepsilon_{\mathbf{p}} \boldsymbol{I} + \int_{Y_c^\star} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \, \boldsymbol{J}_\rho^{-1} j_\rho \right)^{\intercal} \mathrm{d}\boldsymbol{\xi} + \int_{Y_e^\star} \nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star^{\intercal} \, \mathrm{d}\boldsymbol{\xi}$$

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Reformulation on the reference domain $\mathrm{Y}^\star$

- $$\int_{\mathrm{Y}_\mathrm{c}^\star} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \, \boldsymbol{J}_\rho^{-2} j_\rho \right) : \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_\star \, \mathrm{d}\boldsymbol{\xi} + \int_{\mathrm{Y}_\mathrm{e}^\star} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \right) : \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_\star \, \mathrm{d}\boldsymbol{\xi} = - \int_{\Gamma_{\mathrm{sf}}^\star} \boldsymbol{n} \cdot \boldsymbol{v}_\star \, g_\rho \, \mathrm{d}\boldsymbol{\xi}$$
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Condition of ROM efficiency

Separate variables ρ and $\boldsymbol{\xi}$ in $\boldsymbol{J}_\rho^{-1} j_\rho$, $\boldsymbol{J}_\rho^{-2} j_\rho$, j_ρ and g_ρ

Differentiation of $\boldsymbol{\tau}_\rho$

- $\boldsymbol{J}_\rho(\boldsymbol{\xi}) = \beta_\rho \boldsymbol{I} + \alpha_\rho \nabla_{\boldsymbol{\xi}} \boldsymbol{u}(\boldsymbol{\xi})$ où $\boldsymbol{G}_{\boldsymbol{u}(\boldsymbol{\xi})} = \boldsymbol{I} - \frac{1}{\|\boldsymbol{\xi}\|^2} \boldsymbol{\xi} \boldsymbol{\xi}^\mathsf{T}$
- $\boldsymbol{J}_\rho^{-1}(\boldsymbol{\xi}) = \frac{1}{\beta_\rho} \boldsymbol{I} - \frac{\alpha_\rho}{\beta_\rho \left(\beta_\rho \|\boldsymbol{\xi}\| + \alpha_\rho\right)} \boldsymbol{G}_{\boldsymbol{u}(\boldsymbol{\xi})}$
- $j_\rho(\boldsymbol{\xi}) = \beta_\rho \dfrac{\left(\alpha_\rho + \beta_\rho \|\boldsymbol{\xi}\|\right)^{d-1}}{\|\boldsymbol{\xi}\|^{d-1}}$
- $g_\rho = \left(\dfrac{\rho}{\rho_\star}\right)^{d-1}$

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Issue : $\boldsymbol{J}_\rho^{-2} j_\rho$

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Issue : $\boldsymbol{J}_\rho{}^{-2} j_\rho$

Dimension 3 : $\boldsymbol{J}_\rho{}^{-2} j_\rho = \left(\beta_\rho + \dfrac{2\alpha_\rho}{\|\boldsymbol{\xi}\|} + \dfrac{\alpha_\rho{}^2}{\beta_\rho\|\boldsymbol{\xi}\|^2}\right)\boldsymbol{I} - \dfrac{2\alpha_\rho}{\|\boldsymbol{\xi}\|}\boldsymbol{G}_{\boldsymbol{u}(\boldsymbol{\xi})} - \dfrac{\alpha_\rho{}^2}{\beta_\rho\|\boldsymbol{\xi}\|^2}\boldsymbol{G}_{\boldsymbol{u}(\boldsymbol{\xi})}$

Differentiation of $\boldsymbol{\tau}_\rho$

- $\boldsymbol{J}_\rho(\boldsymbol{\xi}) = \beta_\rho \boldsymbol{I} + \alpha_\rho \nabla_{\boldsymbol{\xi}} \boldsymbol{u}(\boldsymbol{\xi})$ où $\boldsymbol{G}_{\boldsymbol{u}(\boldsymbol{\xi})} = \boldsymbol{I} - \frac{1}{\|\boldsymbol{\xi}\|^2} \boldsymbol{\xi}\boldsymbol{\xi}^\mathsf{T}$
- $\boldsymbol{J}_\rho^{-1}(\boldsymbol{\xi}) = \frac{1}{\beta_\rho}\boldsymbol{I} - \frac{\alpha_\rho}{\beta_\rho\left(\beta_\rho\|\boldsymbol{\xi}\|+\alpha_\rho\right)} \boldsymbol{G}_{\boldsymbol{u}(\boldsymbol{\xi})}$
- $j_\rho(\boldsymbol{\xi}) = \beta_\rho \dfrac{\left(\alpha_\rho + \beta_\rho\|\boldsymbol{\xi}\|\right)^{d-1}}{\|\boldsymbol{\xi}\|^{d-1}}$
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Dimension 2 : No exact formula, we develop $\dfrac{1}{(\beta_\rho\|\boldsymbol{\xi}\| + \alpha_\rho)}$ in a power series

Reduced Order Model

$$\begin{cases} \widehat{\chi}_\star(\boldsymbol{\xi};\rho) &= \sum_{j=1}^{n_\chi} a_j(\rho)\,\phi_j(\boldsymbol{\xi})\,, \\ \boldsymbol{A}_\rho\,\boldsymbol{a}(\rho) &= \boldsymbol{c}_\rho\,, \\ \widehat{\boldsymbol{D}}^{\mathrm{hom}}(\rho) &= D\left(\varepsilon_{\mathbf{p}}\boldsymbol{I} + \sum_{k=1}^{n_\chi} a_j(\rho)\boldsymbol{K}_\rho\left(\phi_j\right)\right) \end{cases}$$

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ROM coefficients

Matrix $\boldsymbol{A}_\rho$: $\boldsymbol{A}_\rho = \overline{A} + \Xi_0(\rho)\,A_0 + \Xi_1(\rho)\,A_1 + \ldots$

Vector $\boldsymbol{c}_\rho$: $\boldsymbol{c}_\rho = g_\rho\,\overline{c}$

Tensor $\boldsymbol{K}_\rho$: $\boldsymbol{K}_\rho = \overline{K(\phi_j)} + \Theta_0(\rho)\,K(\phi_j)_0 + \Theta_1(\rho)\,K(\phi_j)_1 + \ldots$

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- $\left[\overline{\boldsymbol{A}}\right]_{jk} = \int_{\mathrm{Y}^\star_{\mathrm{e}}} \nabla_{\boldsymbol{\xi}}\boldsymbol{\phi}_k : \nabla_{\boldsymbol{\xi}}\boldsymbol{\phi}_j \,\mathrm{dY}^\star, \dots \;,\; \left[\boldsymbol{A}_1\right]_{jk} = \int_{\mathrm{Y}^\star_{\mathrm{c}}} \nabla_{\boldsymbol{\xi}}\boldsymbol{\phi}_k\, \boldsymbol{G}_{\boldsymbol{u}(\boldsymbol{\xi})} : \nabla_{\boldsymbol{\xi}}\boldsymbol{\phi}_j \,\mathrm{dY}^\star, \dots$
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Weak equation of r :

$$\epsilon_v \int_{\mathrm{Y}_{\mathrm{f}}} r^{-1}\nabla r\cdot\nabla q\mathrm{dY} + C_{\mathrm{b}}B\int_{\mathrm{Y}_{\mathrm{f}}} rq\mathrm{dY} - C_{\mathrm{b}}B\int_{\mathrm{Y}_{\mathrm{f}}} r^{-1}q\mathrm{dY} + B\int_{\Gamma_{\mathrm{sf}}}\sigma q\mathrm{d}\Gamma_{\mathrm{sf}} = 0 \quad \forall q \in W_\rho$$

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$$\left\{\begin{array}{rcl} \epsilon_v \int\limits_{\mathrm{Y_f}} r^{-1}\nabla r \cdot \nabla q \,\mathrm{dY} + C_{\mathrm{b}} B \int\limits_{\mathrm{Y_f}} r q \mathrm{dY} & = & C_{\mathrm{b}} B \int\limits_{\mathrm{Y_f}} r^{-1} q \mathrm{dY} - B \int\limits_{\Gamma_{\mathrm{sf}}} \sigma q \mathrm{d}\Gamma_{\mathrm{sf}} \\ r r^{-1} & = & 1 \end{array}\right.$$

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Updating with ρ : $\boldsymbol{A}^{*}_{\rho}$, $\boldsymbol{D}_{\rho}$, $\boldsymbol{D}^{*}_{\rho}$, $\boldsymbol{f}_{\rho}$, $\boldsymbol{C}^{*}_{\rho}$ and $\boldsymbol{h}^{*}_{\rho}$ depend explicitely on ρ

ROM for χ knowing $r \simeq \sum_{i=1}^{n_r} b^i \phi_i^r$

$$\text{FOM : } \int_{\mathrm{Y_f}} r \nabla_{\boldsymbol{y}} \boldsymbol{\chi} : \nabla_{\boldsymbol{y}} \boldsymbol{v} \mathrm{dY} = \int_{\mathrm{Y_f}} \nabla_{\boldsymbol{y}} r \cdot \boldsymbol{v} \mathrm{dY} - \int_{\Gamma_{\mathrm{sf}}} r \boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}\Gamma_{\mathrm{sf}}$$

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Explicit dependency of ρ : same thing than the ROM of Boltzmann factor r

Summary of the ROM construction

Offline

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1. Compute snapshots $\boldsymbol{\chi}_\star(\boldsymbol{\xi}, \rho_1), \ldots, \boldsymbol{\chi}_\star(\boldsymbol{\xi}, \rho_{n_{\mathrm{snap}}})$ et $\varphi_\star(\boldsymbol{\xi}, \rho_1), \ldots, \varphi_\star(\boldsymbol{\xi}, \rho_{n_{\mathrm{snap}}})$

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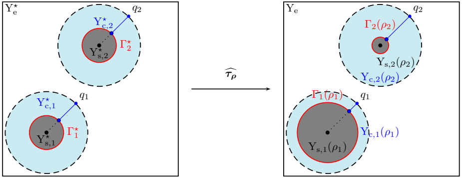
3. Computation in a reduced time of tensor $\widehat{\boldsymbol{D}}^{\mathrm{hom}}(\rho)$ estimated by ROM

Generalization to multiple inclusions

Piecewise defined transformations



a) Reference elementary cell   b) Actual elementary cell

Generalization to multiple inclusions

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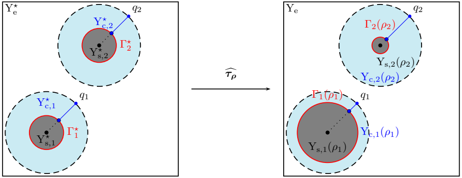

a) Reference elementary cell

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$$\widehat{\boldsymbol{\tau}_{\boldsymbol{\rho}}}(\boldsymbol{\xi}) = \begin{cases} \boldsymbol{\xi}_n + \alpha_{\rho_n}\boldsymbol{u}(\boldsymbol{\xi}-\boldsymbol{\xi}_n) + \beta_{\rho_n}(\boldsymbol{\xi}-\boldsymbol{\xi}_n), & \text{if } \boldsymbol{\xi} \in \mathrm{Y}_{\mathrm{c},n}^\star, \text{ for } \ 1 \le n \le n_{\mathrm{s}}, \\ \boldsymbol{\xi}, & \text{if } \boldsymbol{\xi} \in \overline{\mathrm{Y}_{\mathrm{e}}^\star}, \end{cases}$$

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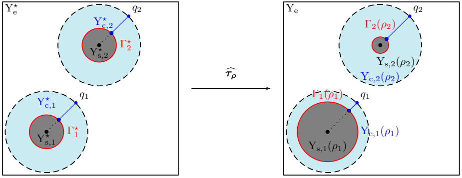

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Affine dependency on the parameters

$$\boldsymbol{A}^{\chi}_{\rho\, jki} = \overline{\boldsymbol{A}} + \quad \Xi_0((\rho)\, \boldsymbol{A}_0 \quad + \quad \Xi_1((\rho)\, \boldsymbol{A}_1 \quad + \dots$$

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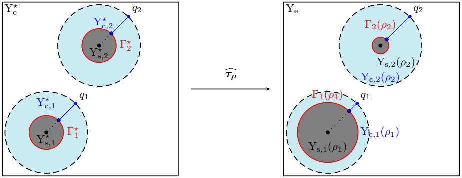

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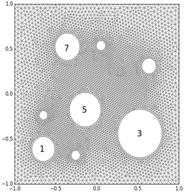
Numerical application : two varying parameters

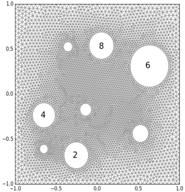
Bidimensional cell : 8 inclusions with two parameters ϱ_1 et ϱ_2

$(\rho_1,\ \rho_2) = (0.1387,\ 0.0547)$



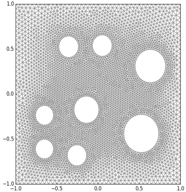
a) Example of geometry

$(\rho_1,\ \rho_2) = (0.0511,\ 0.1485)$



b) Example of geometry

$(\rho_{\star,1}, \rho_{\star,2}) = (0.109,\ 0.109)$

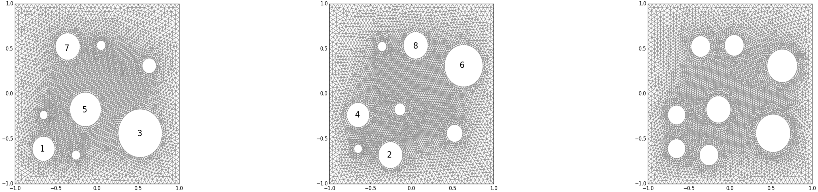


c) Reference geometry

Numerical application : two varying parameters

Bidimensional cell : 8 inclusions with two parameters ϱ_1 et ϱ_2

$(\rho_1,\ \rho_2) = (0.1387,\ 0.0547)$ $(\rho_1,\ \rho_2) = (0.0511,\ 0.1485)$ $(\rho_{\star,1}, \rho_{\star,2}) = (0.109,\ 0.109)$



a) Example of geometry b) Example of geometry c) Reference geometry

- Radii $\rho_n,\ n = 1, 3, 5, 7$ are parametrized by ϱ_1 such as $\rho_n = \varrho_1\, \rho_{\star,n}$;

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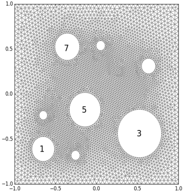

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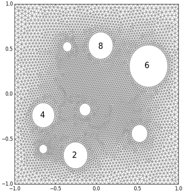

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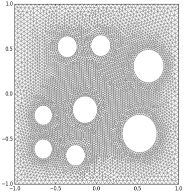
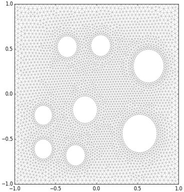


c) Reference geometry

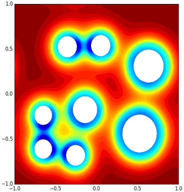
- Radii ρ_n, $n = 1, 3, 5, 7$ are parametrized by ϱ_1 such as $\rho_n = \varrho_1\, \rho_{\star,n}$;
- Radii ρ_n, $n = 2, 4, 6, 8$ are parametrized by ϱ_2 such as $\rho_n = \varrho_2\, \rho_{\star,n}$.

Examples of snapshots

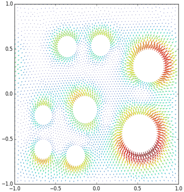
Reference mesh



Field $\varphi_\star$

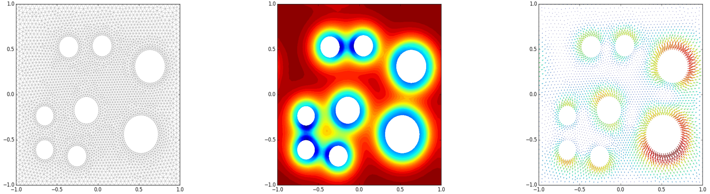


Field $\boldsymbol{\chi}_\star$

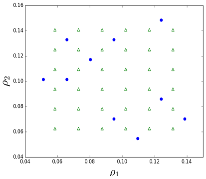


Examples of snapshots

Reference mesh — Field $\varphi_\star$ — Field $\chi_\star$

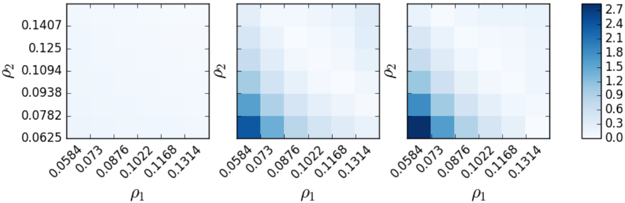


10 snapshots : LHS method (Latin Square Hyperbube)



Results for a bidimensional cell

$$\boldsymbol{D}^{\mathrm{hom}} = s^{\mathrm{hom}}\,\boldsymbol{I} + \boldsymbol{T}^{\mathrm{hom}} \quad \text{where} \quad \begin{cases} s^{\mathrm{hom}} &= \dfrac{D}{|\mathrm{Y}|}\displaystyle\int_{\mathrm{Y_f}} e^{-B\,\varphi}\,\mathrm{dY} \\ \boldsymbol{T}^{\mathrm{hom}} &= \dfrac{D}{|\mathrm{Y}|}\displaystyle\int_{\mathrm{Y_f}} e^{-B\,\varphi}\,\nabla_{\boldsymbol{y}}\,\boldsymbol{\chi}^{\mathsf{T}}\,\mathrm{dY} \end{cases}$$



$a)$ s^{hom} $b)$ T_{11}^{hom} $c)$ T_{22}^{hom}

- Maximum error of 0.13% on s^{hom}

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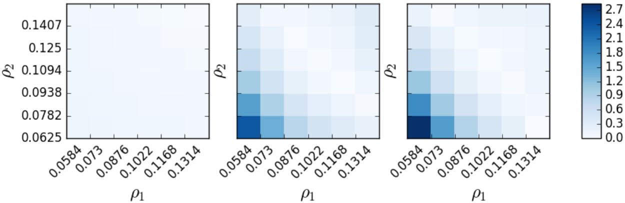

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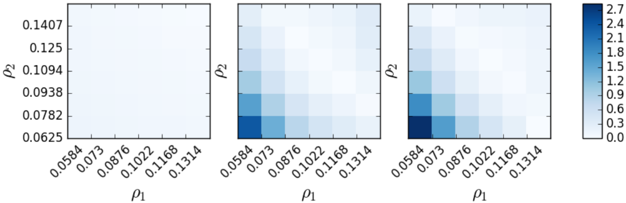

a) s^{hom} b) T_{11}^{hom} c) T_{22}^{hom}

- Maximum error of 0.13% on s^{hom}
- Maximum error of 2.7% on $\boldsymbol{T}^{\mathrm{hom}}$
- The proposed approach is valid for multiparametric domains

ROM with additional parameters

Sensitivity of the EDL's physical parameters

Diffusion tensor

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- ROM is built for fixed values $(C_{\mathrm{b}0}, \sigma_0)$ of each physical parameter

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Issue

- ROM is built for fixed values $(C_{\mathrm{b}0}, \sigma_0)$ of each physical parameter
- A POD basis is only valuable for values (C_{b}, σ) close to the construction ones

Interpolation of POD basis

1. Construction of POD basis for different values $(C_{\mathrm{b}k}, \sigma_k)$ of pair (C_{b}, σ)

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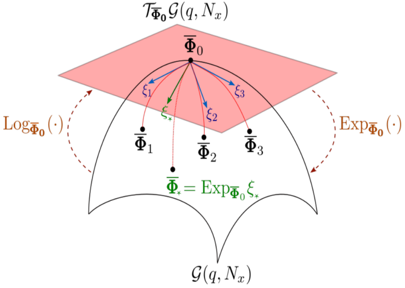
Direct interpolation
Gives bad results

Interpolation on the tangent space
to the Grassmann manifold



Oulghelou et al., 2021

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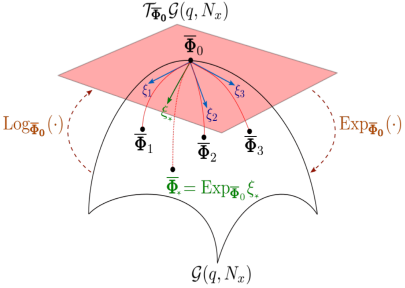
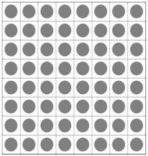


Oulghelou et al., 2021

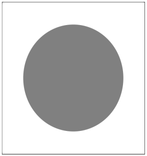
3. Construction and resolution of ROM

Numerical application to a bidimensional monoparametric cell

Periodic microstructure

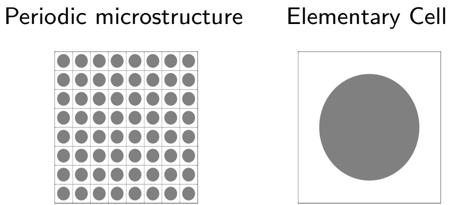


Elementary Cell



Construction of ROMs

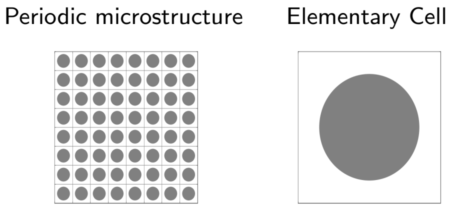
Numerical application to a bidimensional monoparametric cell



Construction of ROMs

- 6 snapshots, with $\rho = 0.45, 0.55, \ldots 0.95$

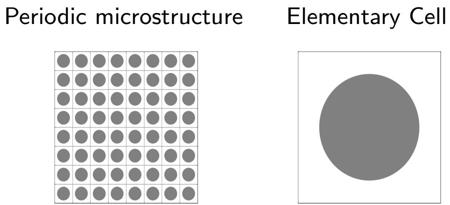
Numerical application to a bidimensional monoparametric cell



Construction of ROMs

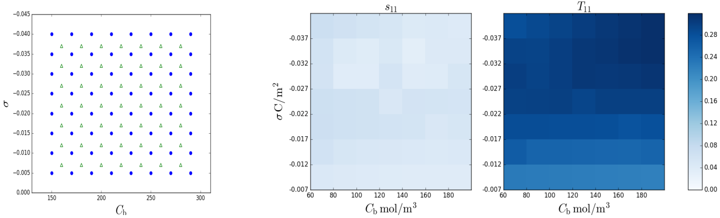
- 6 snapshots, with $\rho = 0.45, 0.55, \ldots 0.95$
- Reference radius $\rho_\star = 0.8$

Numerical application to a bidimensional monoparametric cell

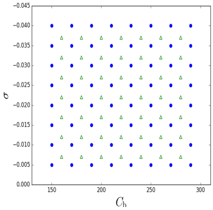


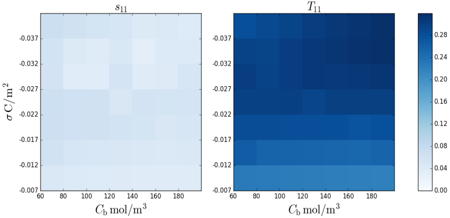
Construction of ROMs

- 6 snapshots, with $\rho = 0.45, 0.55, \ldots 0.95$
- Reference radius $\rho_\star = 0.8$
- 4 POD modes for each (C_b, σ)

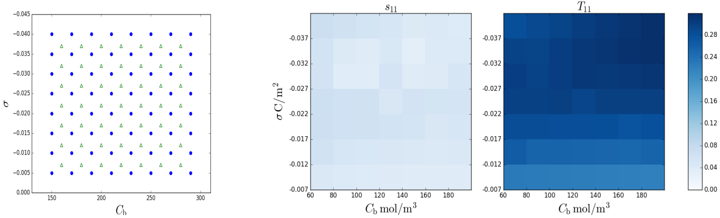


- POD-ROM built for 64 values of pair (C_b, σ)





- POD-ROM built for 64 values of pair (C_b, σ)
- Simulation on 49 values of $(\widetilde{C_b}, \widetilde{\sigma})$



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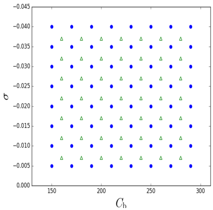

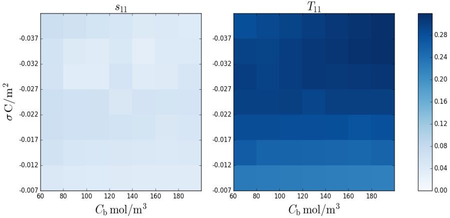

- POD-ROM built for 64 values of pair (C_b, σ)
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- Maximum error : $< 1\%$
- Fast computation compared to FEM

Multiparametric sampling (MPS)

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The key feature: Poisson-Boltzmann equation

$$
\begin{aligned}
\epsilon_v \Delta \varphi - 2F\, C_\mathrm{b} \sinh\left(B\, \varphi\right) &= 0 \\
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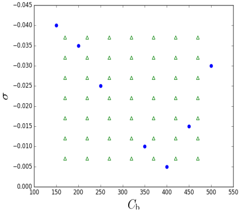
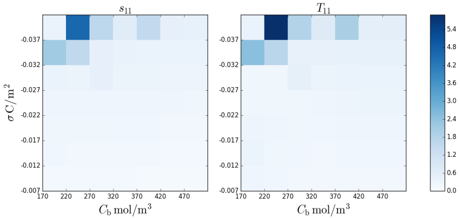
Affine dependency is still possible with σ and C_{b}

Numerical application: a bidimensionnal cell depending on a single geometric parameter





POD-ROM construction :

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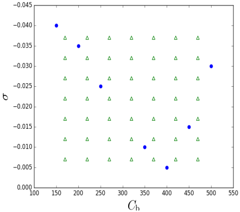

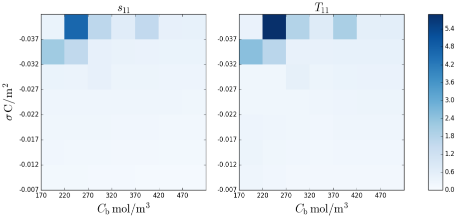

POD-ROM construction :

- 6 radii values $\rho_k = 0.45, 0.55, \ldots 0.95 \ldots$

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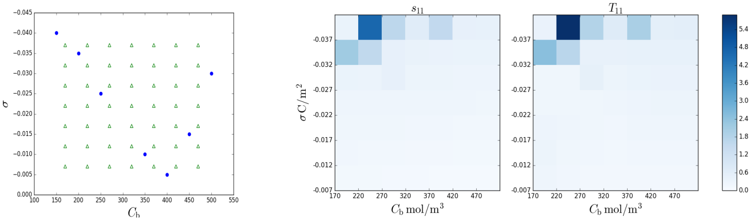

POD-ROM construction :

- 6 radii values $\rho_k = 0.45, 0.55, \ldots 0.95 \ldots$
- ... per 7 valeurs du couple (C_b, σ)

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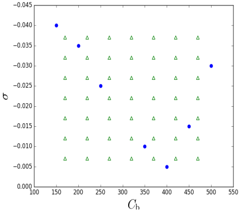

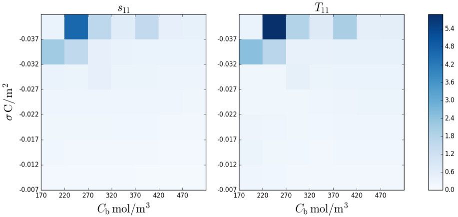

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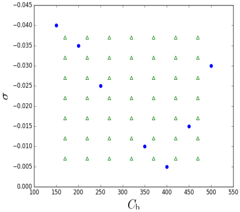

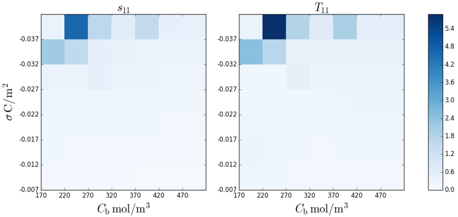

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Validation :

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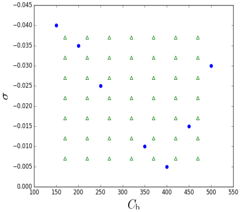

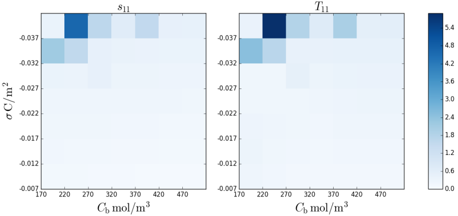

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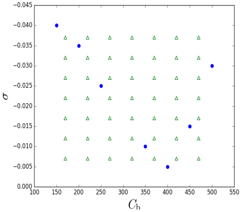

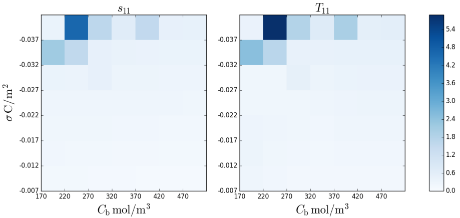

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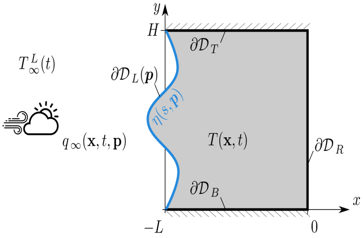
Validation :

- 49 values of $(\widetilde{C_\mathrm{b}}, \widetilde{\sigma})$
- Errors on s^{hom} and T_{11}^{hom} less than 3% for 45 values of (C_b, σ)

Perspectives

The TOPS project (with Julien Berger CR CNRS)

Shape optimization



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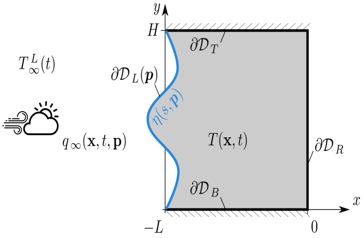
Shape optimization

Wall shape : $\mathcal{D}(\mathbf{p})$ for some $\mathbf{p}$



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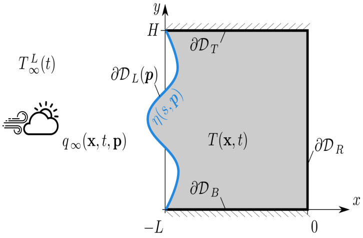

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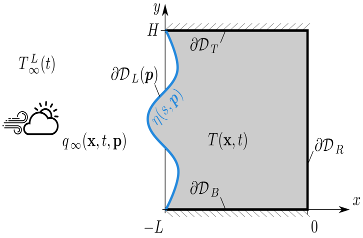

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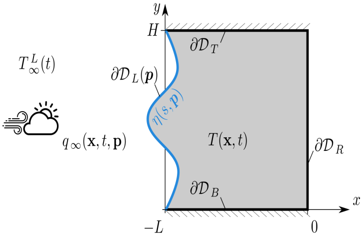

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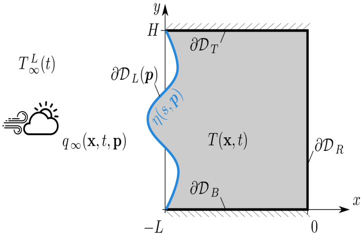

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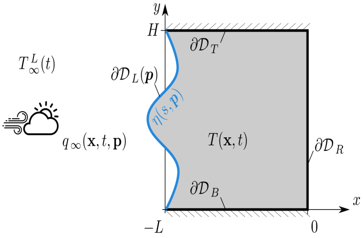

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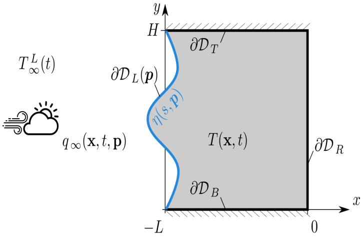

Implementation

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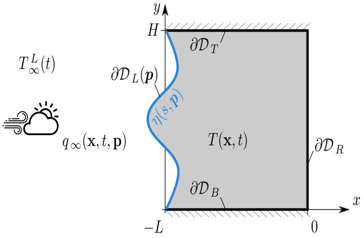

Implementation

- Meteorological model for q_∞, T^L_∞

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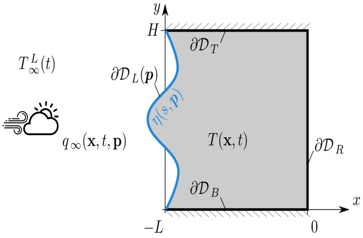

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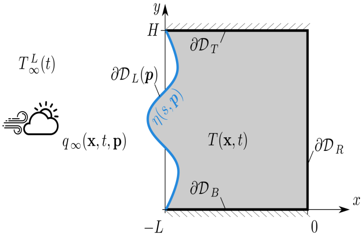

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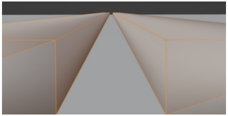
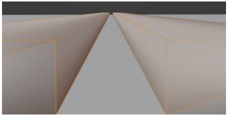
- $T(\mathbf{x}, t)$ computed with the Finite Element Method

Simulation over a month



Simulation over a month

- Urban environment: a canyon street



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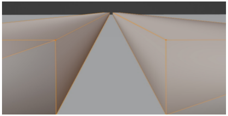
- Urban environment: a canyon street
- $\mathcal{C}(\mathbf{p})$: outdoor heat transfer



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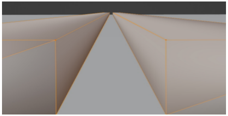
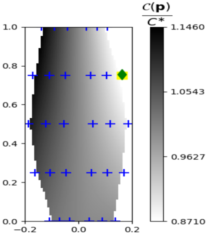
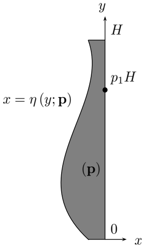


Result : december 2022





Simulation over a month

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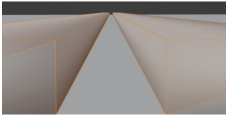

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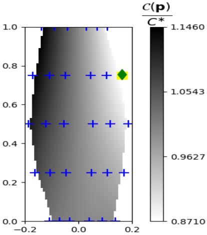

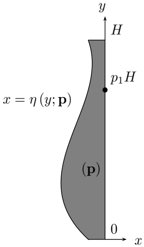

- Outcoming heat flux : $\simeq -13\%$ over a month

Simulation over a month

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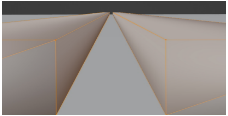

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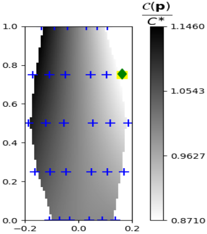

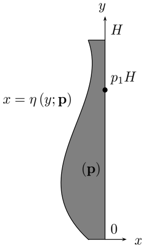

- Outcoming heat flux : $\simeq -13\%$ over a month
- MC simulation of q_∞: 90% of the CPU cost

Real-time shape control

Parametrized POD

Real-time shape control

Parametrized POD

- $T(\mathbf{x}, t; \ , \mathbf{p}) = \sum_{i=1}^{n_{\mathrm{rom}}} a_i(t)\theta_i(\mathbf{x}; \ \mathbf{p})$

Real-time shape control

Parametrized POD

- $T(\mathbf{x}, t; , \mathbf{p}) = \sum_{i=1}^{n_{\text{rom}}} a_i(t)\boldsymbol{\theta}_i(\mathbf{x};\ \mathbf{p})$
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ROM: summary

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ROM: summary

Offline For $\mathbf{p}_j$ in $\mathbf{p}_1, \ldots, \mathbf{p}_N$

Real-time shape control

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- Compute $q_\infty(\mathbf{s}, t; , \mathbf{p}_j)$ and $T(\mathbf{x}, t; , \mathbf{p})$ on a reference domain $\mathcal{D}_{\mathrm{ref}}$

Real-time shape control

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- Compute $q_\infty(\mathbf{s}, t; , \mathbf{p}_j)$ and $T(\mathbf{x}, t; , \mathbf{p})$ on a reference domain $\mathcal{D}_{\mathrm{ref}}$
- Build POD basis $\boldsymbol{\theta}_i(\cdot,\ \mathbf{p}_j)$ on $\mathcal{D}_{\mathrm{ref}}$

Online For a given $\mathbf{p}^{\mathrm{new}}$

Real-time shape control

Parametrized POD

- $T(\mathbf{x}, t; , \mathbf{p}) = \sum_{i=1}^{n_{\mathrm{rom}}} a_i(t)\boldsymbol{\theta}_i(\mathbf{x};\ \mathbf{p})$
- $\mathcal{C}(\mathbf{p})$: computed by the ROM (real time)

ROM: summary

Offline For $\mathbf{p}_j$ in $\mathbf{p}_1, \ldots, \mathbf{p}_N$

- Compute $q_\infty(\mathbf{s}, t; , \mathbf{p}_j)$ and $T(\mathbf{x}, t; , \mathbf{p})$ on a reference domain $\mathcal{D}_{\mathrm{ref}}$
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Online For a given $\mathbf{p}^{\mathrm{new}}$

- Build the new POD basis $\boldsymbol{\theta}_i(\cdot,\ \mathbf{p}^{\mathrm{new}})$

Real-time shape control

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Online For a given $\mathbf{p}^{\mathrm{new}}$

- Build the new POD basis $\boldsymbol{\theta}_i(\cdot, \; \mathbf{p}^{\mathrm{new}})$
- Apply Galerkin projection and compute $a_i(t)$

Real-time shape control

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Remarks

Real-time shape control

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Remarks

- POD basis $\boldsymbol{\theta}_i(\cdot, \; \mathbf{p}^{\mathrm{new}})$ must be interpolated

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- Apply Galerkin projection and compute $a_i(t)$

Remarks

- POD basis $\boldsymbol{\theta}_i(\cdot,\ \mathbf{p}^{\text{new}})$ must be interpolated
- No need of a MC simulation for $q_\infty(\mathbf{s}, t; , \mathbf{p}^{\text{new}})$

Modelling chloride diffusion (with R. Cherif MCF ULR)

The microstructure evolution

Modelling chloride diffusion (with R. Cherif MCF ULR)

The microstructure evolution

$C_{\mathrm{b}}(t)$: Ionic diffusion

Modelling chloride diffusion (with R. Cherif MCF ULR)

The microstructure evolution

$$\begin{aligned} C_{\mathrm{b}}(t) &: \text{ Ionic diffusion} \\ \rho(t) &: \text{ Portlandite melting} \end{aligned}$$

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The microstructure evolution

$$
\begin{aligned}
C_{\mathrm{b}}(t) &: \text{Ionic diffusion} \\
\rho(t) &: \text{Portlandite melting}
\end{aligned}
$$

Multiscale model

Modelling chloride diffusion (with R. Cherif MCF ULR)

The microstructure evolution

$C_{\mathrm{b}}(t)$: Ionic diffusion

$\rho(t)$: Portlandite melting

Multiscale model

Macroscale $\varepsilon_{\mathbf{p}} \dfrac{\partial C_{\mathrm{b}} \left\langle e^{-B\varphi} \right\rangle_{\boldsymbol{y}}}{\partial t} - \mathrm{div}_{\boldsymbol{x}} \left(\boldsymbol{D}^{\mathrm{hom}}(t) \left(\nabla_{\boldsymbol{x}} C_{\mathrm{b}} + B \; C_{\mathrm{b}} \nabla_{\boldsymbol{x}} \psi_{\mathrm{b}} \right) \right) = 0$

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- Provides $C_{\mathrm{b}}(t)$

Portlandite volume : $\boldsymbol{\rho}(t) = f(C_{\mathrm{b}}(t))$, ad-hoc model

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- Depends on $C_{\mathrm{b}}(t)$, $\rho(t)$

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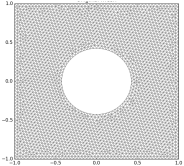
Cell problem : POD-ROM

- Depends on $C_{\mathrm{b}}(t)$, $\rho(t)$
- Provides $\boldsymbol{D}^{\mathrm{hom}}(t)$

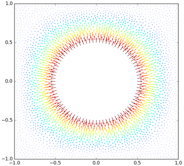
CPU performance

Full-order: macroscale only

Reference



ψ^ρ



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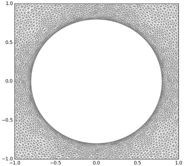
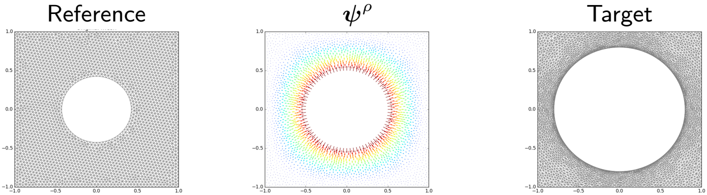
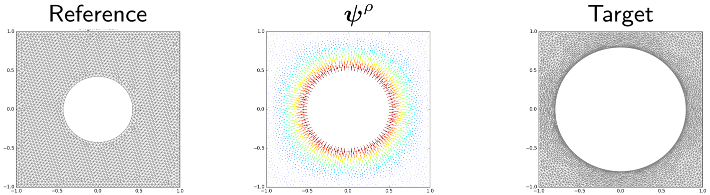


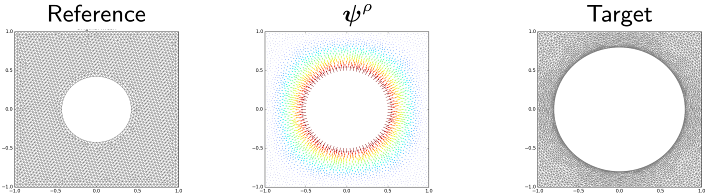


Axiomatic definition :



Axiomatic definition :

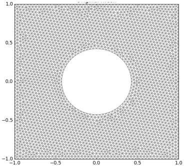
- $\psi^\rho(\Gamma_{\mathrm{sf}}^\star) = \Gamma_{\mathrm{sf}}(\rho)$



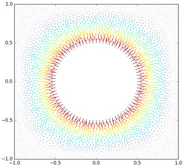
Axiomatic definition :

- $\psi^\rho(\Gamma^\star_{\mathrm{sf}}) = \Gamma_{\mathrm{sf}}(\rho)$
- ψ^ρ preserves other interfaces pointwise

Reference



ψ^ρ



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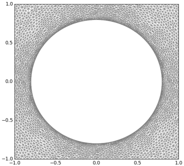
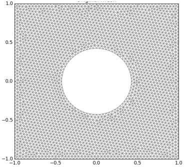


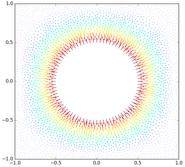
Axiomatic definition :

- $\psi^\rho(\Gamma^\star_{\mathrm{sf}}) = \Gamma_{\mathrm{sf}}(\rho)$
- ψ^ρ preserves other interfaces pointwise
- ψ^ρ is regular enough

Reference



ψ^ρ



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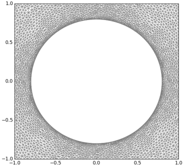
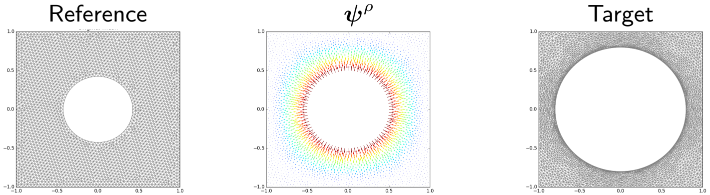


Axiomatic definition :

- $\psi^\rho(\Gamma^\star_{\rm sf}) = \Gamma_{\rm sf}(\rho)$
- ψ^ρ preserves other interfaces pointwise
- ψ^ρ is regular enough

Key : ψ^ρ solution of a PDE, with BC on $\Gamma^\star_{\rm sf}$



Axiomatic definition :

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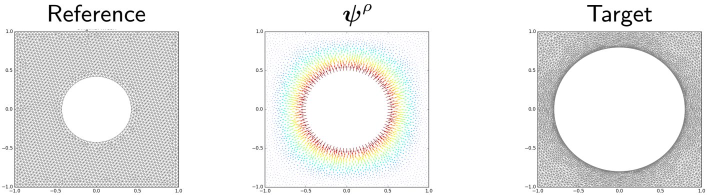
- ψ^ρ preserves other interfaces pointwise
- ψ^ρ is regular enough

Key : ψ^ρ solution of a PDE, with BC on $\Gamma^\star_{\mathrm{sf}}$

The Cell problem

$$\int_{\mathrm{Y}^\star_{\mathrm{f}}} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \, \mathbf{J}^{-1}_{\psi^\rho}\right) : \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{v} \, \mathbf{J}^{-1}_{\psi^\rho}\right) j_{\psi^\rho} \, \mathrm{dY}^\star = - \int_{\Gamma^\star_{\mathrm{sf}}} \boldsymbol{v} \cdot \boldsymbol{n}_\star \, g_\rho \mathrm{d}\Gamma^\star_{\mathrm{sf}}, \; \forall \boldsymbol{v} \in V^d_\star$$



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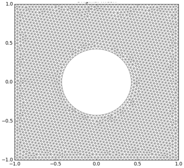
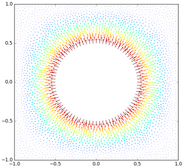
Caveat j_{ψ^ρ} and $\mathbf{J}^{-1}_{\psi^\rho}$ should be computed numerically for each ρ ...

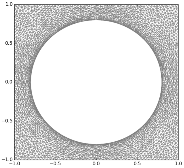
Reference



$\boldsymbol{\psi}^\rho$



Target



Axiomatic definition :

- $\boldsymbol{\psi}^\rho(\Gamma^\star_{\mathrm{sf}}) = \Gamma_{\mathrm{sf}}(\rho)$
- $\boldsymbol{\psi}^\rho$ preserves other interfaces pointwise
- $\boldsymbol{\psi}^\rho$ is regular enough

Key : $\boldsymbol{\psi}^\rho$ solution of a PDE, with BC on $\Gamma^\star_{\mathrm{sf}}$

The Cell problem

$$\int_{\mathrm{Y}^\star_{\mathrm{f}}} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_\star \, \mathbf{J}^{-1}_{\boldsymbol{\psi}^\rho}\right) : \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{v} \, \mathbf{J}^{-1}_{\boldsymbol{\psi}^\rho}\right) j_{\boldsymbol{\psi}^\rho} \, \mathrm{d}\mathrm{Y}^\star = - \int_{\Gamma^\star_{\mathrm{sf}}} \boldsymbol{v} \cdot \boldsymbol{n}_\star \, g_\rho \mathrm{d}\Gamma^\star_{\mathrm{sf}}, \; \forall \boldsymbol{v} \in V^d_\star$$

Caveat $j_{\boldsymbol{\psi}^\rho}$ and $\mathbf{J}^{-1}_{\boldsymbol{\psi}^\rho}$ should be computed numerically for each ρ ...

...involving a full-order problem

A monoparametric cell (A. Falaize, IR ULR)

The Dirichlet problem

$$\begin{cases} \Delta_{\boldsymbol{\xi}} \boldsymbol{\psi}^{\rho} & = & 0 & \text{on } \mathrm{Y}_{\mathrm{f}}^{\star} \\ \boldsymbol{\psi}^{\rho}(\boldsymbol{\xi}) & = & \rho \dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} & \text{on } \Gamma_{\mathrm{sf}}^{\star}(\rho_{\star}) \\ \boldsymbol{\psi}^{\rho}(\boldsymbol{\xi}) & = & \boldsymbol{\xi} & \text{on } \Gamma_{\sharp}^{\star} \end{cases}$$

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A trick for model order reduction

$$\boldsymbol{\psi}^{\rho} = \boldsymbol{\psi}^{\rho_{\mathrm{cust}}} + (\rho - \rho_{\mathrm{cust}}) \boldsymbol{\psi}^{\mathrm{relev}}$$

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$\psi^{\rho_{\mathrm{cust}}}$: particular solution, mesh diffusion for $\rho = \rho_{\mathrm{cust}}$

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ψ^{relev} : solution of $\Delta_{\boldsymbol{\xi}} \psi^{\mathrm{relev}} = 0$ with appropriate boundary conditions

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. . . only two full-order problems to be solved!

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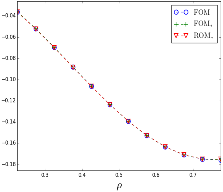
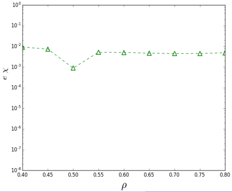
... only two full-order problems to be solved!

Numerical application

T_{11}^{hom}



$e\ \chi$



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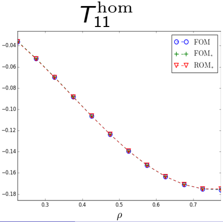
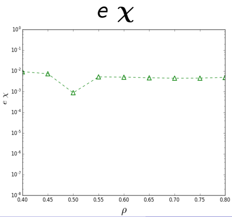
Numerical application





A monoparametric cell (A. Falaize, IR ULR)

The Dirichlet problem

$$\begin{cases} \Delta_{\boldsymbol{\xi}} \boldsymbol{\psi}^{\rho} = 0 & \text{on } \mathrm{Y}_{\mathrm{f}}^{\star} \\ \boldsymbol{\psi}^{\rho}(\boldsymbol{\xi}) = \rho \dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} & \text{on } \Gamma_{\mathrm{sf}}^{\star}(\rho_{\star}) \\ \boldsymbol{\psi}^{\rho}(\boldsymbol{\xi}) = \boldsymbol{\xi} & \text{on } \Gamma_{\sharp}^{\star} \end{cases}$$

A trick for model order reduction

$$\boldsymbol{\psi}^{\rho} = \boldsymbol{\psi}^{\rho_{\mathrm{cust}}} + (\rho - \rho_{\mathrm{cust}}) \boldsymbol{\psi}^{\mathrm{relev}}$$

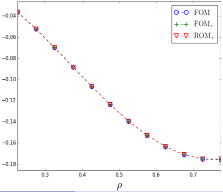
$\boldsymbol{\psi}^{\rho_{\mathrm{cust}}}$: particular solution, mesh diffusion for $\rho = \rho_{\mathrm{cust}}$

$\boldsymbol{\psi}^{\mathrm{relev}}$: solution of $\Delta_{\boldsymbol{\xi}} \boldsymbol{\psi}^{\mathrm{relev}} = 0$ with appropriate boundary conditions

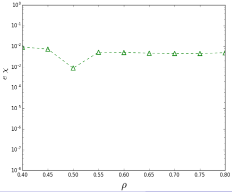
... only two full-order problems to be solved!

Numerical application

T_{11}^{hom}



$e\ \chi$



Obtention of the reduced-order model

Characteristics

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- Explicit dependency of ρ

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- Decomposition of $\boldsymbol{\psi}^\rho$

$$\boldsymbol{\psi}^\rho = \psi^{\rho_{\text{cust}}} + (\rho - \rho_{\text{cust}})\psi^{\text{relev}}$$

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The Cell problem formulated on $\mathrm{Y}^\star$

$$\int_{\mathrm{Y}_\mathrm{f}^\star} \left(\nabla_{\boldsymbol{\xi}}\boldsymbol{\chi}_\star \mathbf{J}_{\boldsymbol{\psi}^\rho}^{-1}\right) : \left(\nabla_{\boldsymbol{\xi}}\boldsymbol{v}_\star \mathbf{J}_{\boldsymbol{\psi}^\rho}^{-1}\right) j_{\boldsymbol{\psi}^\rho}\,\mathrm{dY}^\star = -\int_{\Gamma_\mathrm{sf}^\star} \boldsymbol{n}_\star \cdot \boldsymbol{v}_\star\, g_\rho\,\mathrm{d}\Gamma_\mathrm{sf}^\star$$

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$$\int_{\mathrm{Y}_{\mathrm{f}}^{\star}} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_{\star} \mathbf{J}_{\boldsymbol{\psi}^{\rho}}^{-1}\right) : \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{v}_{\star} \mathbf{J}_{\boldsymbol{\psi}^{\rho}}^{-1}\right) j_{\boldsymbol{\psi}^{\rho}} \, \mathrm{dY}^{\star} = - \int_{\Gamma_{\mathrm{sf}}^{\star}} \boldsymbol{n}_{\star} \cdot \boldsymbol{v}_{\star} \, g_{\rho} \, \mathrm{d}\Gamma_{\mathrm{sf}}^{\star}$$

- $j_{\boldsymbol{\psi}^{\rho}}$: develop $\det\left(\mathbf{J}_{\psi^{\rho_{\mathrm{cust}}}} + (\rho - \rho_{\mathrm{cust}})\mathbf{J}_{\psi^{\mathrm{relev}}}\right)$ through $\det$ multilinearity
- Issue with $\mathbf{J}_{\boldsymbol{\psi}^{\rho}}^{-1}$: nonlinearity
- Solution : POD-ROM $\mathbf{J}_{\boldsymbol{\psi}^{\rho}}^{-1} \simeq \widehat{\mathbf{J}_{\boldsymbol{\psi}^{\rho}}^{-1}}(\boldsymbol{\xi}) = \sum_{k=1}^{n_{\mathrm{iJac}}} b_k(\rho)\, \boldsymbol{\Phi}_k^{\mathrm{inv}}(\boldsymbol{\xi})$ applied to $\mathbf{J}_{\boldsymbol{\psi}^{\rho}}^{-1} \mathbf{J}_{\boldsymbol{\psi}^{\rho}} = \boldsymbol{I}$

Transformations without an explicit dependency in ρ

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Solve fastly $\displaystyle\int_{\mathrm{Y}_{\mathrm{f}}^{\star}} \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\chi}_{\star} \, \mathbf{J}_{\boldsymbol{\psi}^{\rho}}^{-1}\right) : \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{v} \, \mathbf{J}_{\boldsymbol{\psi}^{\rho}}^{-1}\right) j_{\boldsymbol{\psi}^{\rho}} \, \mathrm{dY}^{\star} = - \int_{\Gamma_{\mathrm{sf}}^{\star}} \boldsymbol{v} \cdot \boldsymbol{n}_{\star} \, g_{\rho} \mathrm{d}\Gamma_{\mathrm{sf}}^{\star} \, \ldots$

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... and $$\left\{ \begin{array}{lclcr} \Delta_{\boldsymbol{\xi}} \boldsymbol{\psi}^{\boldsymbol{\rho}} & = & \mathbf{0} & \text{sur} & \mathrm{Y}_{\mathrm{f}}^{\star} \\ \boldsymbol{\psi}^{\boldsymbol{\rho}}(\boldsymbol{\xi}) & = & \boldsymbol{\omega}^{\boldsymbol{\rho}} & \text{sur} & \Gamma_{\mathrm{sf}}^{\star} \cup \Gamma_{\sharp}^{\star} \end{array} \right. .$$

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Suggestion : POD-ROM $\boldsymbol{\psi}^{\boldsymbol{\rho}} \simeq \widehat{\boldsymbol{\psi}^{\boldsymbol{\rho}}} = \sum_{l=1}^{n_\psi} d_l(\boldsymbol{\rho}) \, \boldsymbol{\Psi}_l(\boldsymbol{\xi})$

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Full-order problem : The Nitsche method

$$\mathcal{J} : u \mapsto \frac{1}{2} \int_{Y_f^\star} \nabla_{\boldsymbol{\xi}} \boldsymbol{u} : \nabla_{\boldsymbol{\xi}} \boldsymbol{u} \, \mathrm{d}Y^\star - \int_{\Gamma_{\mathrm{sf}}^\star \cup \Gamma_\sharp^\star} \left(\boldsymbol{u} - \boldsymbol{\omega}^{\boldsymbol{\rho}}\right) \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}_\star} \mathrm{d}\Gamma_{\mathrm{sf}}^\star + \frac{\gamma_0}{2 h_{\mathrm{sf}}} \int_{\Gamma_{\mathrm{sf}}^\star \cup \Gamma_\sharp^\star} \left(\boldsymbol{u} - \boldsymbol{\omega}^{\boldsymbol{\rho}}\right)^2 \mathrm{d}\Gamma_{\mathrm{sf}}^\star .$$

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Bidimensional Cell : $\left[\widetilde{\boldsymbol{A}}\right]_{jilk} = \displaystyle\int_{Y_f^\star} \nabla_{\boldsymbol{\xi}} \phi_i : \left(\nabla_{\boldsymbol{\xi}} \phi_j \, \boldsymbol{\Phi}_k^{\mathrm{inv}} \, \mathrm{com}\left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\Psi}_l\right)\right) \mathrm{d}Y^\star$, since $\mathbf{J}_{\boldsymbol{\psi}^{\boldsymbol{\rho}}}{}^{-\mathsf{T}} j_{\boldsymbol{\psi}^{\boldsymbol{\rho}}} = \mathrm{com}\left(\mathbf{J}_{\boldsymbol{\psi}^{\boldsymbol{\rho}}}\right)$.

Thank you for your attention

Appendix