Milieux de Cosserat de dimension 0, 1, 2 ou 3

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State of the Art

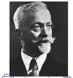
Method of Virtual powers
(Lagrange, 1788)

Generalized continua
(Eugène and François Cosserat, 1909)

Affine mechanics (Souriau, 1997)

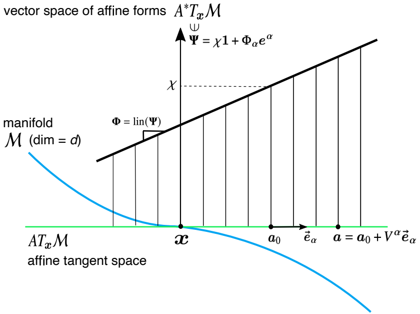
Affine tensors
(Tulczyjew, Urbañski, Grabowski, 1988)

Affine connections
(Élie Cartan, 1923)



Affine tensors

Affine tensors are maps which are affine or linear with respect to their arguments



Affine tensors :	Components :
affine form $\boldsymbol{\Psi}$	(χ, Φ_α) in the affine coframe $f^* = (1, (\boldsymbol{e}^\alpha))$
point $\boldsymbol{a}$	V^α in the affine frame $f = (\boldsymbol{a}_0, (\vec{e}_\alpha))$

torsor $\boldsymbol{\tau}$, bilinear and antisymmetric on $A^*T_{\boldsymbol{x}}\mathcal{M}$ $\quad (T^\alpha, J^{\alpha\beta})$

$\boldsymbol{\tau}(\boldsymbol{\Psi}, \hat{\boldsymbol{\Psi}}) = -\boldsymbol{\tau}(\hat{\boldsymbol{\Psi}}, \boldsymbol{\Psi})$

$\boldsymbol{\tau} = T^\beta (\boldsymbol{a}_0 \otimes \vec{e}_\beta - \vec{e}_\beta \otimes \boldsymbol{a}_0) + J^{\alpha\beta}\, \vec{e}_\alpha \otimes \vec{e}_\beta \qquad J^{\alpha\beta} = -J^{\beta\alpha}$

Transformation laws of (scalar-valued) torsors

Change of affine frames $(\boldsymbol{a}_0, (\vec{\boldsymbol{e}}_\alpha)) \longrightarrow (\boldsymbol{a}'_0, (\vec{\boldsymbol{e}}'_\beta))$ given by

$$\overrightarrow{\boldsymbol{a}'_0 \boldsymbol{a}_0} = C'^{\beta} \vec{\boldsymbol{e}}'_\beta, \qquad \vec{\boldsymbol{e}}'_\beta = P^\alpha_\beta \vec{\boldsymbol{e}}_\alpha, \qquad C = -P\, C'$$

Combining two changes of affine frames, we observe that the set of couples (C, P) is the affine group $\mathbb{GA}(4) = \mathbb{R}^4 \rtimes \mathbb{GL}(4)$

Trick : using the linear representation on $\mathbb{R}^5$ of the affine group of $\mathbb{R}^4$

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix}, \qquad \tilde{\tau} = \begin{pmatrix} 0 & T^T \\ -T & J \end{pmatrix}$$

the law of transformation of torsors is reduced to : $\boxed{\tilde{\tau}' = \tilde{P}^{-1} \tilde{\tau}\, \tilde{P}^{-T}}$

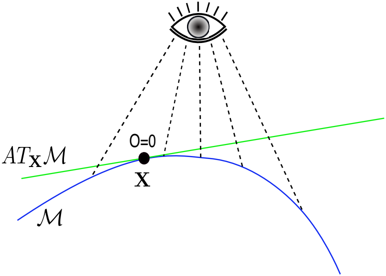
Galilean transformation of the space-time :

$$X = \begin{pmatrix} t \\ x \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

where $u \in \mathbb{R}^3$ is the **Galilean boost** and R is a rotation

Proper frame

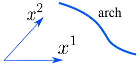
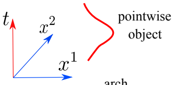
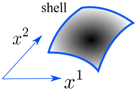
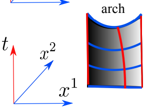
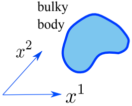
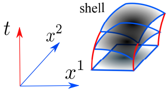
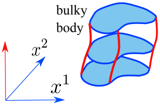
- [Élie Cartan 1923 Sur les variétés à connexion affine...]
  "The affine space at $\boldsymbol{X}$ could be seen as the manifold itself that would be perceived in an affine manner by an observer located at $\boldsymbol{X}$"



- A frame of reference in which an object is **at rest** at the origin $\boldsymbol{O}$ of $AT_{\boldsymbol{X}}\mathcal{M}$ is called a **proper frame** attached to this object

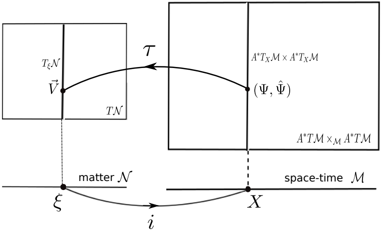
Cosserat media of dimension 0, 1, 2 or 3

Matter manifold

$dim(\mathcal{N})$	*Statics* $dim(\mathcal{M}) = 3$	*Dynamics* $dim(\mathcal{M}) = 4$
1		
2		
3		
4		

The (vector-valued) torsor of a Cosserat medium

- The **matter manifold** $\mathcal{N}$ is the set of material particles ξ
  We describe the motion of the matter by an embedding $i : \xi \mapsto X$
- A **torsor** at X is valued in the tangent vector space to $\mathcal{N}$ at ξ



- Then we defined a bundle map over $\mathcal{N}$
  $i^*(A^*T\mathcal{M} \times_{\mathcal{M}} A^*T\mathcal{M}) \to T\mathcal{N}$
- In an affine frame $(\boldsymbol{a}_0, (\vec{\boldsymbol{e}}_\alpha))$ of $AT_{\boldsymbol{X}}\mathcal{M}$
  and a basis $({}_\gamma\vec{\boldsymbol{\eta}})$ of $T_{\boldsymbol{\xi}}\mathcal{N}$, it is decomposed into

$$\boxed{\boldsymbol{\tau} =_\gamma \vec{\boldsymbol{\eta}} \otimes [\,{}^\gamma T^\beta\,(\boldsymbol{a}_0 \otimes \vec{\boldsymbol{e}}_\beta - \vec{\boldsymbol{e}}_\beta \otimes \boldsymbol{a}_0) + {}^\gamma J^{\alpha\beta}\,\vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta]}$$

Pointwise object (Cosserat medium of dimension 0)

An object in a **proper frame** has a **reduced form**

$$\tilde{\tau}' = \begin{pmatrix} 0 & T'^T \\ -T' & J' \end{pmatrix} = \begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ 0 & 0 & -j(l_0) \end{pmatrix}$$

- Apply a **Galilean boost**

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & v & 1_{\mathbb{R}^3} \end{pmatrix}$$

- The transformation law of torsors gives the **standard form**

$$\tilde{\tau} = \tilde{P}\,\tilde{\tau}'\,\tilde{P}^T = \begin{pmatrix} 0 & m & p^T \\ -m & 0 & -q^T \\ -p & q & -j(l) \end{pmatrix}$$

  where the invariant (mass) m and

  - the linear momentum $p = m\,v$,
  - the quantity of position (or passage) $q = m\,x$
  - the angular momentum $l = l_0 + x \times m\,v$

Dynamics of a Cauchy medium of dimension 3

- As $\dim(\mathcal{N}) = \dim(\mathcal{M}) = 4$, we can choose for easiness $\xi^\alpha = X^\alpha$, then ${}^\gamma T^\beta = T^{\beta\gamma}, \quad {}^\gamma J^{\alpha\beta} = J^{\alpha\beta\gamma}$
- In a **proper frame** attached to the elementary reference volume :
  - If $J^{\alpha\beta\gamma} \neq 0$, we modelize a **3D Cosserat medium**
  - If $J^{\alpha\beta\gamma} = 0$, we modelize a **3D Cauchy medium** (until further notice)

  Moreover, in this proper frame, the reduced form is

  $$T' = \begin{pmatrix} \rho & 0 \\ 0 & -\sigma \end{pmatrix}$$

  where ρ is the mass density and σ is Cauchy's stress tensor
- Applying a **Galilean boost** of velocity v,
  the transformation law of torsors gives the standard form

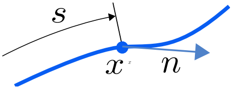
  $$\boxed{T = \begin{pmatrix} \rho & p^T \\ p & -\sigma_\star \end{pmatrix} = \begin{pmatrix} \rho & \rho\, v^T \\ \rho\, v & \rho\, v\, v^T - \sigma \end{pmatrix}}$$

  where $\sigma_\star$ are the dynamical stresses.
  For these reasons, the tensor of components $T^{\beta\gamma}$
  is called **stress-mass tensor**

Dynamics of a Cosserat medium of dimension 1

- The motion of a 1D material body can be described by the embedding of a matter manifold $\mathcal{N}$ into the space-time $\mathcal{M}$

$$i : \xi = \begin{pmatrix} t \\ s \end{pmatrix} \mapsto X = \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ \psi(t,s) \end{pmatrix}$$



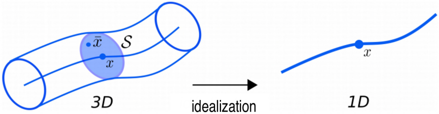
- The components ${}^{\gamma}\bar{T}^{\beta}$ can be stored in a 2-by-4 matrix

$$\boxed{\bar{T} = \begin{pmatrix} \rho_l & p^T \\ p_t & -F_\star^T \end{pmatrix} = \begin{pmatrix} \rho_l & \rho_l v^T \\ \rho_l v_t & (\rho_l v_t v - F)^T \end{pmatrix}}$$

It is called **force-mass tensor** in which appear the tangential velocity v_t, the linear momentum per unit arclength p and the dynamical force $F_\star$

3D-to-1D reduction (1/2)

3D **slender body**
idealized by an 1D body



projection of $T_{\boldsymbol{x}}\mathcal{M}$ into $T_{\boldsymbol{\xi}}\mathcal{N}$: $\qquad \Pi = \begin{pmatrix} 1 & 0 \\ 0 & n^T \end{pmatrix}$

with the convention that Greek indices run from 0 to 3,
Latin indices from 1 to 3 and 0 is the time index,

1D body torsor components are :

$$\boxed{{}^\gamma T^\alpha = \int_{\mathcal{S}} {}^\gamma\Pi_\rho \, \bar{T}^{\alpha\rho} \, d\mathcal{S}}$$

$$\boxed{{}^\gamma J^{\alpha\beta} = \int_{\mathcal{S}} {}^\gamma\Pi_\rho \, \bar{J}^{\alpha\beta\rho} \, d\mathcal{S}}$$

in terms of the 3D body components $\bar{T}^{\alpha\rho}$ and
and $\bar{J}^{i0\rho} = \bar{x}^i \, \bar{T}^{0\rho}$, $\qquad \bar{J}^{ij\rho} = \bar{x}^i \, \bar{T}^{j\rho} - \bar{x}^j \, \bar{T}^{i\rho}$ (for a Cauchy medium)

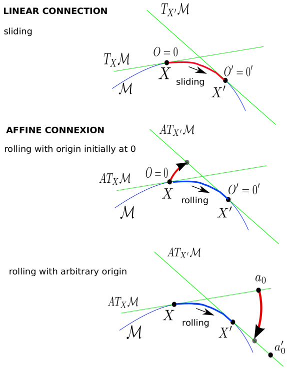
3D-to-1D reduction (2/2)

that gives, putting $q^i = {}^0J^{i0}, \quad l_\star^i = {}^1J^{i0}, \quad l^i = {}^0J^{kl}, \quad M_\star^i = {}^1J^{kl}$

the density	$\rho_l = \int_{\mathcal{S}} \rho \, d\mathcal{S}$
the linear momentum	$p = \int_{\mathcal{S}} \rho \, \bar{v} \, d\mathcal{S}$
the dynamical force	$F = \int_{\mathcal{S}} (\sigma \, n - \rho \, (v_t - \bar{v}_t)(v - \bar{v})) \, d\mathcal{S}$
the position quantity	$q = \int_{\mathcal{S}} \rho \, \bar{x} \, d\mathcal{S}$
the angular momentum	$l = \int_{\mathcal{S}} \bar{x} \times \rho \, \bar{v} \, d\mathcal{S}$
a new momentum	$l_\star = \int_{\mathcal{S}} \rho \, \bar{v}_t \bar{x} \, d\mathcal{S}$
the dynamical moment	$M_\star = \int_{\mathcal{S}} \bar{x} \times (\rho \bar{v}_t \bar{v} - (\sigma \, n)) \, d\mathcal{S}$

Affine connections

[Élie Cartan 1923 Sur les variétés à connexion affine...]



To define a covariant derivative $\tilde{\nabla}$ of an affine tensor,
we consider an infinitesimal affine transformation :
$dX \mapsto (dC, dP) = (\Gamma_A(dX), \Gamma(dX))$

- $\Gamma = (\Gamma^\alpha_{\mu\beta} dX^\mu)$ is the **gravitation**
- $\Gamma_A = (\Gamma^\alpha_{A\mu} dX^\mu)$ represent the **infinitesimal motion of the observer**
  $\Gamma_A(dX) = dX - \nabla_{dX} C$

[Kobayashi & Nomizu 1963 Foundation of Differential Geometry]

Covariant divergence of the torsor

The **affine covariant divergence** of the torsor is decomposed into

$$\tilde{\boldsymbol{Div}}\,\boldsymbol{\tau} = {}_{\gamma}\tilde{\nabla}\,{}^{\gamma}T^{\beta}\,(\boldsymbol{a}_0 \otimes \vec{\boldsymbol{e}}_{\beta} - \vec{\boldsymbol{e}}_{\beta} \otimes \boldsymbol{a}_0) + {}_{\gamma}\tilde{\nabla}\,{}^{\gamma}J^{\alpha\beta}\,\vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta}$$

with

- $${}_{\gamma}\tilde{\nabla}\,{}^{\gamma}T^{\beta} = \frac{\partial({}^{\gamma}T^{\beta})}{\partial({}^{\gamma}\xi)} + {}^{\gamma}_{\gamma\rho}\Gamma^{\rho}\,T^{\beta} + {}^{\gamma}T^{\rho}\,{}_{\gamma}U^{\sigma}\,\Gamma^{\beta}_{\sigma\rho}$$
- $${}_{\gamma}\tilde{\nabla}\,{}^{\gamma}J^{\alpha\beta} = \frac{\partial({}^{\gamma}J^{\alpha\beta})}{\partial({}^{\gamma}\xi)} + {}^{\gamma}J^{\rho\beta}\,{}_{\gamma}U^{\sigma}\,\Gamma^{\alpha}_{\sigma\rho} + {}^{\gamma}J^{\alpha\rho}\,{}_{\gamma}U^{\sigma}\,\Gamma^{\beta}_{\sigma\rho} + {}^{\gamma}_{\gamma\rho}\Gamma^{\rho}\,J^{\alpha\beta} + {}_{\gamma}U^{\sigma}\,\Gamma^{\alpha}_{A\sigma}\,{}^{\gamma}T^{\beta} - {}^{\gamma}T^{\alpha}\,{}_{\gamma}U^{\sigma}\,\Gamma^{\beta}_{A\sigma}$$

Gravitation in relativity :
40 Christoffel symbols

Galilean gravitation :
6 non vanishing Christoffel symbols

- $g^i = -\Gamma^i_{00}$ is the classical **gravity**
- $\Omega^i_j = -\Omega^j_i = \Gamma^i_{0j}$ is the **spinning**, responsible of Coriolis' force

Dynamics of 3D Cauchy media and 0D Cosserat media

To be compatible with Galilei principle of relativity,
the physical laws must be covariant

Principle : the **covariant divergence** of the torsor vanishes $\boxed{\tilde{\boldsymbol{Div}}\,\boldsymbol{\tau} = 0}$

dynamics of 3D Cauchy media : **balance equations**

- of the $\boldsymbol{T}$ components. Euler's equations of fluid : **balance equations**
  - of the mass $\dfrac{\partial \rho}{\partial t} + div\,(\rho\, v) = 0$
  - of the linear momentum $\rho\,\left[\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x}\,v\right] = div\ \sigma + \rho(g - 2\,\Omega \times v)$
- of the $\boldsymbol{J}$ components. $T^{\alpha\beta} = T^{\beta\alpha}$

dynamics of a 0D Cosserat media : **balance equations**

- of the mass $\dot{m} = 0$
- of the linear momentum $\dot{p} = m\,(g - 2\,\Omega \times v)$
- of the position quantity $\dot{q} = p$
- of the angular momentum $\dot{l} + \Omega \times l_0 = x \times m\,(g - 2\,\Omega \times v)$

Dynamics of 1D and 2D Cosserat media

From the principle $\boxed{\tilde{Div}\,\tau = 0}$, we deduce

the **dynamics of 1D Cosserat media** : **balance equations**

- of the mass $\dfrac{\partial \rho_l}{\partial t} + \dfrac{\partial}{\partial s}(\rho_l v_t) = 0$
- of the linear momentum $\rho_l \left[\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial s}\, v_t\right] = \dfrac{\partial F}{\partial s} + \rho_l (g - 2\,\Omega \times v)$
- of the position quantity $\dfrac{\partial q}{\partial t} + \dfrac{\partial l_\star}{\partial s} = \rho_l v$
- of the angular momentum
$$\frac{\partial l}{\partial t} + \Omega \times l + l_\star \times (\Omega \times n) = -\frac{\partial M_\star}{\partial s} + n \times F$$

Remark : the extension to the **dynamics of 2D Cosserat media** was published in [de Saxcé & Vallée, JTAM, 2003 Affine tensors in shell theory]

Dynamics of 3D Cosserat media

The relevancy of Cosserat theory for the statics of 3D media is not clear. Extending it to the dynamic, the range of application is much more open.

With engineering notations : $q^i = J^{i00},\ l^i = J^{kl0},\ l_\star^{ir} = J^{i0r},\ M_\star^{ir} = J^{klr}$,

we find the **balance equations**

- of mass : $\dfrac{\partial T^{00}}{\partial t} + \dfrac{\partial T^{0i}}{\partial x^i} = 0$
- of linear momentum : $\dfrac{\partial T^{i0}}{\partial t} + \dfrac{\partial T^{ij}}{\partial x^j} - (T^{00} g^i - \Omega_j^i (T^{0j} + T^{j0})) = 0$
- of position quantity : $\dfrac{\partial q^i}{\partial t} + \Omega_j^i q^j + \dfrac{\partial l_\star^{ir}}{\partial x^r} + T^{0i} - T^{i0} = 0$
- of angular momentum :
  $$\frac{\partial l^k}{\partial t} + \frac{\partial M_\star^{km}}{\partial x^m} - (q \times g)^k + \Omega_p^j l^q - \Omega_r^i l^s + \Omega_r^j l_\star^{ir} - \Omega_r^i l_\star^{jr} + T^{ji} - T^{ij} = 0$$
  where (ijk), (ipq) and (jrs) are cyclic permutations of (123) and summations have to be performed on all triples (ipq) and (jrs)

To know more...

[de Saxcé & Vallée, 2016,
Galilean Mechanics and Thermodynamics of Continua]

[de Saxcé, Int. J. of Eng. Sci., 2025,
Cosserat media in dynamics]

Thank you very much for your attention !