Reduced Order Modeling of Nonlinear Vibration Dynamics Using Shell Finite Elements and Invariant Manifold

Zixu Xia*, Cyril Touzé, Yu Cong***

* LMEE, Université d'Evry

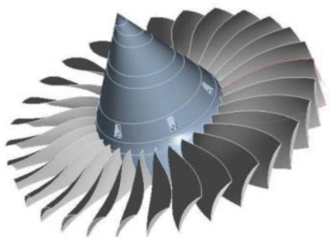
** IMSIA, ENSTA Paris

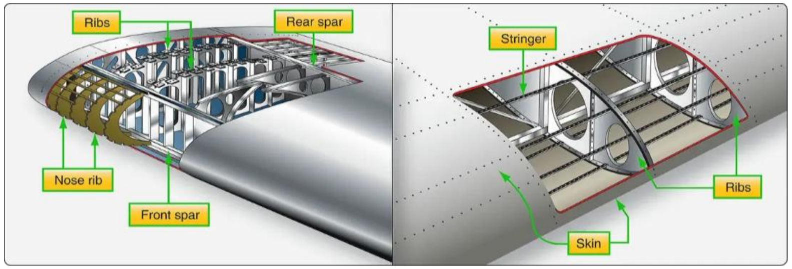
Goal of this study :

Combine **shell finite element** and **reduced order modeling** (using **DPIM** : Direct Parametrization of Invariant Manifold*) to model **nonlinear vibration dynamics** of thin structures.

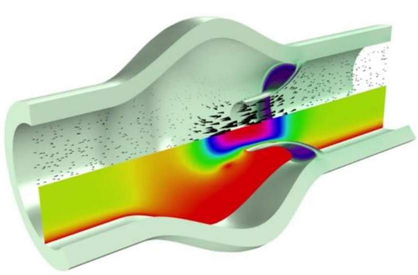
* Touzé et al. 2021

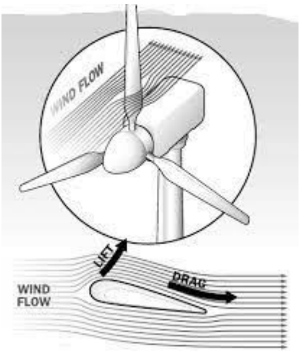
Engineering applications



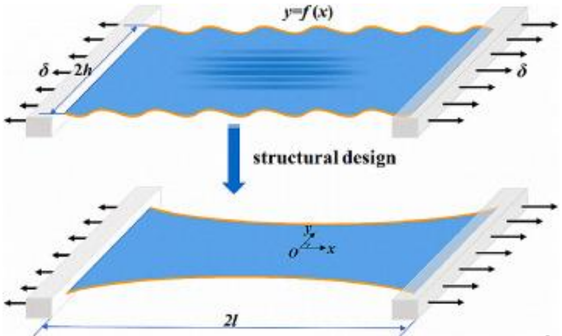


Fluid-structure interaction





Thin films system



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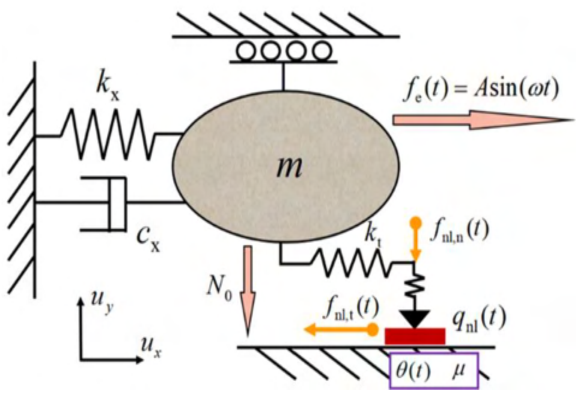
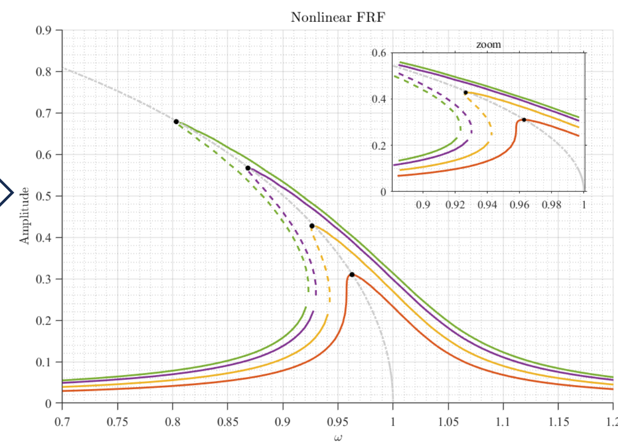
Common methodology

Structure



Simplified model



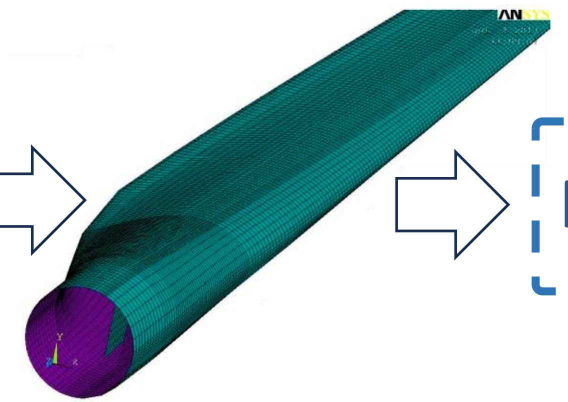
Nonlinear vibration analysis (**Harmonic Balance Method**)

Goal of this study :

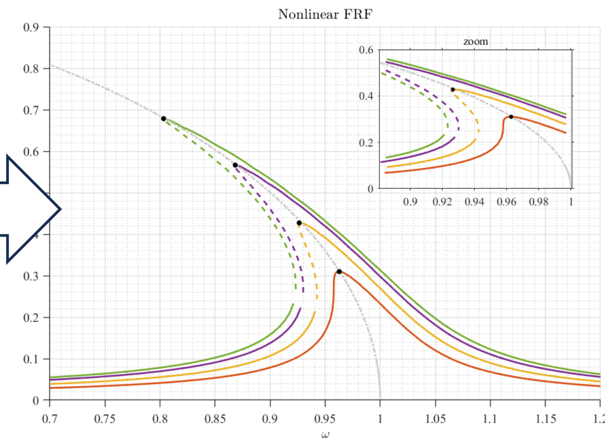
Combine **shell finite element** and **reduced order modeling** (using **DPIM** : Direct Parametrization of Invariant Manifold*) to model **nonlinear vibration dynamics** of thin structures.

* Touzé et al. 2021

Main idea of the work



ROM



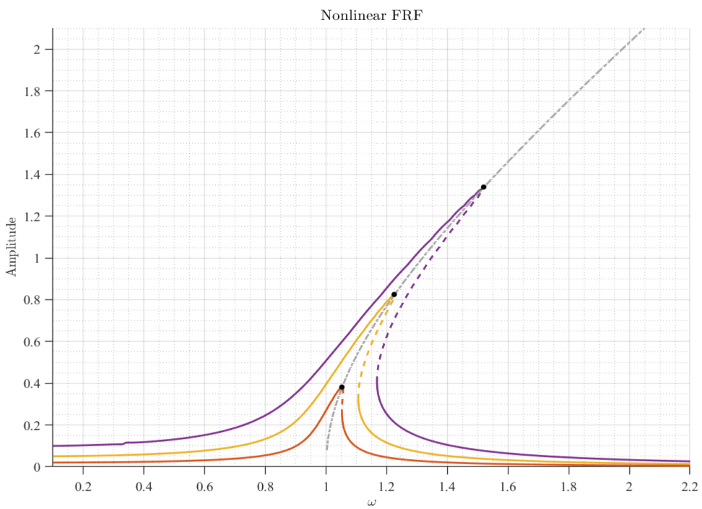
Structure

Shell FEM model

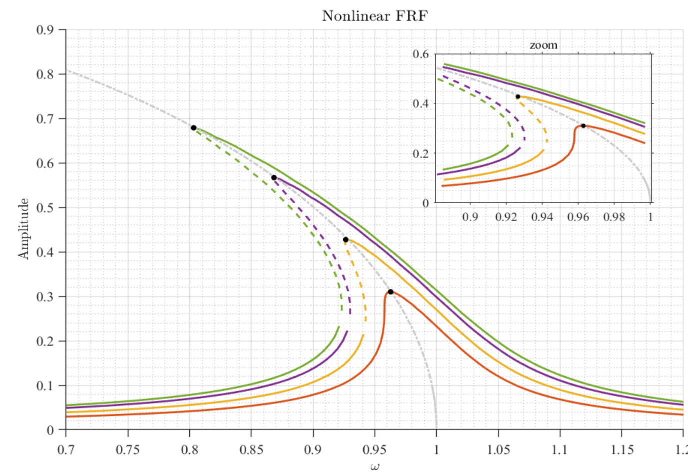
ROM + Nonlinear vibration analysis

(ROM and Harmonic Balance Method)

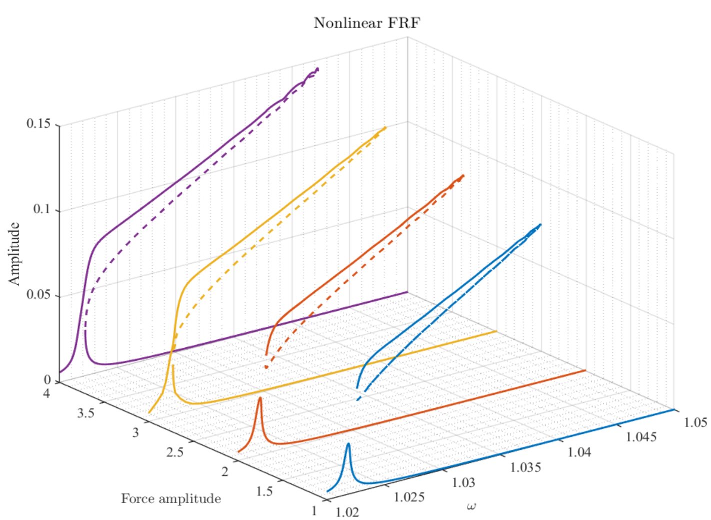
Critical nonlinear vibration dynamics phenomena



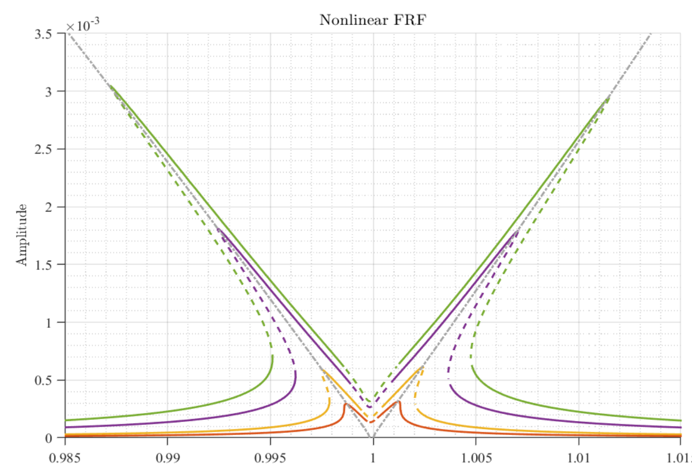
Backbone bending (hardening)



Backbone bending (softening)



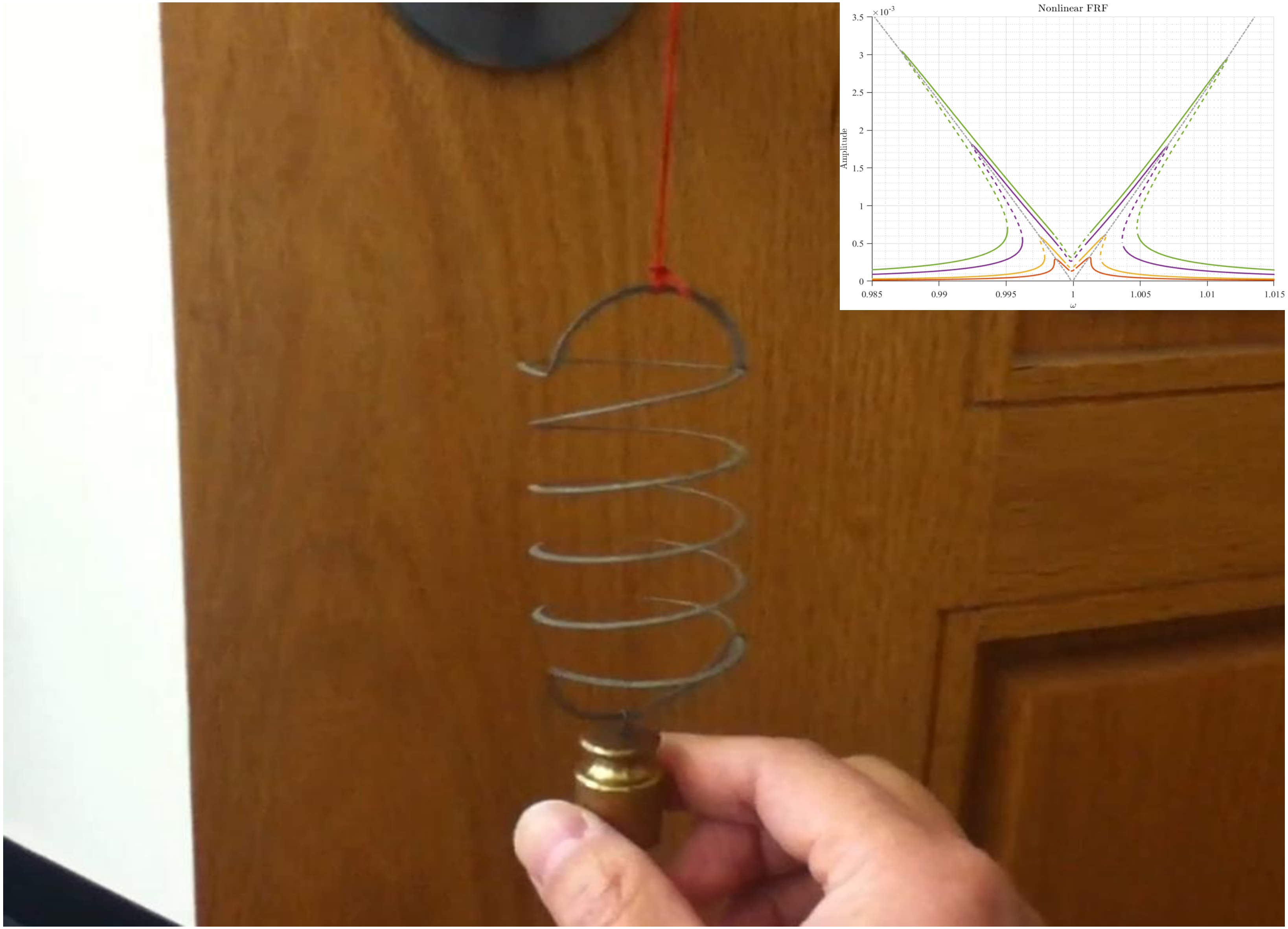
Isolated solution



Internal resonance

Contexte et objectifs

Example of internal resonance (energy transfer from modes, double natural frequency)



Problem: A Large Algebraic Challenge

A second-order nonlinear dynamical system with N degrees of freedom can be written as

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{F}_{nl}(\mathbf{U}, \cdots) = \mathbf{F}e^{i\Omega t}$$

Introducing the state-space formulation, it can be expressed as

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \underbrace{\mathbf{Q}(\mathbf{y}, \cdots)}_{nonlinear} + \underbrace{\boldsymbol{\Upsilon}(\Omega)}_{external\ force}$$

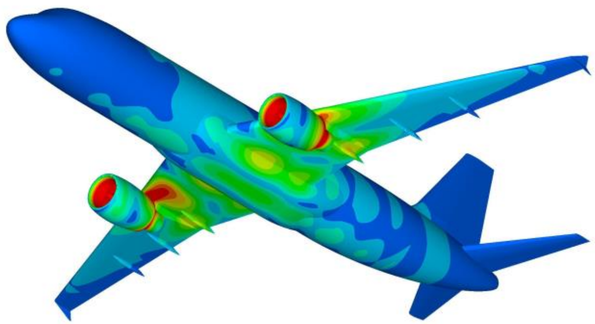
where

$$\mathbf{y} = \begin{bmatrix} \mathbf{U} \\ \dot{\mathbf{U}} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{K} & -\mathbf{C} \end{bmatrix}$$

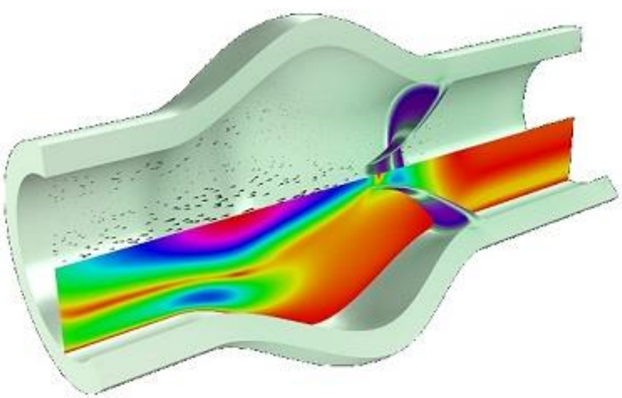
- $\mathbf{y} \in \mathbb{R}^{2N}$: the system state
- $\mathbf{A}, \mathbf{B}$: $2N \times 2N$ matrices (linear parts)
- $\mathbf{Q}(\mathbf{y}, \cdots)$: nonlinear term (e.g., quadratic)

Issue : When N becomes very large, it becomes difficult to directly solve this dynamical system

Solution: Develop a **reliable** and **robust** model reduction method



Aircraft aerodynamics



Cardiovascular FSI

ROM by Invariant Manifolds

From Theory to Numerical method

1. What is the Invariant Manifolds?

2. The Relationship between Invariant Manifolds and ROM

3. Direct parametrisation of invariant manifolds

Initial Idea: Linear Subspace

In the linear world $\mathbf{Q}(\mathbf{y}, \cdots) = \mathbf{0}$: $\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$

- Solve the generalized eigenproblem : $(\lambda_i \mathbf{B} - \mathbf{A})\boldsymbol{\phi}_i = \mathbf{0}$
- Express the solution as a linear combination of eigenvectors

$$\mathbf{y}(t) = \sum_{i=1}^{N} z_i(t)\boldsymbol{\phi}_i$$

- **Reduced order modeling (ROM) :** keep d-th master mode

$$\mathbf{y}(t) \approx \sum_{i=1}^{d} z_i(t)\boldsymbol{\phi}_i$$

This is mathematically valid because the dynamics are decoupled!

Here, we first consider a simplified two-degree-of-freedom model as follows

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{Y}_1 \\ \dot{Y}_2 \end{pmatrix} = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -\omega_1^2 & & & \\ & -\omega_2^2 & & \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix}$$

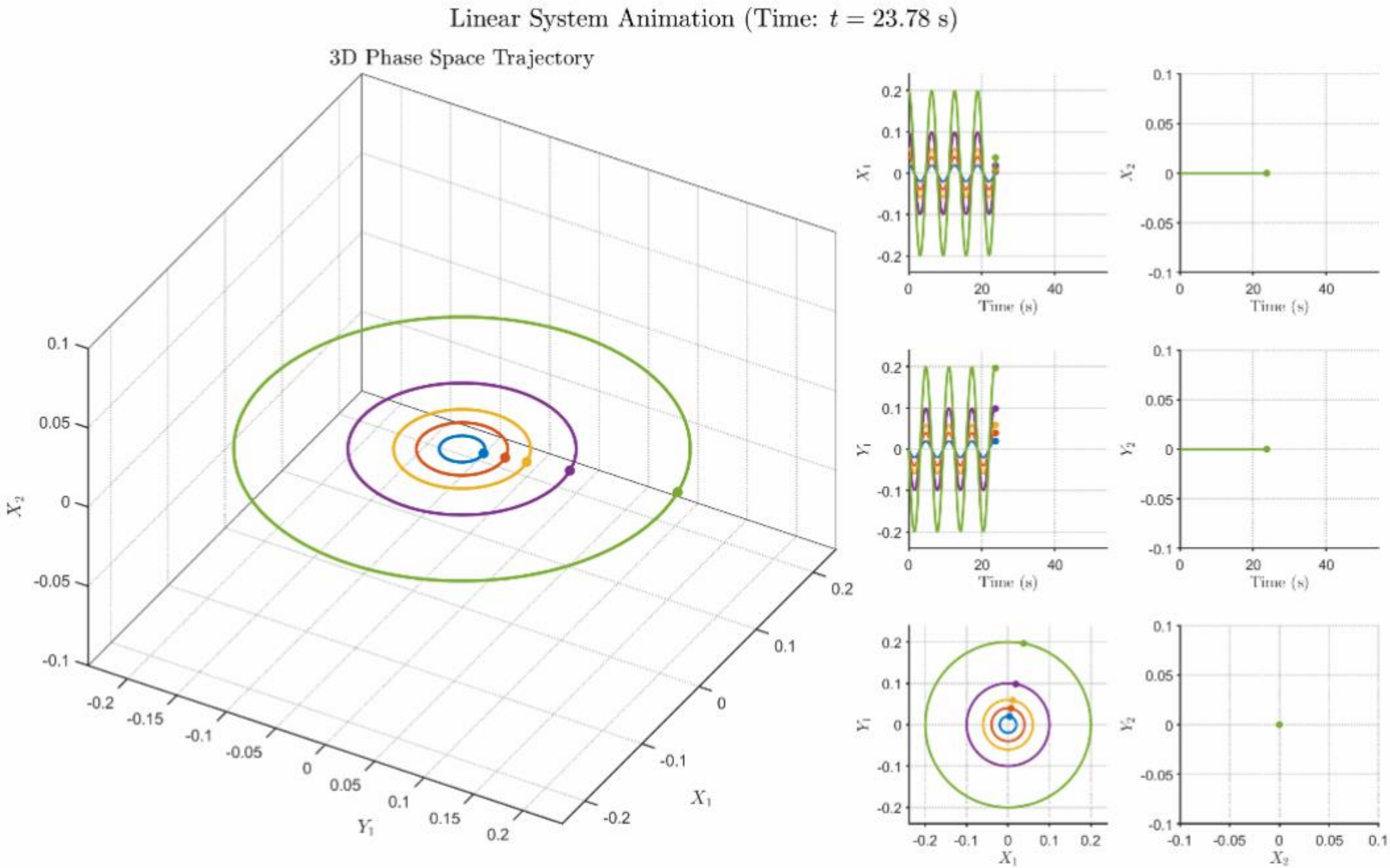
C. Touzé (2004)

where $\dot{X}_1 = Y_1 \quad \dot{X}_2 = Y_2 \quad \omega_1 = 1 \quad \omega_2 = \sqrt{3}$

At time t = 0, $\mathbf{y}_0 = (\Delta \;\; 0 \;\; 0 \;\; 0)^T, \; \Delta = 0.02, 0.04, 0.06, 0.1, 0.2$ **Gradually increasing the amplitude**

Initial Idea: Linear Subspace

At time t = 0 $\mathbf{y}_0 = (\Delta \quad 0 \quad 0 \quad 0)^T,\ \Delta = 0.02, 0.04, 06, 0.1, 0.2$



The linear subspace fully decouples the modal responses, leading to successful model reduction

Initial Idea: Linear Subspace

In the linear world $\mathbf{Q}(\mathbf{y},\cdots)=\mathbf{0}$: $\mathbf{B}\dot{\mathbf{y}}=\mathbf{A}\mathbf{y}$

- Solve the generalized eigenproblem : $(\lambda_i\mathbf{B}-\mathbf{A})\boldsymbol{\phi}_i=\mathbf{0}$
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This is mathematically valid because the dynamics are decoupled!

We now introduce **nonlinear terms** into the governing equations

$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{pmatrix}\dot{X}_1\\\dot{X}_2\\\dot{Y}_1\\\dot{Y}_2\end{pmatrix}=\begin{bmatrix}&&1&\\&&&1\\-\omega_1^2&&&\\&-\omega_2^2&&\end{bmatrix}\begin{pmatrix}X_1\\X_2\\Y_1\\Y_2\end{pmatrix}-\underbrace{\begin{bmatrix}&&1&\\&&&1\\\frac{3\omega_1^2}{2}X_1&\omega_2^2X_1&&\\\omega_1^2X_2&\frac{3\omega_2^2}{2}X_2&&\end{bmatrix}\begin{pmatrix}X_1\\X_2\\Y_1\\Y_2\end{pmatrix}}_{quatratic}-\underbrace{\frac{(\omega_1^2+\omega_2^2)}{2}\begin{bmatrix}&&1&\\&&&1\\(X_1^2+X_2^2)&&&\\&(X_1^2+X_2^2)&&\end{bmatrix}\begin{pmatrix}X_1\\X_2\\Y_1\\Y_2\end{pmatrix}}_{cubic}$$

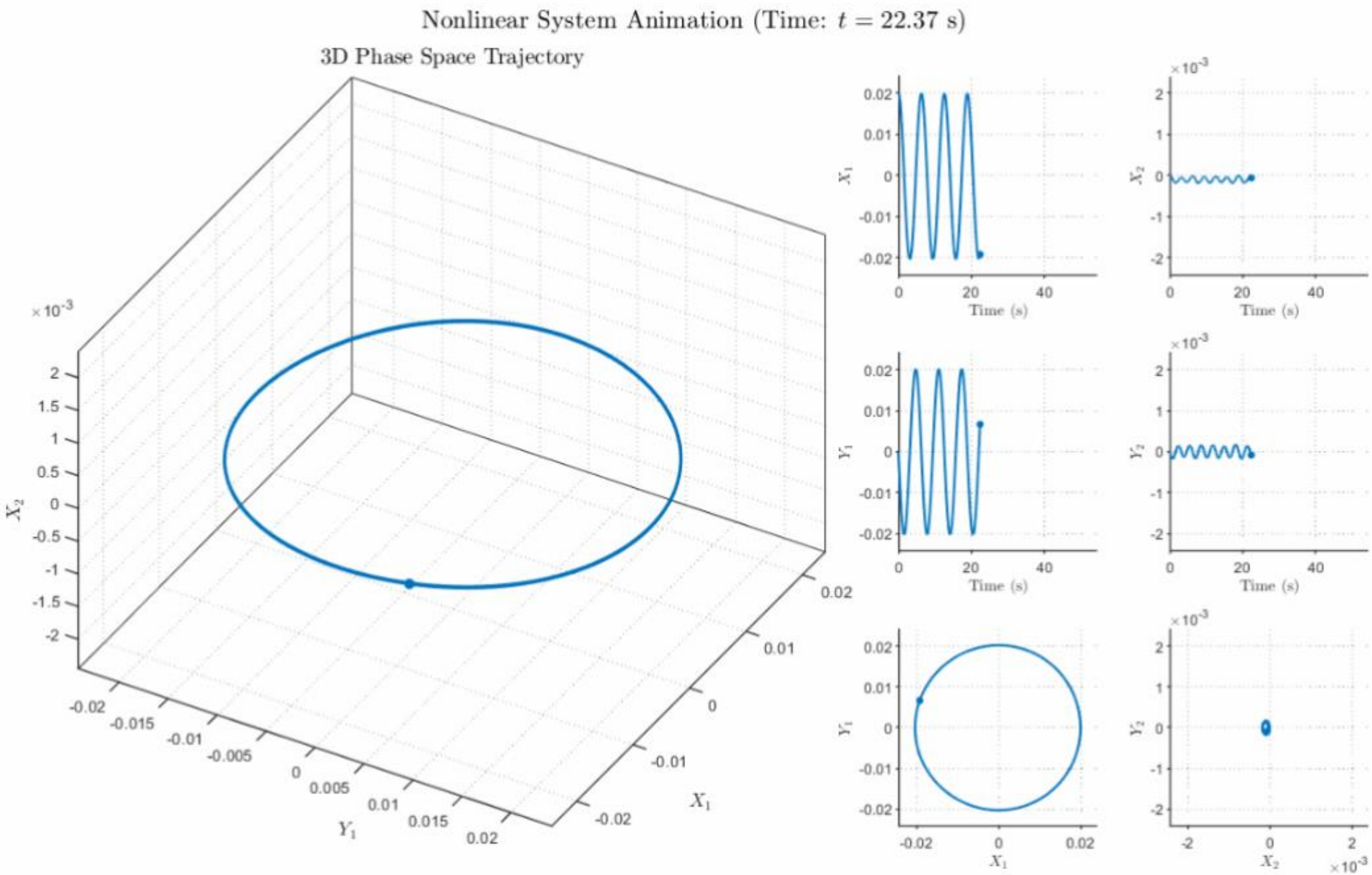
C. Touzé (2004)

where $\dot{X}_1=Y_1 \quad \dot{X}_2=Y_2 \quad \omega_1=1 \quad \omega_2=\sqrt{3}$

At time t = 0 $\quad \mathbf{y}_0=(\varDelta\ \ 0\ \ 0\ \ 0)^T,\ \varDelta=0.02, 0.04, 0.06, 0.1, 0.2$ **increasing the amplitude**

Initial Idea: Linear Subspace

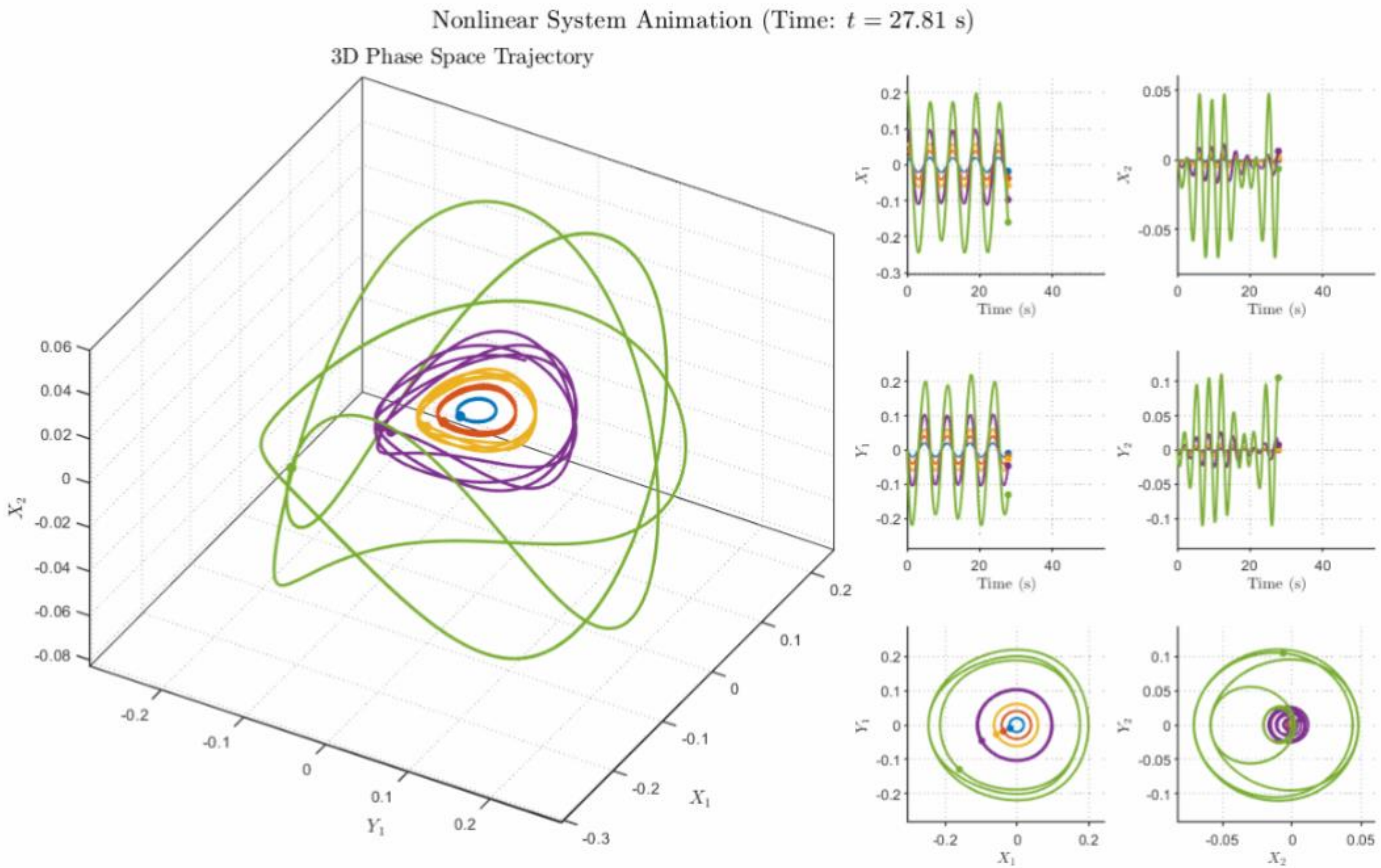
At time t = 0 $\mathbf{y}_0 = (\Delta \;\; 0 \;\; 0 \;\; 0)^T,\ \Delta = 0.02$



At small amplitudes, the linear subspace remains effective in decoupling the system

Initial Idea: Linear Subspace

At time t = 0 $\mathbf{y}_0 = (\Delta \ \ 0 \ \ 0 \ \ 0)^T$, $\Delta = 0.02, 0.04, 06, 0.1, 0.2$



Increasing amplitude, the motion deviates from the linear subspace

Initial Idea: Linear Subspace

In the linear world $\mathbf{Q}(\mathbf{y},\cdots)=\mathbf{0}$ **:** $\mathbf{B}\dot{\mathbf{y}}=\mathbf{A}\mathbf{y}$

- Solve the generalized eigenproblem : $(\lambda_i\mathbf{B}-\mathbf{A})\boldsymbol{\phi}_i=\mathbf{0}$
- Express the solution as a linear combination of eigenvectors

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- **Reduced order modeling (ROM) :** keep d-th master mode

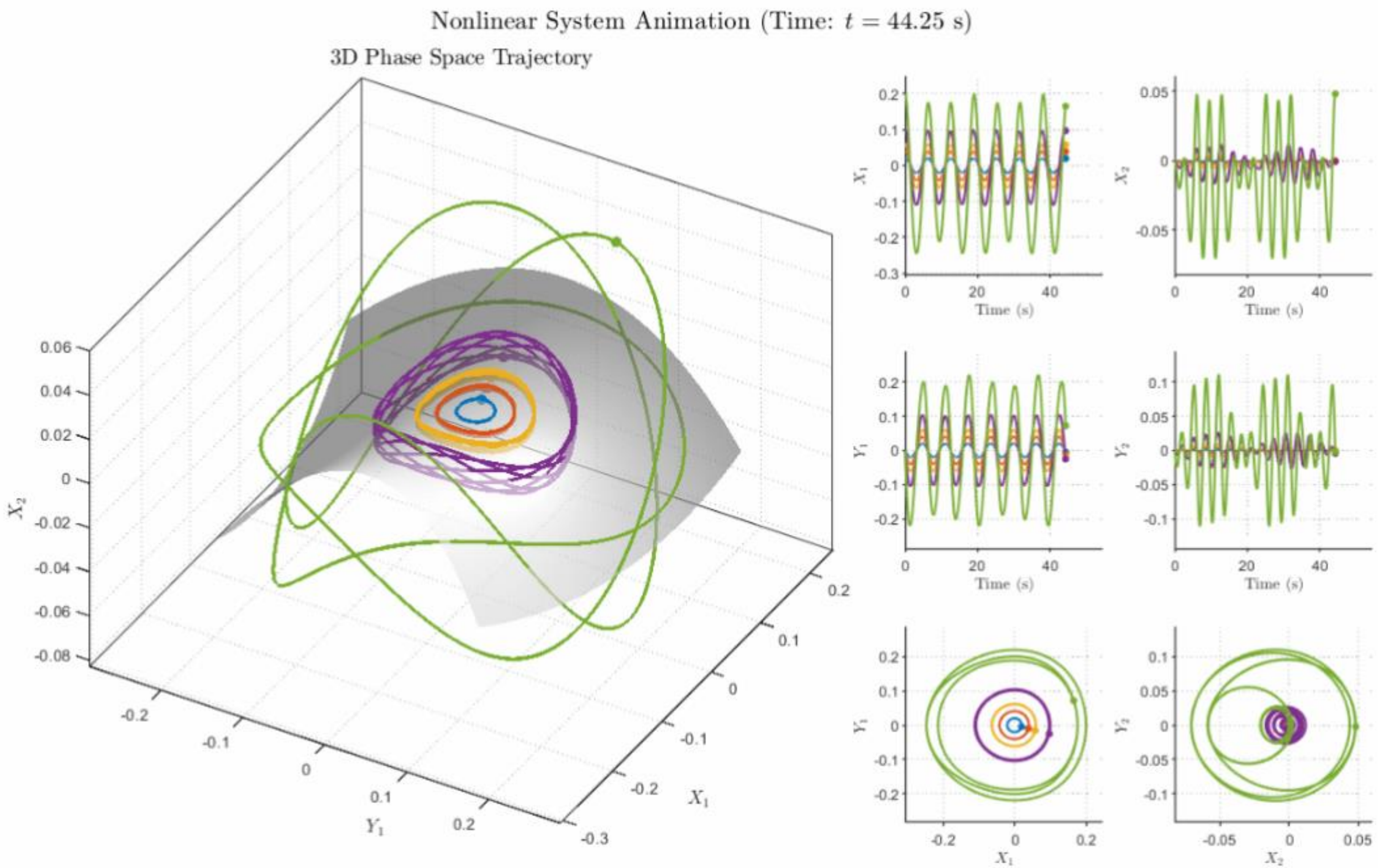
$$\mathbf{y}(t)\approx\sum_{i=1}^{d} z_i(t)\boldsymbol{\phi}_i$$

This is mathematically valid because the dynamics are decoupled!

In the nonlinear world $\mathbf{Q}(\mathbf{y},\cdots)\neq\mathbf{0}$**:**

- Nonlinear terms lead to **coupling among the modes**
- Even if the system is excited only in mode 1, the energy will still **transfer to other modes**
- **Conclusion:** The dynamics **no longer lie on a linear subspace** spanned by eigenvectors

Initial Idea: Linear Subspace

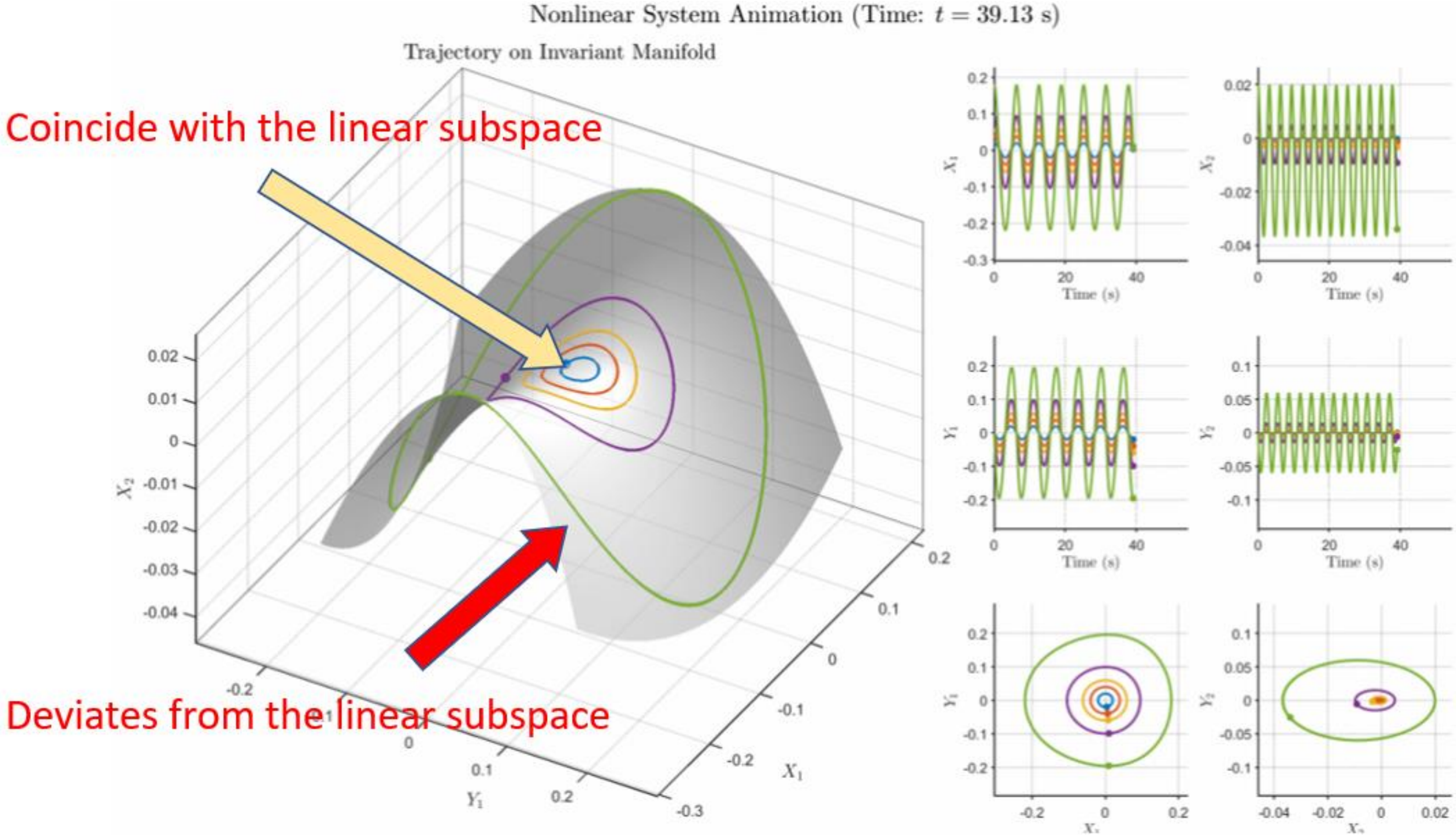


Problem: For **nonlinear systems**, does there exist a geometric manifold that constrains the system trajectories to **evolve within the same surface**?

New Perspective: A d-Dimensional Invariant Manifold

The dynamics are confined to a d-dimensional invariant manifold $\mathcal{M}$

- "**Invariant**" means any trajectory starting on $\mathcal{M}$ stays on $\mathcal{M}$ forever
- The manifold $\mathcal{M}$ is tangent at the origin $\mathbf{y}_0 = \mathbf{0}$ to the linear subspace spanned by the 2d-th dominant eigenvector
- Away from the origin, the nonlinear terms $\mathbf{Q}(\mathbf{y}, \cdots)$ bend this manifold



How Do We Find Invariant Manifold?

The Parametrization Method

Core idea: assume the d-dimensional manifold $\mathcal{M}$ can be parametrised by a set of d-dimensional "**modal coordinates**" $\mathbf{z} \in \mathbb{C}^d$

We seek an (unknown) nonlinear mapping $\mathbf{W}$:

$$\mathbf{y} = \mathbf{W}(\mathbf{z}) \text{ with } \mathbf{y} \in \mathbb{R}^{2N},\ \mathbf{z} \in \mathbb{C}^d$$

Simultaneously, we seek the (unknown) reduced dynamics $\mathbf{f}$ on the manifold:

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$$

This is our ROM!

Two unknowns : W and f, to construct in the form of polynomial series :

We look for two polynomial series:

- **Manifold map:** $\mathbf{y} = \mathbf{W}(\mathbf{z}) = \mathbf{W}^{(1)}\mathbf{z} + \mathbf{W}^{(2)}\mathbf{z}^2 + \cdots$
- **Reduced dynamic:** $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) = \mathbf{f}^{(1)}\mathbf{z} + \mathbf{f}^{(2)}\mathbf{z}^2 + \cdots$

Question: How do we solve for $\mathbf{W}$ and $\mathbf{f}$ simultaneously?

Polynomial Expansions: A Key Step

Let us recall the governing equations we have established

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{Q}(\mathbf{y}, \cdots) \qquad \mathbf{y} = \mathbf{W}(\mathbf{z}) \qquad \dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$$

Invariance Equation

$$\mathbf{B} \cdot \nabla_{\mathbf{z}} \mathbf{W}(\mathbf{z}) \cdot \mathbf{f}(\mathbf{z}) = \mathbf{A}\mathbf{W}(\mathbf{z}) + \mathbf{Q}(\mathbf{W}(\mathbf{z}), \cdots)$$

Multi-index notation

A monomial in $\mathbf{z}$ is written as

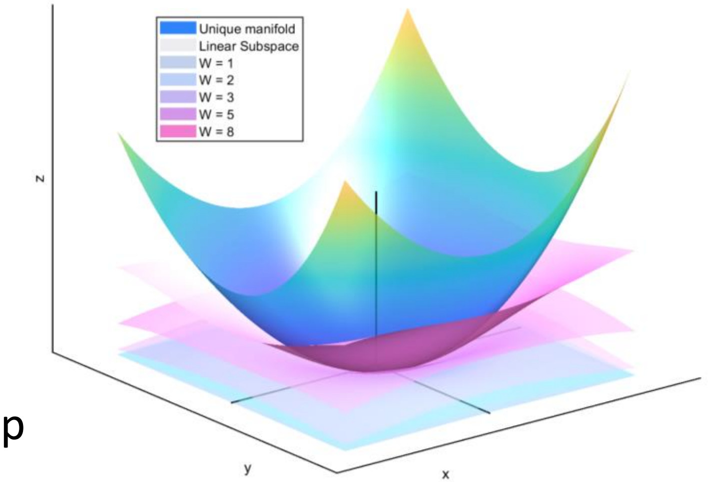
$$\mathbf{z}^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}, \ \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$$

with total degree $|\alpha| = \sum_{j=1}^{d} \alpha_j = p$

We expand both unknowns:

$$\mathbf{W}(\mathbf{z}) = \sum_{p=1}^{\infty} \sum_{|\alpha|=p} \mathbf{W}^{(p,\alpha)} \mathbf{z}^{\alpha}$$

$$\mathbf{f}(z) = \sum_{p=1}^{\infty} \sum_{|\alpha|=p} \mathbf{f}^{(p,\alpha)} \mathbf{z}^{\alpha}$$



Interpretation

- $\mathbf{W}^{(p,\alpha)}$ tells how the manifold **bends** at order p
- $\mathbf{f}^{(p,\alpha)}$ provides the **nonlinear coupling** in the ROM

Homological Equations via Order-by-Order Matching

Substitute the polynomial into the invariance equation and **collect terms by monomial** $\mathbf{z}^{\alpha}$

$$\mathbf{B} \cdot \nabla_{\mathbf{z}} \mathbf{W}(\mathbf{z}) \cdot \mathbf{f}(\mathbf{z}) = \mathbf{A}\mathbf{W}(\mathbf{z}) + \mathbf{Q}(\mathbf{W}(\mathbf{z}), \cdots)$$

$$\text{with } \mathbf{W}(\mathbf{z}) = \sum_{p=1}^{\infty} \sum_{|\alpha|=p} \mathbf{W}^{(p,\alpha)} \mathbf{z}^{\alpha}, \quad \mathbf{f}(z) = \sum_{p=1}^{\infty} \sum_{|\alpha|=p} \mathbf{f}^{(p,\alpha)} \mathbf{z}^{\alpha}, \textit{ with } \mathbf{z}^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}, \ \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$$

Each monomial yields a **separate linear problem**. For every degree **p** and multi-index α :

$$\left(\sigma^{(p,\alpha)} \mathbf{B} - \mathbf{A}\right) \mathbf{W}^{(p,\alpha)} + \mathbf{B}\mathbf{\Phi}^{R} \mathbf{f}^{(p,\alpha)} = \mathbf{R}^{(p,\alpha)}$$

where $\sigma^{(p,\alpha)} = \alpha_1 \lambda_1 + \cdots + \alpha_d \lambda_d$ (combination frequency)

- $\mathbf{R}^{(p,\alpha)}$: known right-hand side from lower-order terms
- $\mathbf{W}^{(p,\alpha)}$: manifold coefficient at order **p**
- $\mathbf{f}^{(p,\alpha)}$: ROM coefficient at order **p**

Key Point

A complicated nonlinear problem becomes a sequence of **linear systems**, one for each multi-index α

Eigenproperties of First-Order System

Spectral properties of first order system

$$\mathbf{B}=\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix} \quad \mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ -\mathbf{K} & -\mathbf{C}\end{bmatrix}$$

Generalized Eigenvalue Problem

$$(\lambda_j\mathbf{B}-\mathbf{A})\mathbf{\Phi}_j^R=\mathbf{0}$$

$$(\lambda_j\mathbf{B}^T-\mathbf{A}^T)\mathbf{\Phi}_j^L=\mathbf{0}$$

Ideal Decoupling

$$(\mathbf{\Phi}^L)^\dagger\mathbf{A}\mathbf{\Phi}^R=\mathbf{\Lambda}$$

$$(\mathbf{\Phi}^L)^\dagger\mathbf{B}\mathbf{\Phi}^R=\mathbf{I}$$

Reality:

- **Symmetry is lost** in the first-order equation
- Direct computation of left eigenvectors from the eigenvalue problem is **not feasible**

Loss of bi-orthogonality

Spectral properties of second order system

$$(\mathbf{K}-\omega_k^2\mathbf{M})\boldsymbol{\psi}_k=\mathbf{0}$$

Considering the case of **Rayleigh damping,** based on the orthogonality relationship

$$\boldsymbol{\psi}_k^T\mathbf{M}\boldsymbol{\psi}_l=\delta_{kl} \quad \boldsymbol{\psi}_k^T\mathbf{K}\boldsymbol{\psi}_l=\omega_k^2\delta_{kl}$$

$$\boldsymbol{\psi}_k^T\mathbf{C}\boldsymbol{\psi}_l=2\xi_k\omega_k\delta_{kl}$$

The **conjugate eigenvalues** can be expressed

$$\lambda_k,\bar{\lambda}_k=-\xi_k\omega_k\pm i\omega_k\sqrt{1-\xi_k^2}$$

Right eigenvectors are constructed directly from the **displacement** and **velocity**

$$\mathbf{\Phi}_k^R=\begin{bmatrix}\boldsymbol{\psi}_k & \lambda_k\boldsymbol{\psi}_k & \boldsymbol{\psi}_k & \bar{\lambda}_k\boldsymbol{\psi}_k\end{bmatrix}^T$$

The left eigenvectors are constructed using the **bi-orthogonality property**

$$\mathbf{\Phi}_k^L=\begin{bmatrix}\frac{\bar{\lambda}_k\boldsymbol{\psi}_k}{\lambda_k-\bar{\lambda}_k} & \frac{\boldsymbol{\psi}_k}{\lambda_k-\bar{\lambda}_k} & \frac{\lambda_k\boldsymbol{\psi}_k}{\lambda_k-\bar{\lambda}_k} & \frac{\boldsymbol{\psi}_k}{\lambda_k-\bar{\lambda}_k}\end{bmatrix}^T$$

The left and right eigenvectors satisfy

$$(\mathbf{\Phi}^L)^\dagger\mathbf{B}\mathbf{\Phi}^R=\mathbf{I} \quad (\mathbf{\Phi}^L)^\dagger\mathbf{A}\mathbf{\Phi}^R=\mathbf{\Lambda}$$

Homological Equations via Order-by-Order Matching

Substitute the polynomial into the invariance equation and **collect terms by monomial** $\mathbf{z}^{\alpha}$

$$\mathbf{B} \cdot \nabla_{\mathbf{z}} \mathbf{W}(\mathbf{z}) \cdot \mathbf{f}(\mathbf{z}) = \mathbf{A}\mathbf{W}(\mathbf{z}) + \mathbf{Q}(\mathbf{W}(\mathbf{z}), \cdots)$$

with $\mathbf{W}(\mathbf{z}) = \sum_{p=1}^{\infty} \sum_{|\alpha|=p} \mathbf{W}^{(p,\alpha)} \mathbf{z}^{\alpha}$, $\mathbf{f}(z) = \sum_{p=1}^{\infty} \sum_{|\alpha|=p} \mathbf{f}^{(p,\alpha)} \mathbf{z}^{\alpha}$, *with* $\mathbf{z}^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$

Each monomial yields a **separate linear problem**. For every degree **p** and multi-index α :

$$(\sigma^{(p,\alpha)}\mathbf{B} - \mathbf{A})\,\mathbf{W}^{(p,\alpha)} + \mathbf{B}\mathbf{\Phi}^{R}\,\mathbf{f}^{(p,\alpha)} = \mathbf{R}^{(p,\alpha)}$$

where $\sigma^{(p,\alpha)} = \alpha_1\lambda_1 + \cdots + \alpha_d\lambda_d$ (combination frequency)

- $\mathbf{R}^{(p,\alpha)}$: known right-hand side from lower-order terms
- $\mathbf{W}^{(p,\alpha)}$: manifold coefficient at order **p**
- $\mathbf{f}^{(p,\alpha)}$: ROM coefficient at order **p**

Key Point

A complicated nonlinear problem becomes a sequence of **linear systems**, one for each multi-index α

Central Issue: Resonance ⟷ Singular Matrix

$$\underbrace{\left(\sigma^{(p,\alpha)}\mathbf{B}-\mathbf{A}\right)}_{\mathbf{L}}\mathbf{W}^{(p,\alpha)}+\mathbf{B}\mathbf{\Phi}^{R}\mathbf{f}^{(p,\alpha)}=\mathbf{R}^{(p,\alpha)}$$

Key insight 1: Underdetermined equation

Both variables $\mathbf{W}^{(p,\alpha)}$ and $\mathbf{f}^{(p,\alpha)}$ are unknown, but there is only one independent equation, leading to **an underdetermined problem**

Key insight 2: When is the matrix $\mathbf{L}=\left(\sigma^{(p,\alpha)}\mathbf{B}-\mathbf{A}\right)$ singular?

- $\sigma^{(p,\alpha)}$ is the combination frequency of the current monomial $\mathbf{z}^{\alpha}$ at order *p* $(e.g.,\ \text{for}\ p=3,\ \text{case}\ \mathbf{z}^{\alpha}=z_1^2 z_2,\ \text{then}\ \sigma=2\lambda_1+\lambda_2\ )$
- $\mathbf{L}$ is singular if $\sigma^{(p,\alpha)}$ equals an eigenvalue λ_r of the system. $\sigma^{(p,\alpha)}\approx\lambda_r$

Physics ⟷ Algebra

A **physical resonance** $(\sigma^{(p,\alpha)}\approx\lambda_r)$ lead to an **algebraic singularity** ($\mathbf{L}$ is non-invertible)

Final Dilemma

At resonance $(\sigma^{(p,\alpha)}\approx\lambda_r)$, how do we solve a system that is **underdetermined** (two unknowns) and **singular** $\mathbf{L}$?

Resolution: Make a Choice (Parametrization Styles)

We must impose a **constraint** to ensure **uniqueness**. This choice defines the **"style"**

$$(\sigma^{(p,\alpha)}\mathbf{B}-\mathbf{A})\mathbf{W}^{(p,\alpha)}+\mathbf{B}\boldsymbol{\Phi}^{R}\mathbf{f}^{(p,\alpha)}=\mathbf{R}^{(p,\alpha)}$$

The most direct approach is to impose the **simplest reduced-order constraint** on the system

$$\begin{bmatrix}\sigma^{(p,\alpha)}\mathbf{B}-\mathbf{A} & \mathbf{B}\boldsymbol{\Phi}^{R}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{pmatrix}\mathbf{W}^{(p,\alpha)}\\ \mathbf{f}^{(p,\alpha)}\end{pmatrix}=\begin{pmatrix}\mathbf{R}^{(p,\alpha)}\\ \mathbf{0}\end{pmatrix}$$

singular?

No ROM ?

- At physical resonance $(\sigma^{(p,\alpha)}\approx\lambda_r)$, the first row of the system becomes singular, causing the equations to **remain underdetermined**
- Because a **null space exists**, there exist infinite solutions $\mathbf{W}^{(p,\alpha)}+\epsilon\,\boldsymbol{\Phi}_r^{R}$

Key issue: how to eliminate the null space?

Constrain $\mathbf{W}^{(p,k)}$ to have no free part along the **dominant modal direction** $(\boldsymbol{\Phi}_r^{L})^{\dagger}\mathbf{B}\mathbf{W}^{(p,\alpha)}=\mathbf{0}$

$$\begin{bmatrix}\sigma^{(p,\alpha)}\mathbf{B}-\mathbf{A} & \mathbf{B}\boldsymbol{\Phi}_{\mathfrak{R}}^{R} & \mathbf{0}\\ (\boldsymbol{\Phi}_{\mathfrak{R}}^{L})^{\dagger}\mathbf{B} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{pmatrix}\mathbf{W}^{(p,\alpha)}\\ \mathbf{f}_{\mathfrak{R}}^{(p,\alpha)}\\ \mathbf{f}_{\not\mathfrak{R}}^{(p,\alpha)}\end{pmatrix}=\begin{pmatrix}\mathbf{R}^{(p,\alpha)}\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}$$

Normal Form Style

$$\begin{bmatrix}\sigma^{(p,\alpha)}\mathbf{B}-\mathbf{A} & \mathbf{B}\boldsymbol{\Phi}^{R}\\ (\boldsymbol{\Phi}^{L})^{\dagger}\mathbf{B} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\mathbf{W}^{(p,\alpha)}\\ \mathbf{f}^{(p,\alpha)}\end{pmatrix}=\begin{pmatrix}\mathbf{R}^{(p,\alpha)}\\ \mathbf{0}\end{pmatrix}$$

Graph Style

Resolution: Make a Choice (Parametrization Styles)

Normal Form style

Choice: we want the **dynamics** $\mathbf{f}^{(p,\alpha)}$ to be as simple as possible.

Constraint: enforce $\mathbf{f}^{(p,\alpha)} = \mathbf{0}$ **unless** there is resonance

$$f_s^{(p,\alpha)} = 0 \quad (\text{if } s \notin \mathscr{R})$$

Advantage:

- facilitating analytical computation
- able to bypass the **geometric folding**

Disadvantage: It may easily overlook potential resonant terms

Graph style

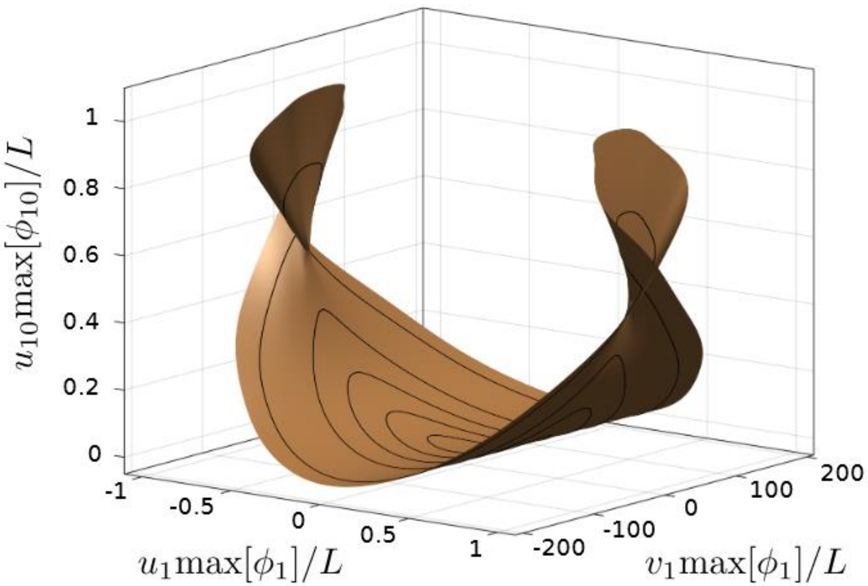
Choice: we want $\mathbf{W}^{(p,\alpha)}$ to be as simple as possible

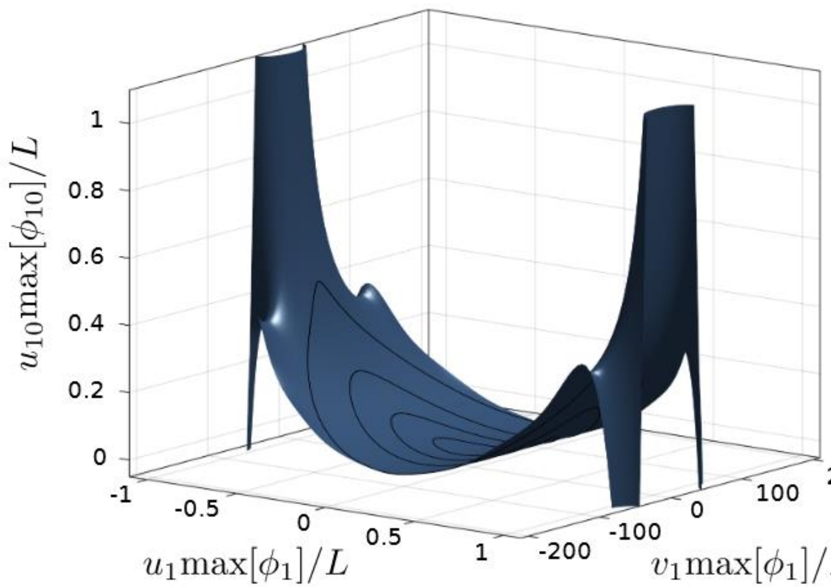
Constraint: force the components of $\mathbf{W}^{(p,\alpha)}$ in the master subspace to be zero

$$(\mathbf{\Phi}_r^L)^\dagger \mathbf{B}\mathbf{W}^{(p,\alpha)} = \mathbf{0} \quad (\text{for all } r)$$

Advantage: all resonance are considered

Disadvantage: The one-to-one mapping between the high-dimensional and reduced spaces inevitably leads to **geometric folding**





Final Challenge: How to Handle External Forces?

"**Invariant**" means any trajectory starting on $\mathcal{M}$ stays on $\mathcal{M}$ forever

$$\mathbf{z}(t_0) \in \mathcal{M} \Rightarrow \mathbf{z}(t_1) \in \mathcal{M},\ \forall t,\ \mathcal{M} \subset \mathbb{C}^d \quad \Rightarrow \quad \phi_{t+s} = \phi_s \circ \phi_t$$ Semigroup property

The **semigroup property** requires the system to remain **autonomous** $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$

Question: what if the system is externally forced?

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{Q}(\mathbf{y}, \cdots) + \boldsymbol{\Upsilon} e^{i\Omega t} \quad \Rightarrow \quad \dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) + \boldsymbol{\tau}(t)$$

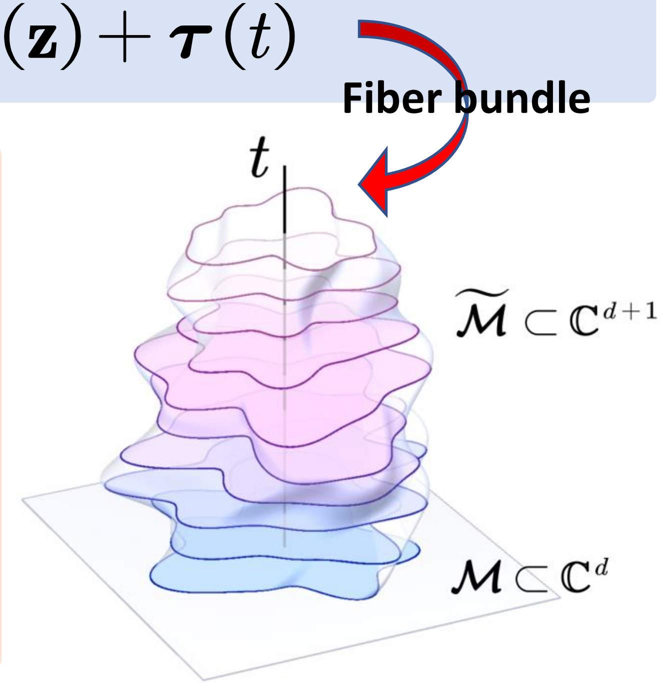
The extended invariant manifold:

incorporating the time $\widetilde{\mathcal{M}} = \cup_t (\mathcal{M} \times t),\ \widetilde{\mathcal{M}} \subset \mathbb{C}^{d+1}$

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{Q}(\mathbf{y}, \cdots) + \boldsymbol{\Upsilon} \tilde{z}$$

$$\begin{pmatrix} \dot{\mathbf{z}} \\ \dot{\tilde{z}} \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{z}) \\ i\Omega \tilde{z} \end{pmatrix}$$

- Generalized autonomous system
- restoring the semigroup property



We have now completed the theoretical development and numerical implementation of the entire parametrized model reduction framework!

Nonlinear dynamic for thin structure

From Single to Multi-Mode Resonance

1. Nonlinear dynamics in the frequency domain

2. Thin-structure modeling based on continuum shell elements

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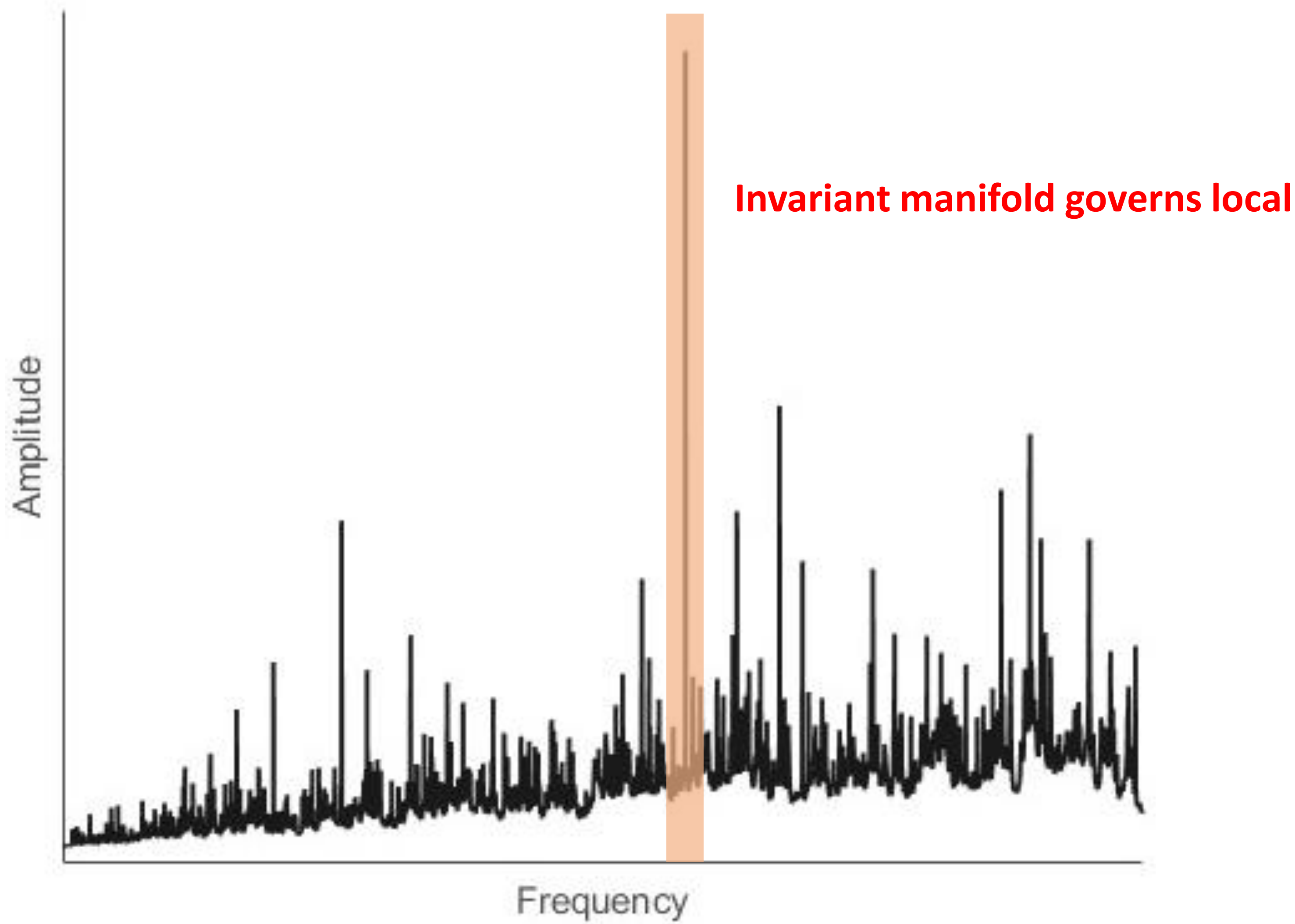
Nonlinear Dynamics in Frequency Domain

Global Frequency Response： illustrates the relationship between the external excitation frequency and the structural response amplitude



Question: Why is it sufficient to focus only on the local nonlinear dynamical behavior?

Nonlinear Dynamics in Frequency Domain

Why Focusing on Local Frequency Response Instead of Global Spectrum

1. **Key idea:** *Nonlinear dynamics live on the peak, not the whole spectrum*

2. Critical dynamic phenomena occur locally
   - Backbone bending (hardening/softening)
   - Bifurcations and multi-stability
   - Internal resonances between modes

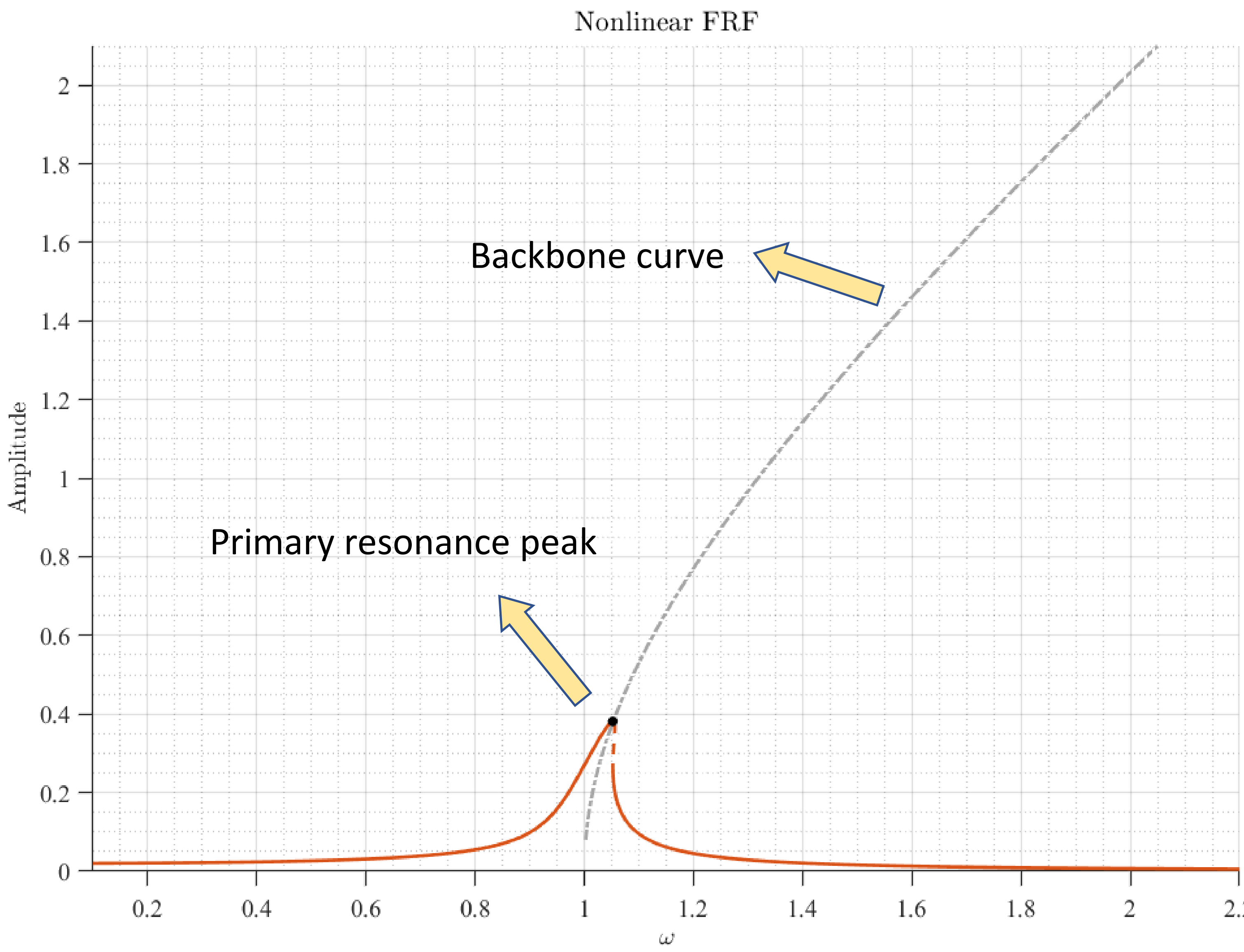
3. Engineering interest lies at peak response
   - Peak amplitude determines **safety & failure risk**
   - Stability issues and flutter arise near resonance
   - Control & design decisions depend on local behavior

4. Computational efficiency
   - Full-spectrum nonlinear continuation is expensive
   - Local reduced-order models capture essential dynamics
   - Focused analysis = lower cost, clearer physics

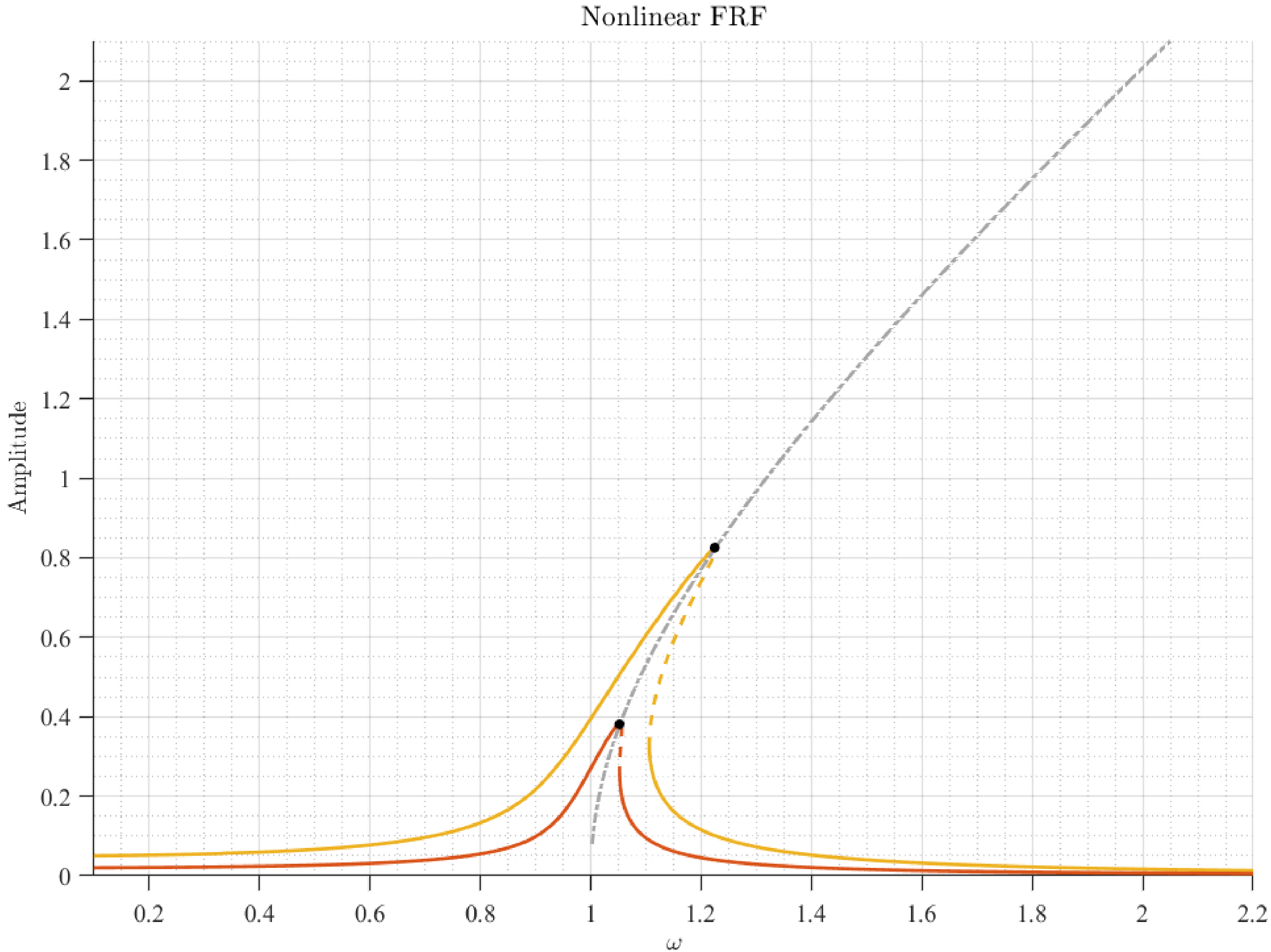
Global FRF tells us where to look,
Local FRF reveals what actually happens.

Some Typical Nonlinear Characteristics

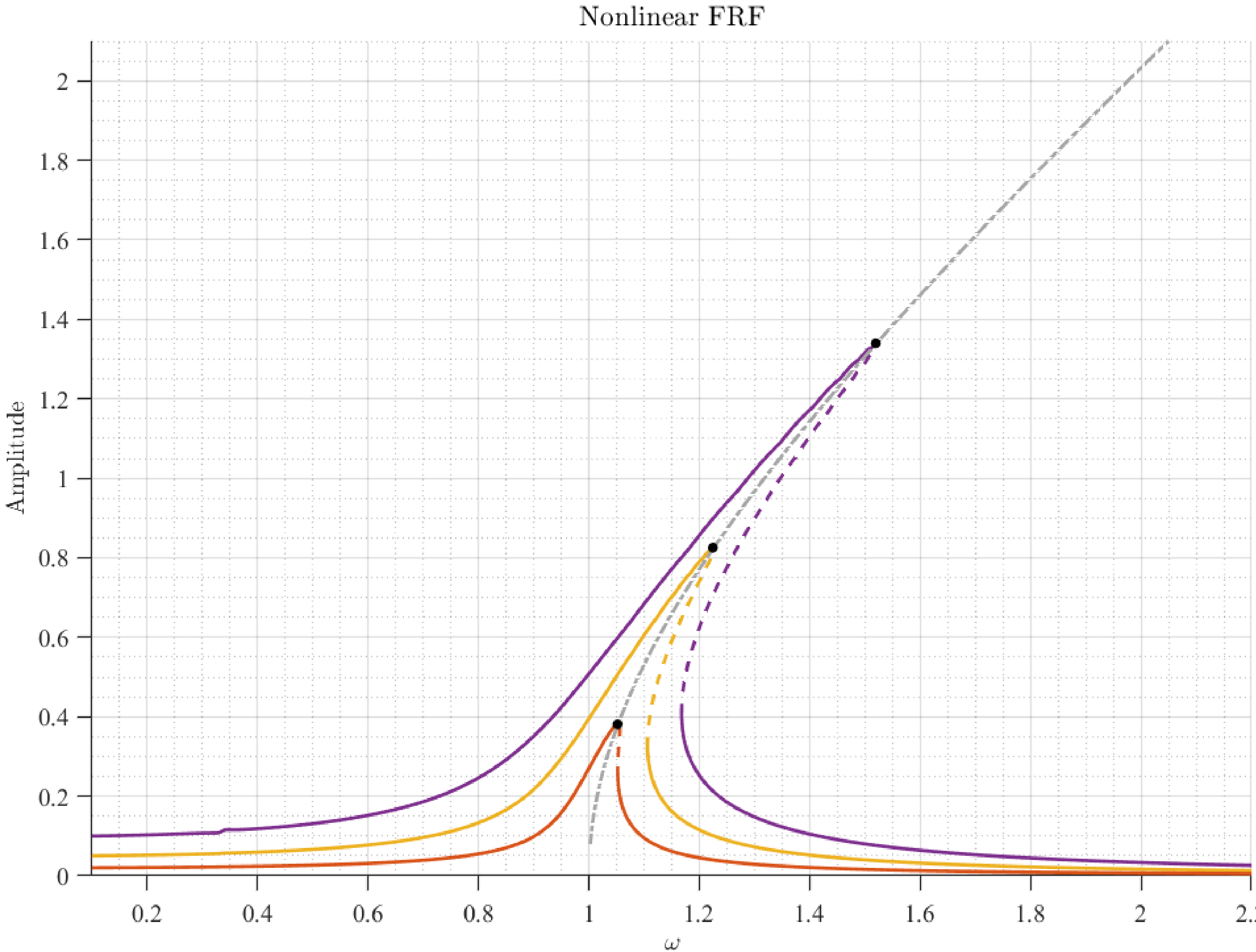
Hardening behavior: The maximum response amplitude lags behind the natural frequency



Some Typical Nonlinear Characteristics

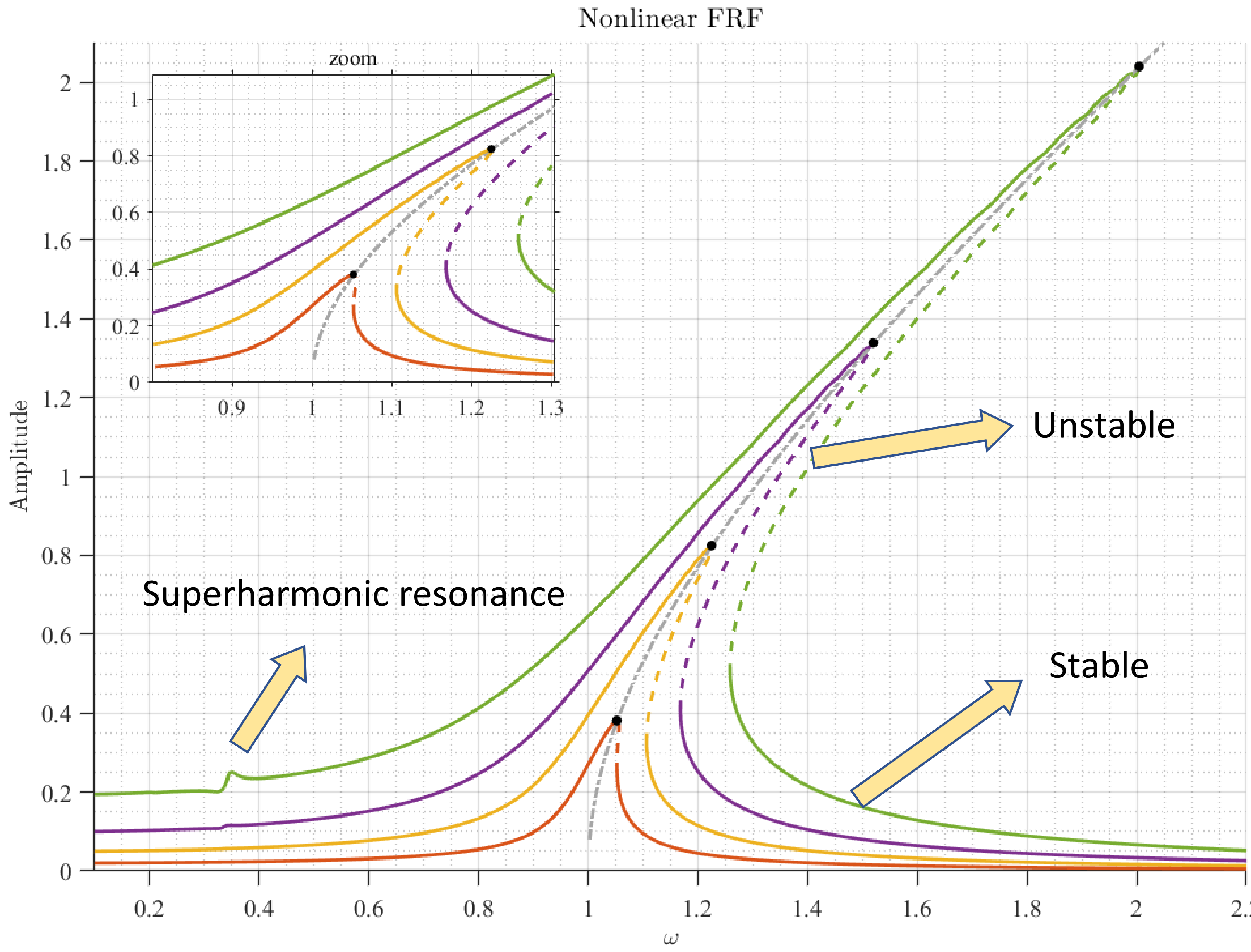


Some Typical Nonlinear Characteristics



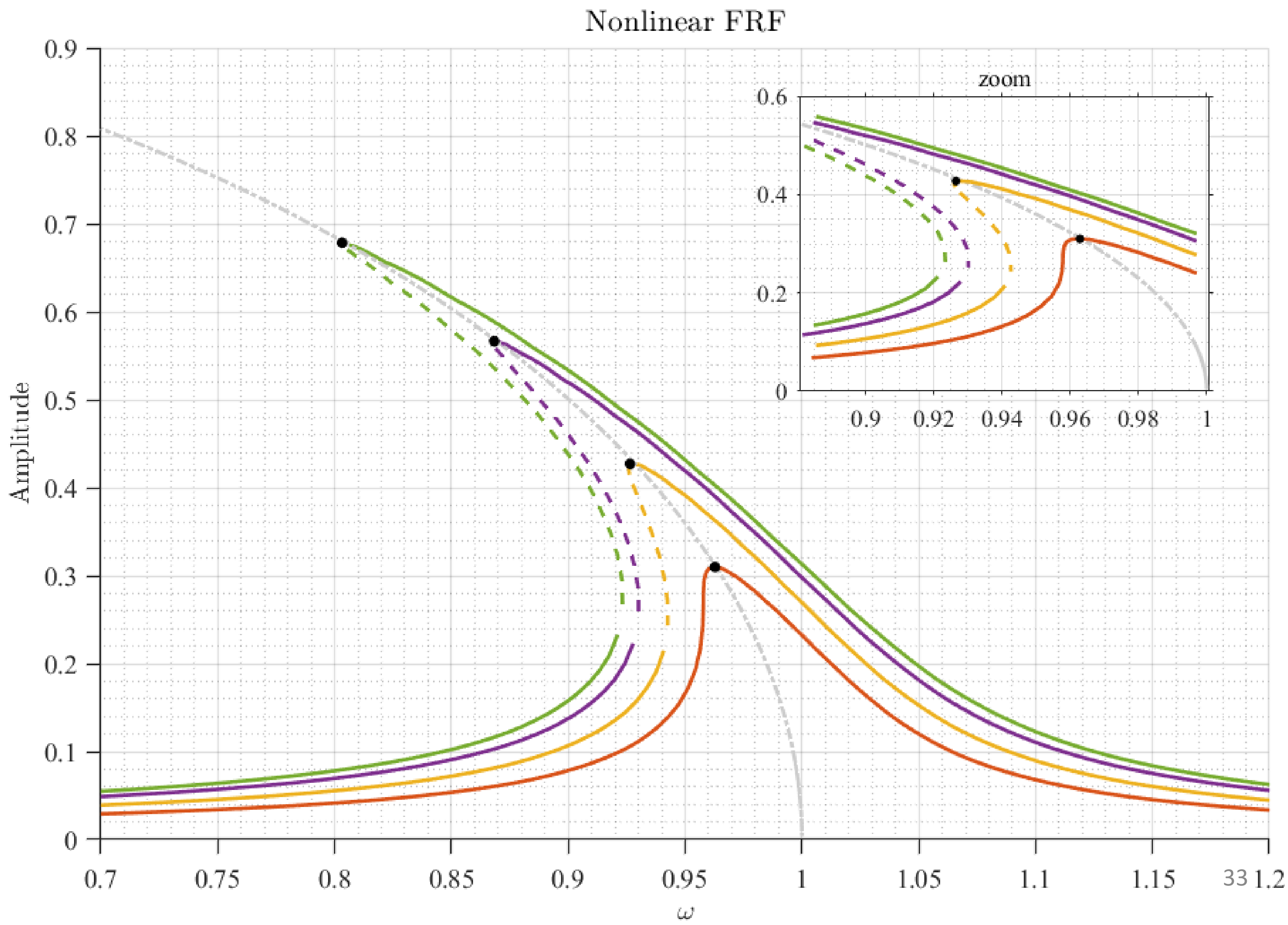
Some Typical Nonlinear Characteristics

Hardening behavior: The maximum response amplitude lags behind the natural frequency



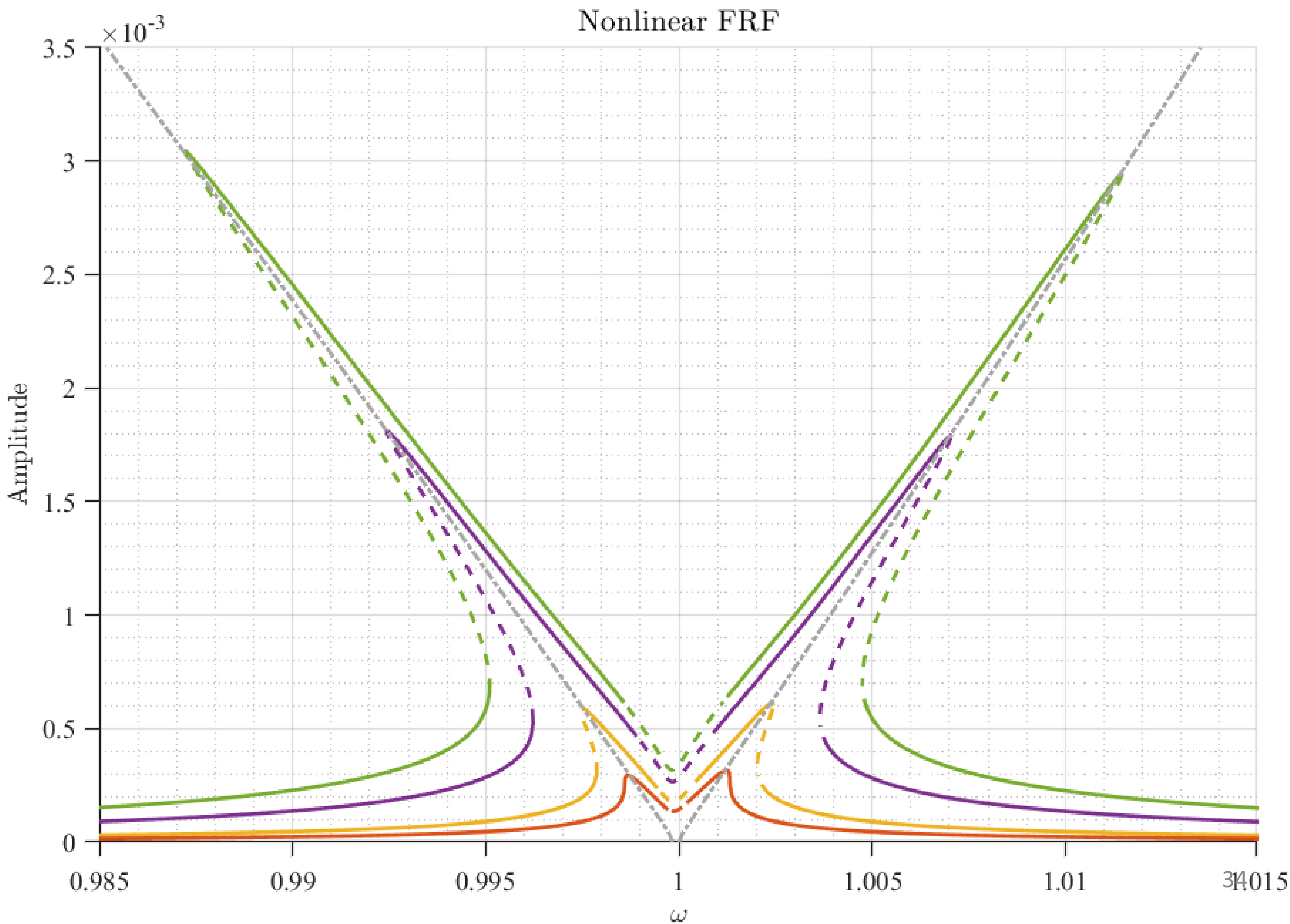
Some Typical Nonlinear Characteristics

Softening behavior: The maximum response amplitude leads the natural frequency



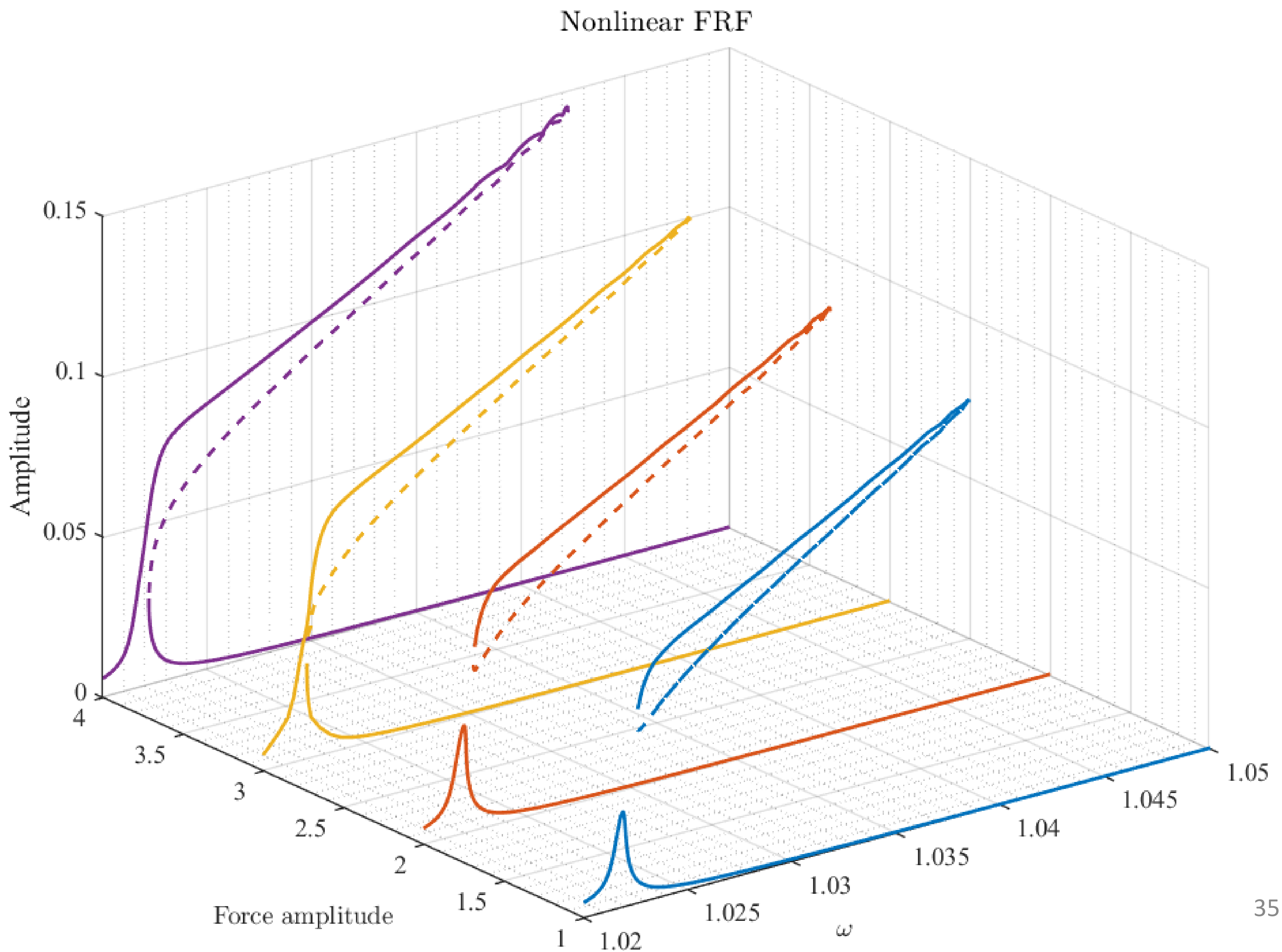
Some Typical Nonlinear Characteristics

Internal resonance: Integer ratio relationships between the natural frequencies $\omega_i : \omega_j = p : q$



Some Typical Nonlinear Characteristics

Isolated solution: symmetry breaking of the system



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Curved Shell Structure Modeling: Governing Equations

Positional relationship

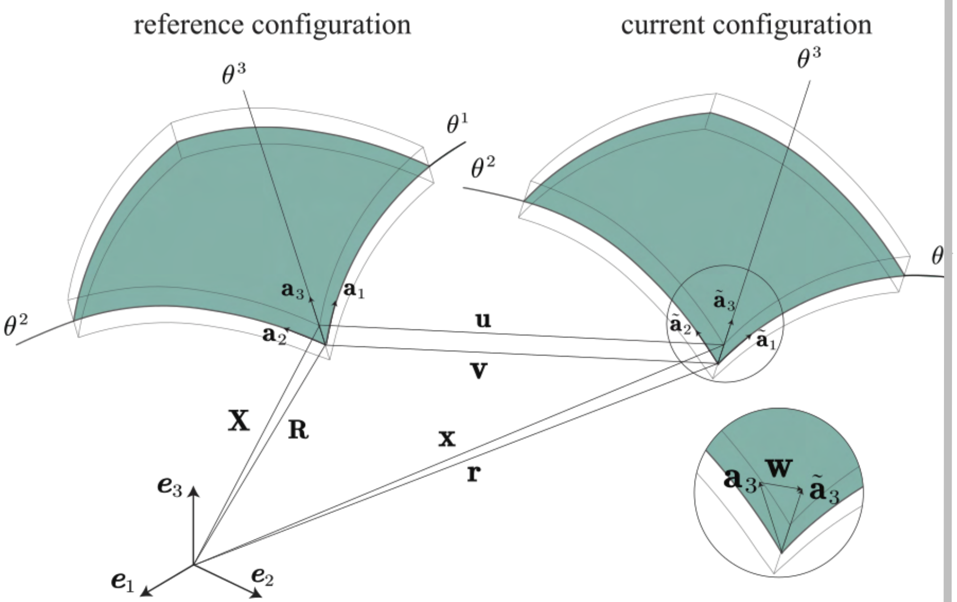
$$\mathbf{X}(\theta^\alpha,\theta^3)=\mathbf{R}(\theta^\alpha)+\theta^3\mathbf{a}_3(\theta^\alpha),\alpha=1,2$$

$$\mathbf{x}(\theta^\alpha,\theta^3)=\mathbf{r}(\theta^\alpha)+\theta^3\tilde{\mathbf{a}}_3(\theta^\alpha),\alpha=1,2$$

Covariant base tensor

$$\mathbf{G}_\alpha=\mathbf{X}_{,\alpha}=\mathbf{a}_\alpha+\theta^3\mathbf{a}_{3,\alpha} \qquad \mathbf{g}_\alpha=\mathbf{x}_{,\alpha}=\tilde{\mathbf{a}}_\alpha+\theta^3\tilde{\mathbf{a}}_{3,\alpha}$$

$$\mathbf{G}_3=\mathbf{X}_3=\mathbf{a}_3 \qquad \mathbf{g}_3=\mathbf{x},3=\tilde{\mathbf{a}}_3$$



Green-Lagrange strain

$$\begin{cases} E_{ij}=\frac{1}{2}(\mathbf{g}_i\cdot\mathbf{g}_j-\mathrm{G}_i\cdot\mathrm{G}_j) \\ E_{ij}=\frac{1}{2}\left(\mathbf{G}_i\cdot\frac{\partial\mathbf{u}}{\partial\theta^j}+\mathbf{G}_j\cdot\frac{\partial\mathbf{u}}{\partial\theta^i}+\frac{\partial\mathbf{u}}{\partial\theta^i}\frac{\partial\mathbf{u}}{\partial\theta^j}\right) \end{cases}$$

Constitutive relation

$$\mathbf{S}=\mathbb{D}:\mathbf{E}$$ **Lead to locking issues**

$$E_{33}^{(0)}+\theta^3E_{33}^{(1)}\simeq-\frac{D^{33ij}}{D^{3333}}(E_{ij}^{(0)}+\theta^3E_{ij}^{(1)})$$

Enhanced assumed strain

$$\mathbf{E}^{\text{full}}=\mathbf{E}+\tilde{\mathbf{E}} \quad \text{with} \quad \int_\Omega \mathbf{S}:\delta\tilde{\mathbf{E}}\mathrm{d}\Omega=0$$

Hu-Washizu functional

$$\Pi_{HW}(\mathbf{u},\mathbf{E}^{\text{full}},\mathbf{S})=\int_\Omega\frac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\mathrm{d}\Omega+\int_\Omega\frac{1}{2}\mathbf{E}^{\text{full}}:\mathbb{D}:\mathbf{E}^{\text{full}}\mathrm{d}\Omega-\int_\Omega\mathbf{S}:(\mathbf{E}^{\text{full}}-\mathbf{E})\mathrm{d}\Omega-\mathbf{F}(t)\cdot\mathbf{u}$$

$$\begin{cases} \int_\Omega\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\mathrm{d}\Omega+\int_\Omega\mathbf{S}:\delta\mathbf{E}\mathrm{d}\Omega-\mathbf{F}(t)\cdot\delta\mathbf{u}=0 \\ \int_\Omega\mathbf{S}:\delta\tilde{\mathbf{E}}\mathrm{d}\Omega=0 \end{cases}$$

Extend equations to avoid locking issues

After finite element discretization, the **second-order dynamical system** can be written as

$$\begin{bmatrix} \mathbf{m}^{(e)} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}^{(e)} \\ \ddot{\boldsymbol{\alpha}}^{(e)} \end{Bmatrix} + \begin{bmatrix} \mathbf{k}_{uu}^{(e)} & \mathbf{k}_{u\alpha}^{(e)} \\ (\mathbf{k}_{u\alpha}^{(e)})^T & \mathbf{k}_{\alpha\alpha}^{(e)} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{(e)} \\ \boldsymbol{\alpha}^{(e)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{ext}^{(e)}(t) \\ 0 \end{Bmatrix}$$

EAS terms enhanced by static compression

$$[\mathbf{m}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\} + [\mathbf{k}^{*(e)}(\mathbf{u}^{(e)})]\{\mathbf{u}^{(e)}\} = \{\mathbf{f}_{ext}^{(e)}(t)\}$$

$$[\mathbf{k}^{*(e)}(\mathbf{u}^{(e)})] = [\mathbf{k}_{uu}^{(e)}(\mathbf{u}^{(e)})] - [\mathbf{k}_{u\alpha}^{(e)}(\mathbf{u}^{(e)})][\mathbf{k}_{\alpha\alpha}^{(e)}]^{-1}([\mathbf{k}_{u\alpha}^{(e)}(\mathbf{u}^{(e)})])^T$$

After **assembling the elements** and categorizing the order according to the **displacement**

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\} + [\mathbf{C}]\{\dot{\mathbf{U}}\} + [\mathbf{K}_L]\{\mathbf{U}\} + \{\mathbf{G}(\mathbf{U},\mathbf{U})\} + \{\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})\} = \{\mathbf{F}(t)\}$$

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{Q}(\mathbf{y}, \cdots) + \boldsymbol{\Upsilon} e^{i\Omega t}$$

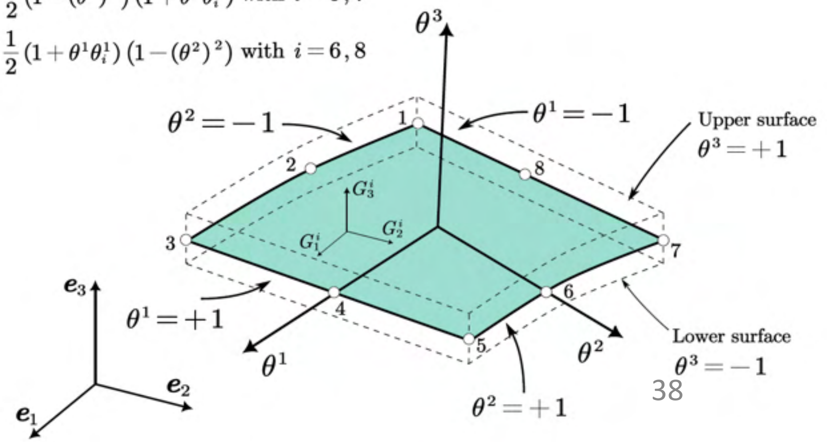
$$\mathbf{y} = \mathbf{W}(\mathbf{z}) \ \text{ with } \mathbf{y} \in \mathbb{R}^{2N}, \ \mathbf{z} \in \mathbb{C}^{d+1}$$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$$

$$N^i = \frac{1}{4}(1+\theta^1\theta^1_i)(1+\theta^2\theta^2_i)(\theta^1\theta^1_i+\theta^2\theta^2_i-1) \ \text{ with } i=1,2,3,4$$

$$N^i = \frac{1}{2}(1-(\theta^1)^2)(1+\theta^2\theta^2_i) \ \text{ with } i=5,7$$

$$N^i = \frac{1}{2}(1+\theta^1\theta^1_i)(1-(\theta^2)^2) \ \text{ with } i=6,8$$



Everything is now ready!

Nonlinear dynamic for thin structure

From Single to Multi-Mode Resonance

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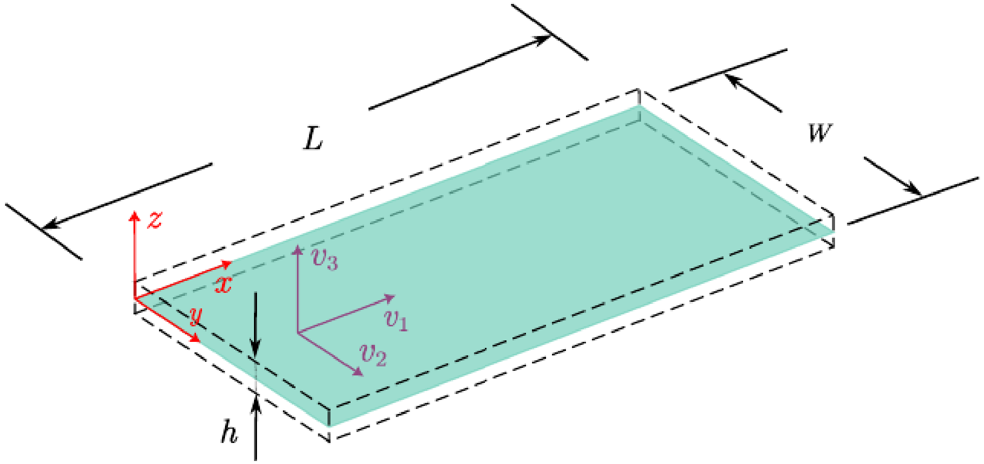
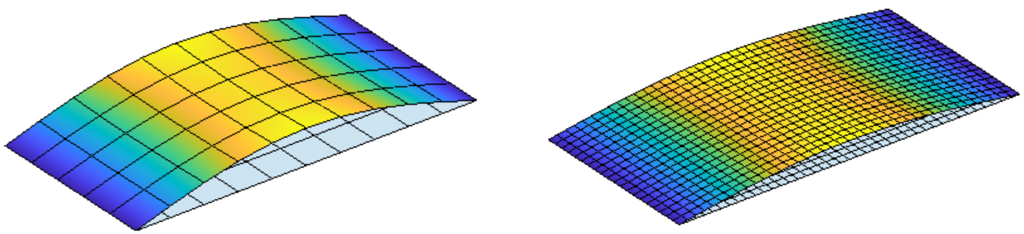
Shell vs. Solid Elements: Single Mode Reduction



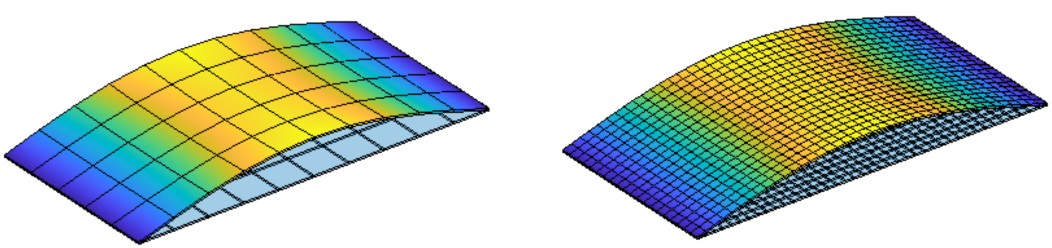
$$v_1 = v_2 = v_3 = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L$$

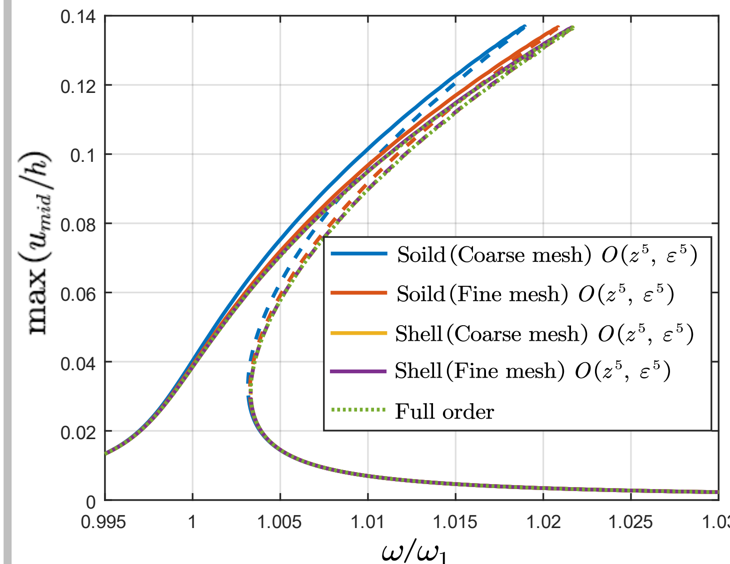
$$\theta_1 = \theta_2 = \theta_3 = free, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L$$

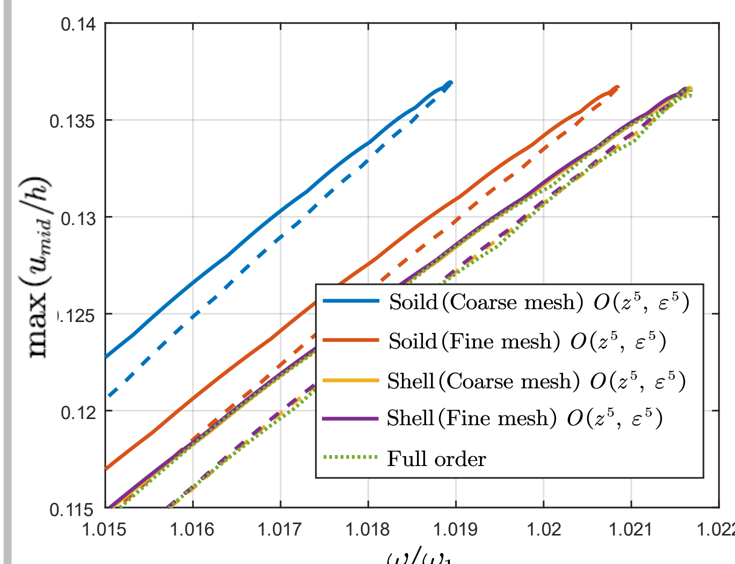
Ramme Shell (from coarse mesh to fine mesh)



Solid element (from coarse mesh to fine mesh)

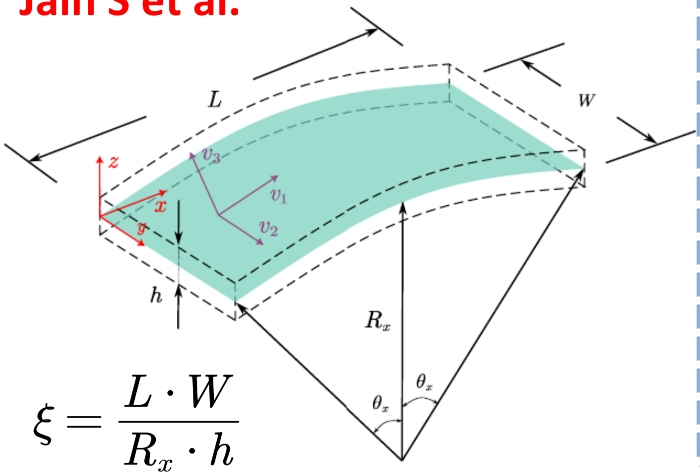






From Flat Plates to Curved Shells: Single Mode Reduction

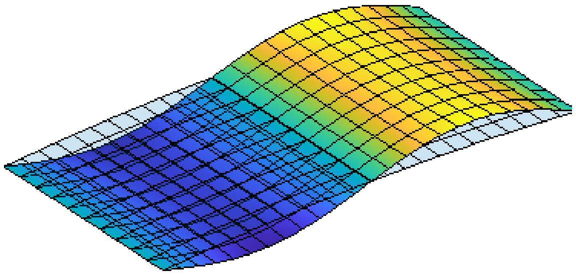
Jain S et al.

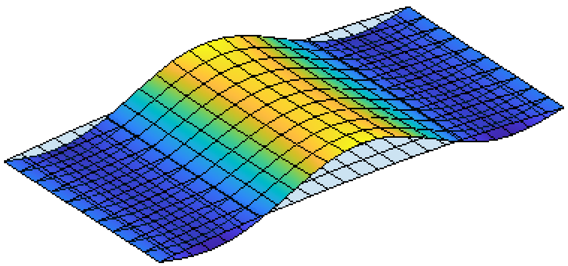


$$\xi = \frac{L \cdot W}{R_x \cdot h}$$

Exact 1:2 internal resonance $\xi_0 = 16.4609$

$\omega_1 = 23.71136$ $\omega_2 = 47.36333$





Normal Form-Based Single-Mode ROM

$$\dot{z}_1 = z_2 \cdot \Omega(z_1^2 + z_2^2)$$
$$\dot{z}_2 = -z_1 \cdot \Omega(z_1^2 + z_2^2)$$

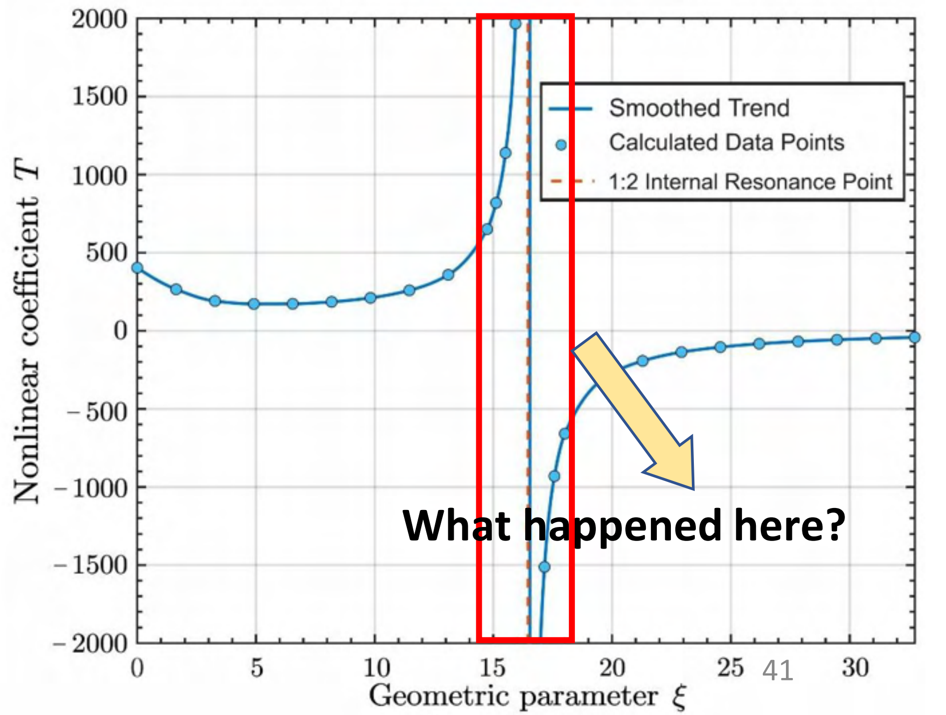
Stabile et al.

Polynomial Expansion of Frequency

$$\Omega = K_0 + K_1 A^2 + K_2 A^4 + \cdots$$

Backbone Curve Expression

$$\omega_{NL} = \Omega(A^2) = K_0 \left(1 + \underbrace{\frac{K_1}{K_0}}_{T} A^2 + \frac{K_2}{K_0} A^4 + \cdots \right)$$



From Flat Plates to Curved Shells: Dual Mode Reduction

Dual-Master-Mode Reduced-Order Model

$$\dot{z}_1 = i\omega_1 z_1 + i\alpha_{23} z_2 z_3 + i\beta_{113} z_1^2 z_3 + i\beta_{124} z_1 z_2 z_4$$

$$\dot{z}_2 = i\omega_2 z_2 + i\alpha_{11} z_1^2 + i\beta_{123} z_1 z_2 z_3 + i\beta_{224} z_2^2 z_4$$

$$\dot{z}_3 = -i\omega_1 z_3 - i\alpha_{23} z_4 z_1 - i\beta_{113} z_3^2 z_1 - i\beta_{124} z_3 z_4 z_2$$

$$\dot{z}_4 = -i\omega_2 z_4 - i\alpha_{11} z_3^2 - i\beta_{123} z_3 z_4 z_1 - i\beta_{224} z_4^2 z_2$$

Express the ROM coordinates in polar form

$$z_1 = \frac{1}{2}\rho_1 e^{i\theta_1}, \quad z_2 = \frac{1}{2}\rho_2 e^{i\theta_2}$$

Substitute into the reduced system

$$\dot{\rho}_1 = -\frac{1}{2}\alpha_{23}\rho_1\rho_2\sin(\theta_2 - 2\theta_1)$$

$$\dot{\theta}_1 = \omega_1 + \frac{1}{2}\alpha_{23}\rho_2\cos(\theta_2 - 2\theta_1) + \frac{1}{4}\beta_{113}\rho_1^2 + \frac{1}{4}\beta_{124}\rho_2^2$$

$$\dot{\rho}_2 = \frac{1}{2}\alpha_{11}\rho_1^2\sin(\theta_2 - 2\theta_1) \qquad \gamma(t) = \theta_2(t) - 2\theta_1(t)$$

$$\dot{\theta}_2 = \omega_2 + \frac{1}{2}\alpha_{11}\frac{\rho_1^2}{\rho_2}\cos(\theta_2 - 2\theta_1) + \frac{1}{4}\beta_{123}\rho_1^2 + \frac{1}{4}\beta_{224}\rho_2^2$$

The backbone curve of the first mode $\dot{\theta}_1 = 0$

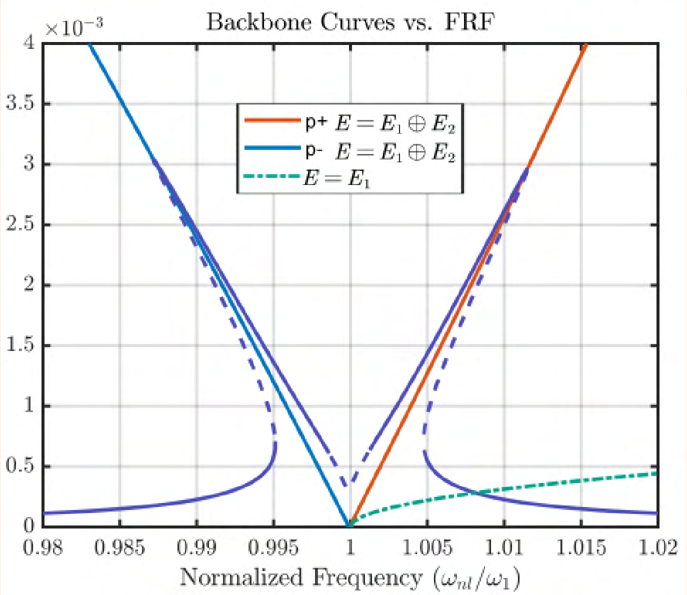
A fixed point is defined as $\dot{\rho}_1 = \dot{\rho}_2 = \dot{\gamma} = 0$

The backbone curve is defined by a hyperbolic

$$\omega_{NL}^{p+} = \omega_1 + \frac{\alpha_{23}}{2}\rho_2 + \frac{\beta_{113}}{4}\rho_1^2 + \frac{\beta_{124}}{4}\rho_2^2$$

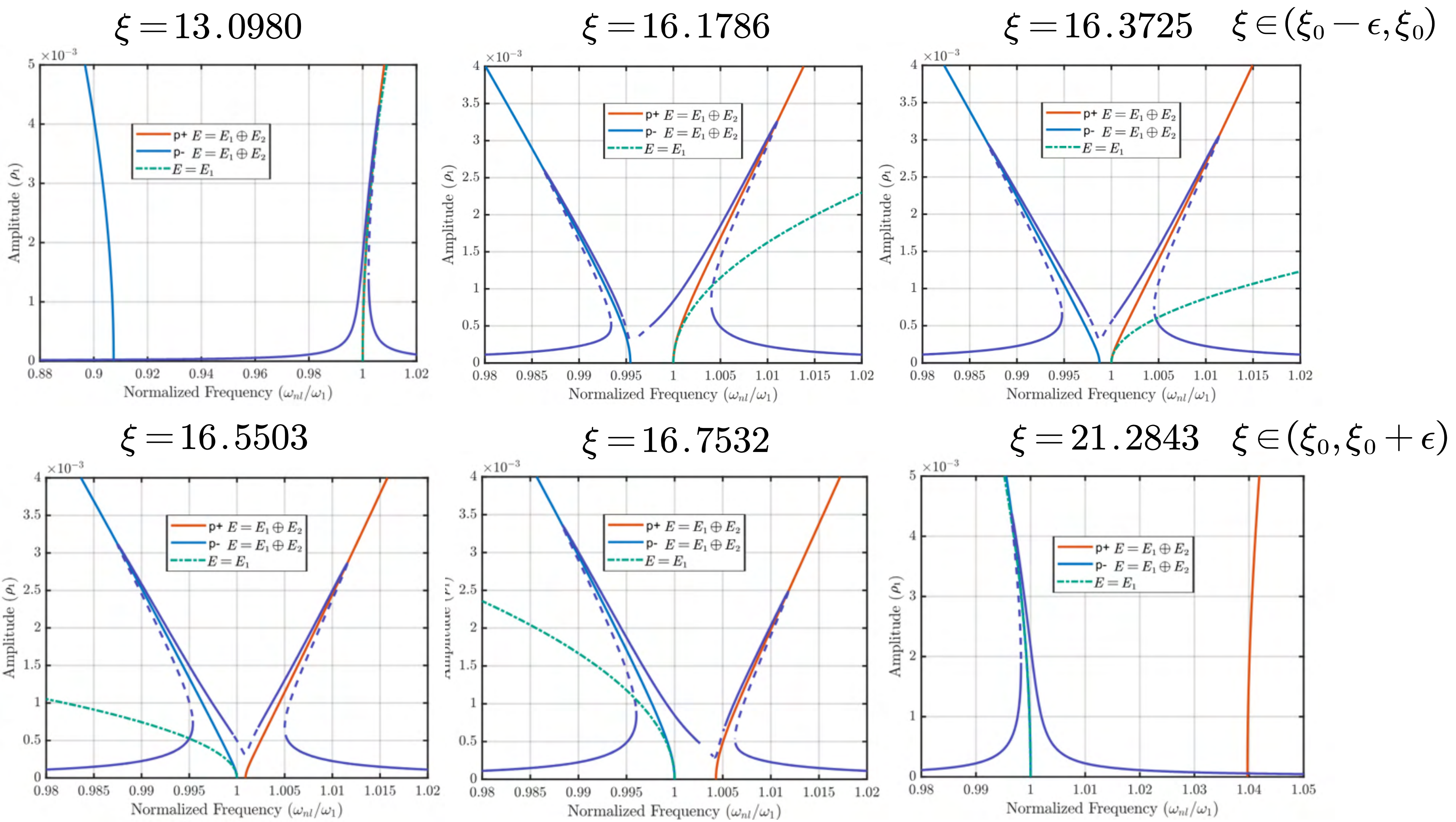
$$\omega_{NL}^{p-} = \omega_1 - \frac{\alpha_{23}}{2}\rho_2 + \frac{\beta_{113}}{4}\rho_1^2 + \frac{\beta_{124}}{4}\rho_2^2$$

The amplitudes are subject to a constraint



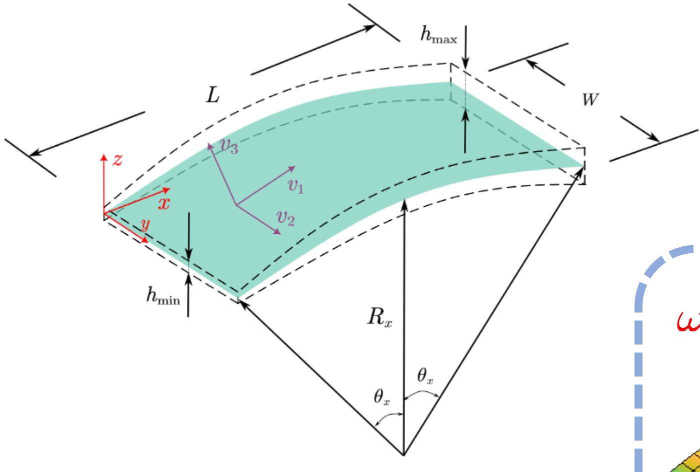
$$\frac{(\beta_{224} - 2\beta_{124})}{4}\rho_2^3 - p\alpha_{23}\rho_2^2 + \left[\frac{(\beta_{123} - 2\beta_{113})}{4}\rho_1^2 + \omega_2 - 2\omega_1\right]\rho_2 + p\frac{\alpha_{11}}{2}\rho_1^2 = 0$$

From Flat Plates to Curved Shells: Dual Mode Reduction



- The **single-mode reduced-order model** captures the **dominant resonance behavior**
- The dominant resonance **switches between p^+ and p^-**
- With moving away from the resonance, influence of internal resonance **gradually weakens**

Symmetry breaking: Uniform → Non-Uniform Thickness

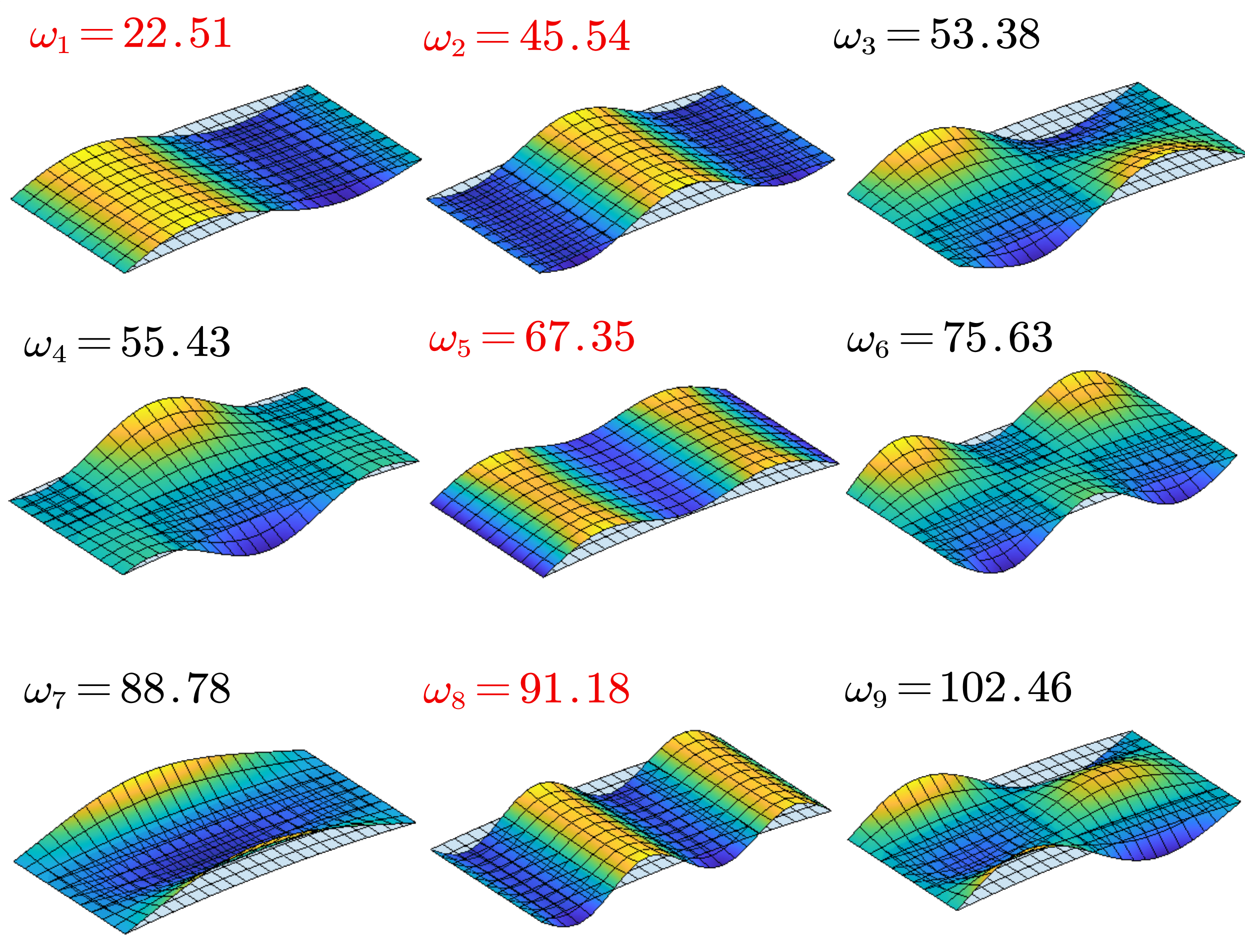


$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} + \mathbf{G}(\mathbf{U},\ \mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = \kappa\mathbf{M}\boldsymbol{\phi}_{B1}\cos(\Omega t)$$

$$h_{\min} = 0.009 \quad h_{\max} = 0.01$$

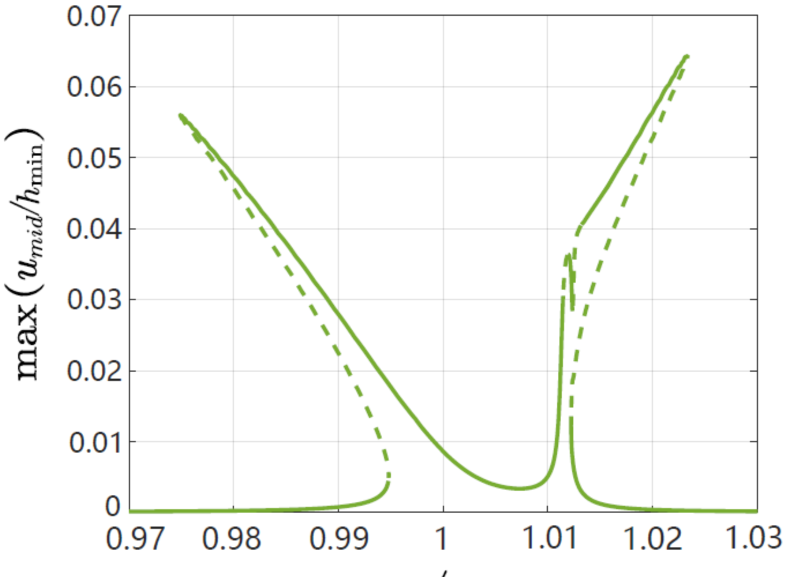
$$h(x) = h_{\min} + \frac{h_{\max} - h_{\min}}{x_{\max} - x_{\min}}(x - x_{\min})$$

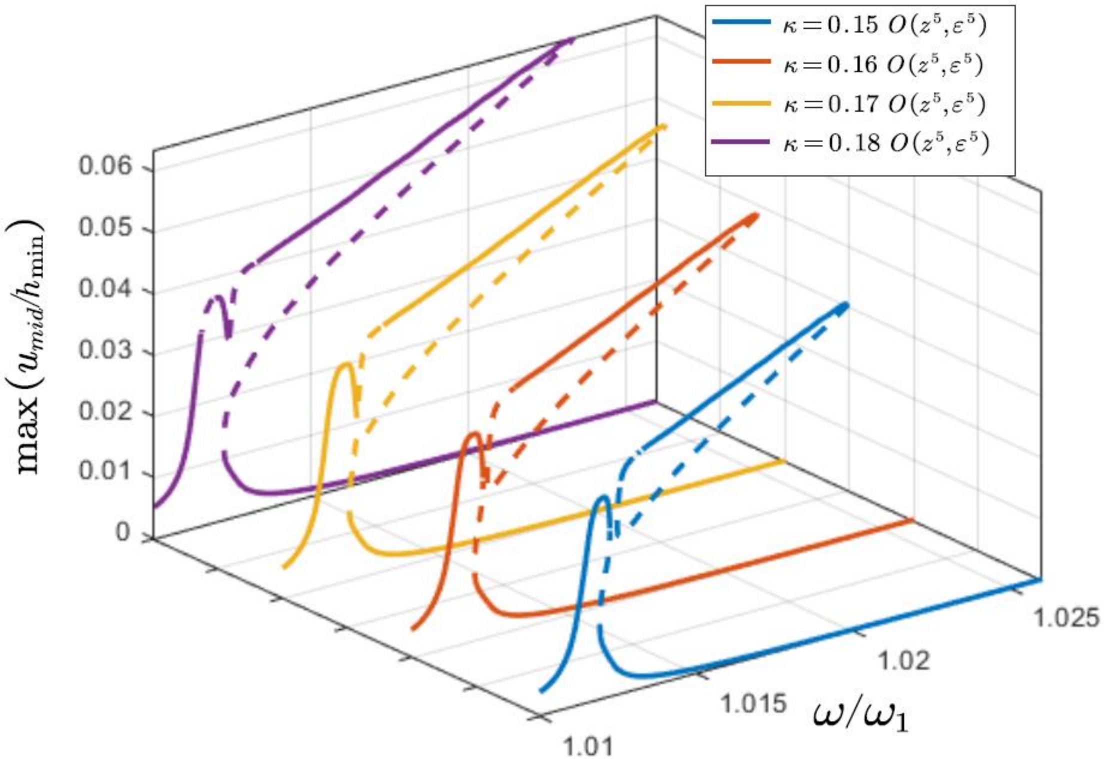
- It has been confirmed that capturing the system's dynamics accurately in the reduced-order model necessitates the inclusion of the first four bending modes

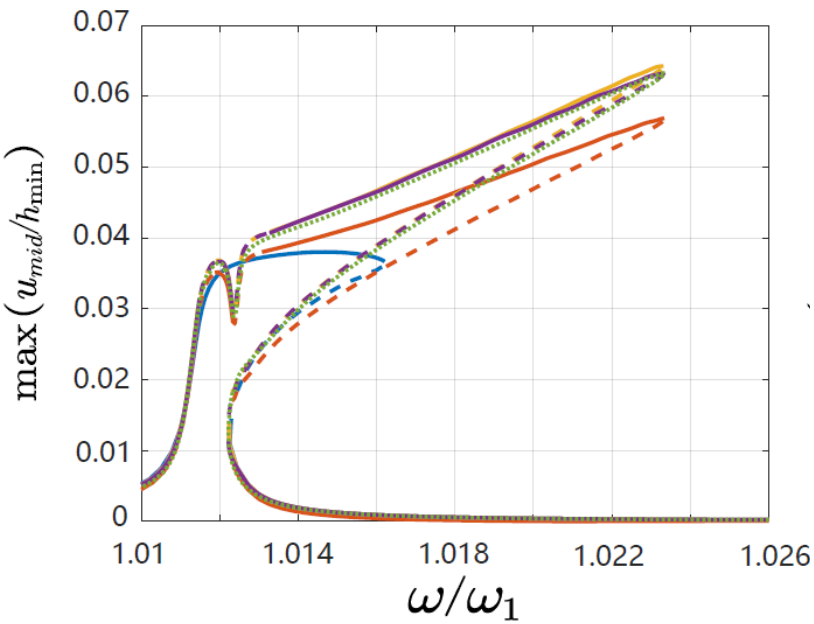


Symmetry breaking: Uniform → Non-Uniform Thickness

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U},\ \mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = 0.18\mathbf{M}\boldsymbol{\phi}_{\mathrm{B1}}\cos(\Omega \mathrm{t})$$

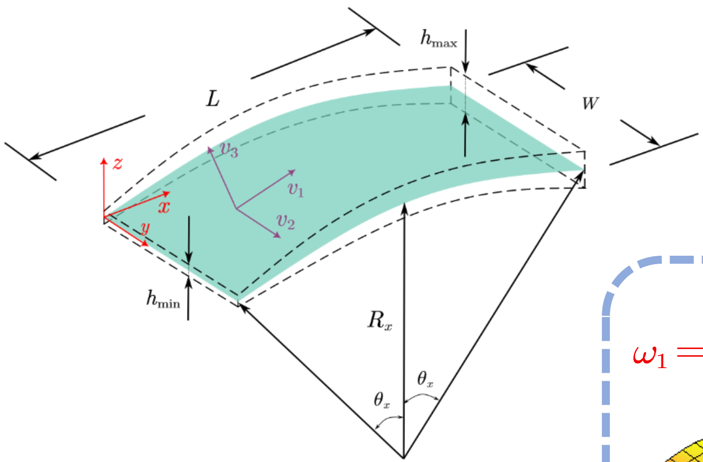






- $\mathcal{O}(5)\ \ E = E_1 \oplus E_2$
- $\mathcal{O}(5)\ \ E = E_1 \oplus E_2 \oplus E_8$
- $\mathcal{O}(3)\ \ E = E_1 \oplus E_2 \oplus E_5 \oplus E_8$
- $\mathcal{O}(5)\ \ E = E_1 \oplus E_2 \oplus E_5 \oplus E_8$
- Full order

Symmetry breaking: Uniform → Non-Uniform Thickness

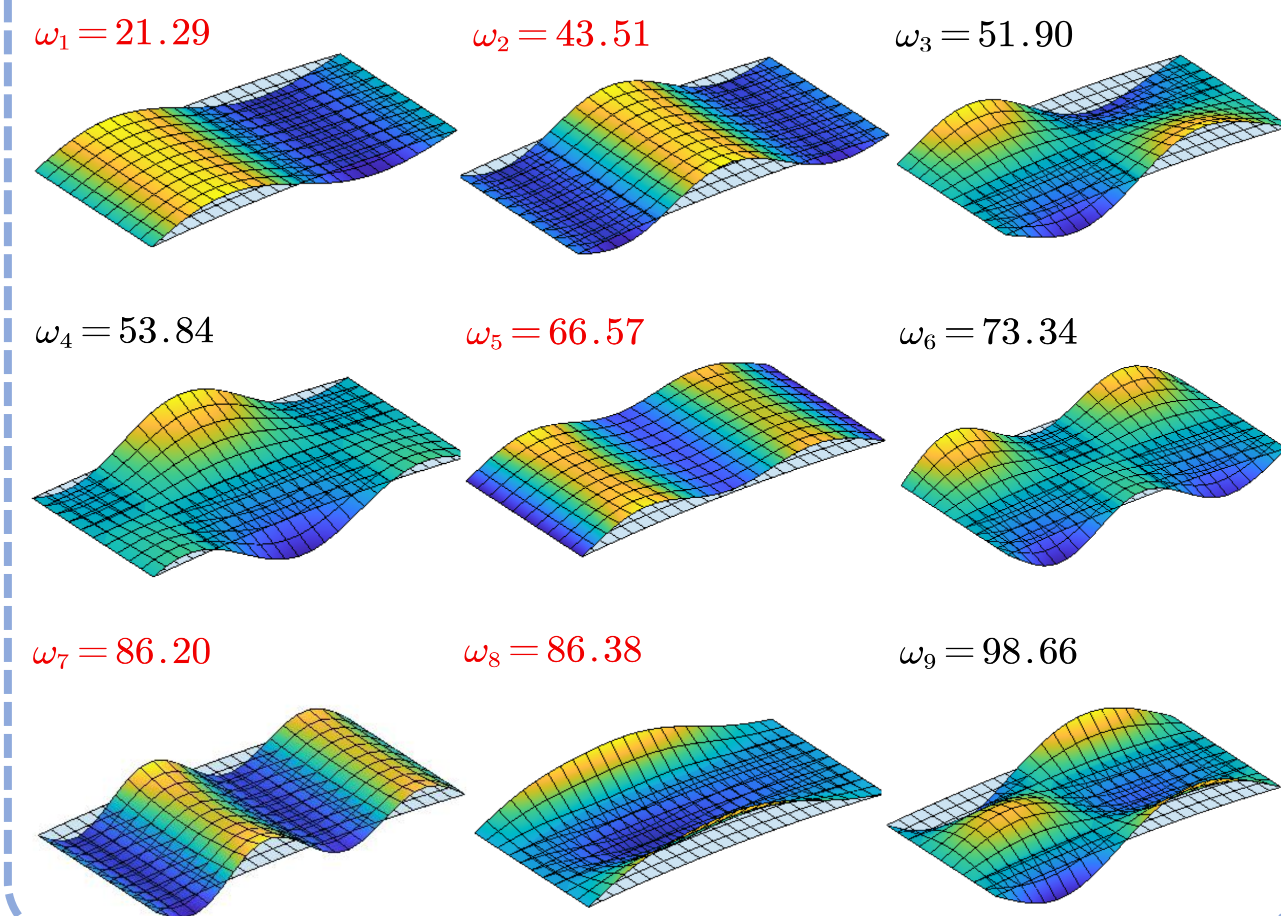


$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{C}\dot{\mathbf{U}}+\mathbf{K}\mathbf{U}+\mathbf{G}(\mathbf{U},\ \mathbf{U})+\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})=\kappa\mathbf{M}\boldsymbol{\phi}_{\mathrm{B}1}\cos(\Omega t)$$

$$h_{\min}=0.008 \quad h_{\max}=0.01$$

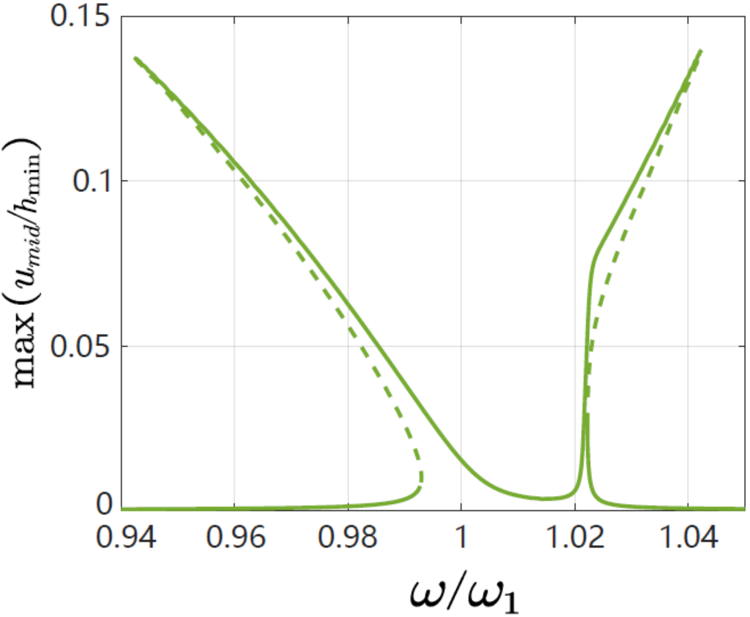
$$h(x)=h_{\min}+\frac{h_{\max}-h_{\min}}{x_{\max}-x_{\min}}(x-x_{\min})$$

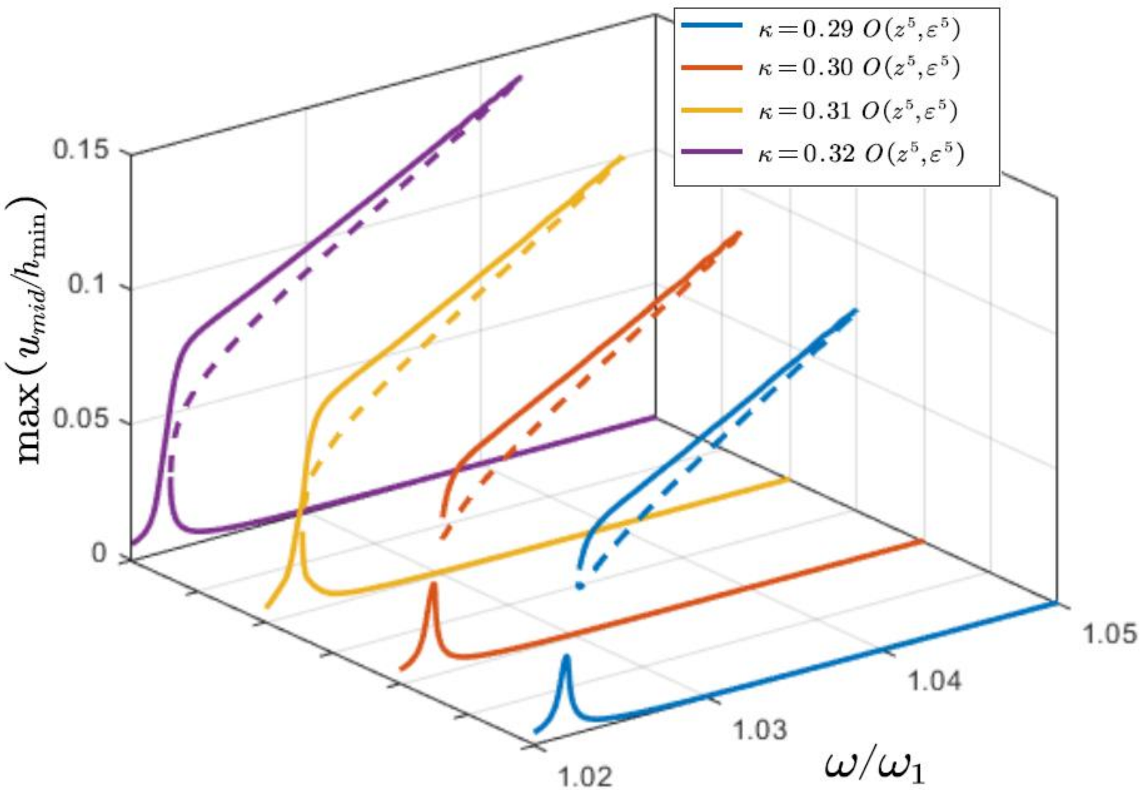
- **With a further reduction in the thickness of the thinner region, the frequency ratio between the first and second modes increases to 2.04**

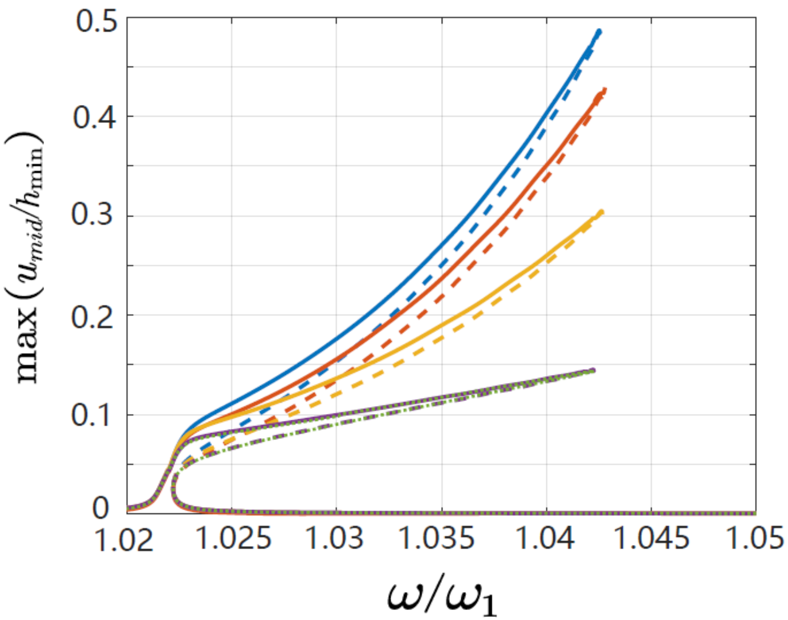


Symmetry breaking: Uniform → Non-Uniform Thickness

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U},\ \mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = 0.32\mathbf{M}\boldsymbol{\phi}_{\mathrm{B1}}\cos(\Omega t)$$







- $\mathcal{O}(5)\ \ E = E_1 \oplus E_2$
- $\mathcal{O}(5)\ \ E = E_1 \oplus E_2 \oplus E_5$
- $\mathcal{O}(5)\ \ E = E_1 \oplus E_2 \oplus E_5 \oplus E_7$
- $\mathcal{O}(5)\ \ E = E_1 \oplus E_2 \oplus E_5 \oplus E_7 \oplus E_8$
- Full order

Master Mode Selection: A User Guideline

Master modes	Order of DPIM	CPU time (s) ROM	CPU time (s) FOM	Time radio
$E = E_1$	$\mathcal{O}(3)$	6.42	92788	14 452.95
	$\mathcal{O}(5)$	49.39		1878.70
$E = E_1 \oplus E_2$	$\mathcal{O}(3)$	17.07	129639	7594.55
	$\mathcal{O}(5)$	376.68		344.16
$E = E_1 \oplus E_2 \oplus E_5 \oplus E_8$	$\mathcal{O}(3)$	72.52	172950	2384.86
	$\mathcal{O}(5)$	3685.24		46.93
$E = E_1 \oplus E_2 \oplus E_5 \oplus E_7 \oplus E_8$	$\mathcal{O}(3)$	122.89	152196	1238.47
	$\mathcal{O}(5)$	6507.12		23.39

The choice of master modes is a key determinant of the performance and efficiency of a ROM !

The guiding principle is to pre-identify the key resonant terms !

Master Mode Selection: A User Guideline

For the conventional CNF method $\lambda_k = \sum_{i=1}^{n} m_i \lambda_i,\ \lambda_i = \omega_i \text{ or } \bar{\omega}_i,\ \text{with } m_i \geqslant 0 \text{ and } \sum_{i=1}^{n} m_i = p$

For $h_{\min} = 0.009 \qquad h_{\max} = 0.01$

Order 2 of DPIM

$$2\omega_1 \approx \omega_2, \omega_1 + \omega_2 \approx \omega_5, \omega_1 + \omega_3 \approx \omega_6, \omega_1 + \omega_5 \approx \omega_7, \omega_1 + \omega_5 \approx \omega_8,$$

$$\omega_1 + \bar{\omega}_5 \approx \bar{\omega}_2, \omega_1 + \bar{\omega}_6 \approx \bar{\omega}_3, \omega_1 + \bar{\omega}_7 \approx \bar{\omega}_5, \omega_1 + \bar{\omega}_8 \approx \bar{\omega}_5$$

$$2\omega_2 \approx \omega_8, \omega_2 + \omega_4 \approx \omega_9, \omega_2 + \bar{\omega}_8 \approx \bar{\omega}_2, \omega_3 + \bar{\omega}_6 \approx \bar{\omega}_1$$

Order 3 of DPIM

$$3\omega_1 \approx \omega_5, 2\omega_1 + \omega_2 \approx \omega_8, 2\omega_1 + \omega_4 \approx \omega_9, 2\omega_1 + \bar{\omega}_5 \approx \bar{\omega}_1,$$

$$\omega_1 + \omega_6 + \bar{\omega}_3 \approx \omega_2, \omega_1 + \omega_7 + \bar{\omega}_4 \approx \omega_4, \omega_1 + \omega_8 + \bar{\omega}_2 \approx \omega_5,$$

$$\omega_1 + \bar{\omega}_2 + \bar{\omega}_3 \approx \bar{\omega}_6, \omega_1 + \bar{\omega}_2 + \bar{\omega}_5 \approx \bar{\omega}_7, \omega_1 + \bar{\omega}_2 + \bar{\omega}_5 \approx \bar{\omega}_8, \cdots$$

Identifying the invariance-breaking terms is an efficient way to find the master modes !

Master modes selected as $E = E_1 \oplus E_2 \oplus E_5 \oplus E_8$

Master Mode Selection: A User Guideline

For $h_{\min} = 0.008 \quad h_{\max} = 0.01$

Order 2 of DPIM

$$2\omega_1 \approx \omega_2, \omega_1 + \omega_2 \approx \omega_5, \omega_1 + \omega_3 \approx \omega_6, \omega_1 + \omega_4 \approx \omega_6, \omega_1 + \omega_5 \approx \omega_7$$

$$\omega_1 + \omega_5 \approx \omega_8, \omega_1 + \omega_6 \approx \omega_9, \omega_1 + \bar{\omega}_2 \approx \bar{\omega}_1, \omega_1 + \bar{\omega}_5 \approx \bar{\omega}_2, \omega_1 + \bar{\omega}_6 \approx \bar{\omega}_3$$

$$\omega_1 + \bar{\omega}_6 \approx \bar{\omega}_4, \omega_1 + \bar{\omega}_7 \approx \bar{\omega}_5, \omega_1 + \bar{\omega}_8 \approx \bar{\omega}_5$$

$$2\omega_2 \approx \omega_7, 2\omega_2 \approx \omega_8 \cdots$$

Order 3 of DPIM

$$3\omega_1 \approx \omega_5, 2\omega_1 + \omega_2 \approx \omega_7, 2\omega_1 + \omega_2 \approx \omega_8, 2\omega_1 + \omega_3 \approx \omega_9$$

$$2\omega_1 + \omega_4 \approx \omega_9, 2\omega_1 + \bar{\omega}_7 \approx \bar{\omega}_2, 2\omega_1 + \bar{\omega}_8 \approx \bar{\omega}_2$$

$$2\omega_1 + \bar{\omega}_9 \approx \bar{\omega}_4, \omega_1 + \omega_2 + \bar{\omega}_7 \approx \bar{\omega}_1, \omega_1 + \omega_2 + \bar{\omega}_8 \approx \bar{\omega}_1 \cdots$$

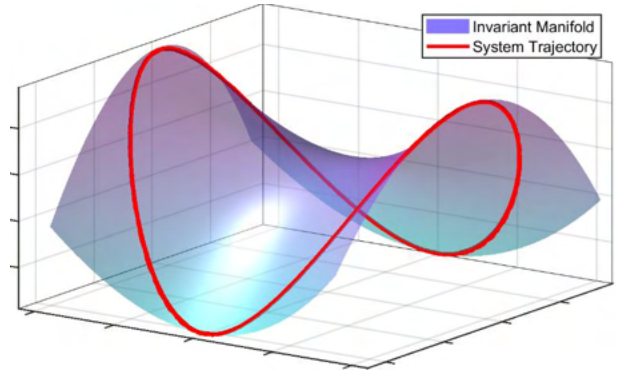
Master modes selected as $E = E_1 \oplus E_2 \oplus E_5 \oplus E_7 \oplus E_8$

We have previously validated that these selected master modes are sufficient to reconstruct the FOM results

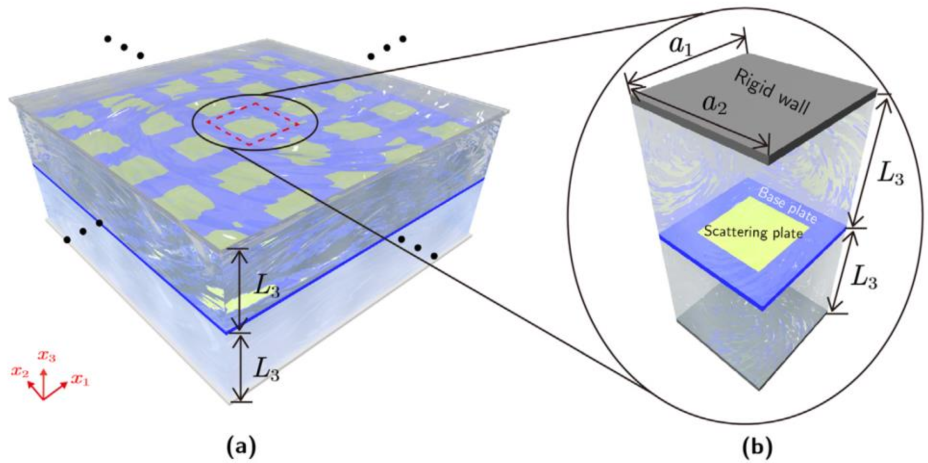
Open-Source Code & Resources

[evry-dynamics (Structure dynamics research group)]

DPIM ROM based on solid-shell element



Metamaterial with hydro-elastic effect



- Invariant-manifold implementation in **MATLAB and Julia**
- nonlinear equations solved via **Harmonic Balance** for both ROM and FOM

Thanks for your listening