Composing exponentials and intrinsic numerical integration on manifolds

Adrien Busnot Laurent - INRIA MINGuS
Joint work with E. Grong and H. Munthe-Kaas

Project MaStoC - Manifolds and Stochastic Computations

Motivation: ODE $y' = F(y)$, $F \in \mathfrak{X}(\mathcal{M})$

Dynamics on a manifold $\mathcal{M}$: .

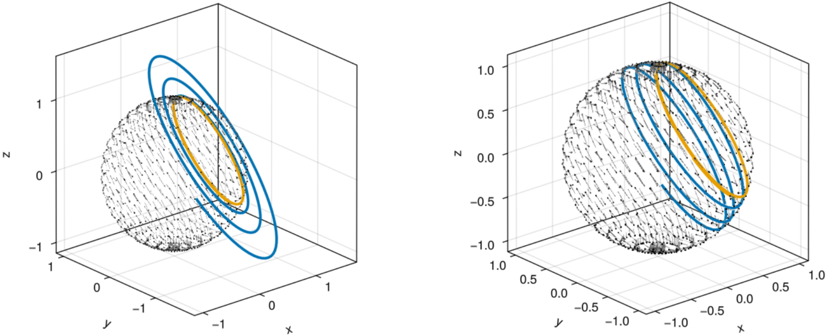


Figure: Non-geometric versus geometric methods for $y' = A(y)y$, $A^T = -A$.

Idea: dynamics come with geometric invariants and the **numerical methods should try to preserve invariants as much as possible**.

Challenge: a geometry is not just a manifold. The numerical approaches have to satisfy that **their definition, convergence analysis, and implementation all rely on the same geometric framework as the model.**

Motivation: ODE $y' = F(y)$, $F \in \mathfrak{X}(\mathcal{M})$

Figure: Numerical simulations of a Brownian motion on the sphere.

Idea: dynamics come with geometric invariants and the **numerical methods should try to preserve invariants as much as possible**.
Challenge: a geometry is not just a manifold. The numerical approaches have to satisfy that **their definition, convergence analysis, and implementation all rely on the same geometric framework as the model.**

Contents

References of this talk:

- K. Beauchard, A. BL, F. Marbach, Control theory and splitting methods, arXiv:2407.02127.
- E. Bronasco, A. BL, B. Huguet, High order integration of stochastic dynamics on Riemannian manifolds with frozen flow methods, arXiv:2503.21855.
- A. BL, E. Grong, H. Munthe-Kaas, General RKMK methods, *Ongoing*.

Contents

Lie algebras and adjoint map

Definition

A **Lie algebra** is

- A vector space $\mathfrak{g}$
- A bilinear $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying

$$[a,b]=-[b,a],\quad [a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0.$$

The smallest Lie algebra spanned by a set A is $\mathrm{Lie}(A)$. The **adjoint representation** is

$$\mathrm{ad}_\omega(x)=[\omega,x].$$

Lie algebras and adjoint map

Definition

A **Lie algebra** is

- A vector space $\mathfrak{g}$
- A bilinear $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying

$$[a,b]=-[b,a],\quad [a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0.$$

The smallest Lie algebra spanned by a set A is $\mathrm{Lie}(A)$. The **adjoint representation** is

$$\mathrm{ad}_\omega(x)=[\omega,x].$$

Example

Let an alphabet $A=\{a,b,c,\dots\}$ and an associative product $\cdot$. Then $(\mathrm{Span}(A),[-,-])$ is a Lie algebra, where

$$[a,b]=a\cdot b-b\cdot a.$$

The words algebra $\mathcal{W}=T(A)=\mathcal{U}(\mathrm{Lie}(A))$ is generated by words: $abc=a\cdot b\cdot c$.

The BCH formula

Theorem (Baker-Campbell-Hausdorff formula)

Let $a,\ b \in \mathrm{Lie}(A)$*, then the* ***product of exponentials*** *satisfies*

$$\exp^{\cdot}(a)\cdot\exp^{\cdot}(b)=\exp^{\cdot}(\mathrm{BCH}(a,b)),\quad \exp^{\cdot}(x)=\sum\frac{1}{n!}x^{\cdot n}$$

where $\mathrm{BCH}\colon\ \mathrm{Lie}(A)\times\mathrm{Lie}(A)\to\mathrm{Lie}(A)$ *is the solution* $\mathrm{BCH}(a,b)=\omega(1)$ *of the ODE*

$$\omega'(t)=d_{\omega(t)}\exp^{\cdot -1}(ta),\quad \omega(0)=b.$$

Example

The first terms are

$$\begin{aligned}BCH(a,b)=a+b+\frac{1}{2}[a,b]+\frac{1}{12}([a,[a,b]]+[b,[b,a]])-\frac{1}{24}[b,[a,[a,b]]]\\ -\frac{1}{720}([b,[b,[b,[b,a]]]]+[a,[a,[a,[a,b]]]])\\ +\frac{1}{360}([a,[b,[b,[b,a]]]]+[b,[a,[a,[a,b]]]])+\cdots\end{aligned}$$

The BCH formula

Theorem (Baker-Campbell-Hausdorff formula)

Let a*,* $b \in \mathrm{Lie}(A)$*, then the* ***product of exponentials*** *satisfies*

$$\exp^{\cdot}(a) \cdot \exp^{\cdot}(b) = \exp^{\cdot}(\mathrm{BCH}(a, b)), \quad \exp^{\cdot}(x) = \sum \frac{1}{n!} x^{\cdot n}$$

where $\mathrm{BCH}\colon \mathrm{Lie}(A) \times \mathrm{Lie}(A) \to \mathrm{Lie}(A)$ *is the solution* $\mathrm{BCH}(a, b) = \omega(1)$ *of the ODE*

$$\omega'(t) = d_{\omega(t)} \exp^{\cdot -1}(ta), \quad \omega(0) = b.$$

Proof.

Let $\omega(t) = \mathrm{BCH}(ta, b)$ satisfy $\exp^{\cdot}(\omega(t)) = \exp^{\cdot}(ta) \cdot \exp^{\cdot}(b)$. Then we find

$$d_{\omega(t)} \exp^{\cdot}(\omega'(t)) \cdot \exp^{\cdot}(\omega(t)) = d_{ta} \exp^{\cdot}(a) \cdot \exp^{\cdot}(ta) \cdot \exp^{\cdot}(b) = ta \cdot \exp^{\cdot}(\omega(t)).$$

Hence the result. □

A word on the map $d\exp$

The **derivative of the exponential** $d\exp$ satisfies (see e.g. Reutenauer, 1993)

$$d_\omega \exp^{\cdot} = \sum_{n=0} \frac{1}{(n+1)!} \mathrm{ad}_\omega^n = \frac{e^z - 1}{z}\Big|_{z=\mathrm{ad}_\omega}.$$

A word on the map $d\exp$

The **derivative of the exponential** $d\exp$ satisfies (see e.g. Reutenauer, 1993)

$$d_\omega \exp^{\cdot} = \sum_{n=0} \frac{1}{(n+1)!} \mathrm{ad}_\omega^n = \frac{e^z - 1}{z}\Big|_{z=\mathrm{ad}_\omega}.$$

Moreover, $d_\omega \exp^{\cdot}$ is invertible as a formal series and satisfies

$$d_\omega \exp^{\cdot -1} = \sum_{n=0} \frac{B_n}{n!} \mathrm{ad}_\omega^n = \frac{z}{e^z - 1}\Big|_{z=\mathrm{ad}_\omega},$$

where the B_n are the Bernoulli numbers:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \ldots$$

A word on the map $d\exp$

The **derivative of the exponential** $d\exp$ satisfies (see e.g. Reutenauer, 1993)

$$d_\omega \exp^{\cdot} = \sum_{n=0} \frac{1}{(n+1)!} \mathrm{ad}_\omega^n = \frac{e^z - 1}{z}\Big|_{z=\mathrm{ad}_\omega}.$$

Moreover, $d_\omega \exp^{\cdot}$ is invertible as a formal series and satisfies

$$d_\omega \exp^{\cdot -1} = \sum_{n=0} \frac{B_n}{n!} \mathrm{ad}_\omega^n = \frac{z}{e^z - 1}\Big|_{z=\mathrm{ad}_\omega},$$

where the B_n are the Bernoulli numbers:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \dots$$

Remark

The BCH formula is used in the high order theory of splitting methods (see, for instance, Blanes, Casas, 2024), in control theory, ...

Lie group methods - RKMK approach[1]

Proposition

Consider the ***ODE on a matrix Lie group***

$$y'(t) = A(y(t))y(t), \quad A\colon G \to \mathfrak{g}, \quad y_0 \in G.$$

Then, $y(t) = \operatorname{Exp}(\Omega(t))y_0$*, where the dynamics on* $\mathfrak{g}$ *is*

$$\Omega'(t) = d\operatorname{Exp}_{\Omega(t)}^{-1}(A(y(t))), \quad \Omega(0) = 0. \tag{*}$$

Proof.

We find on one hand

$$y'(t) = A(y(t))y(t) = A(y(t))\operatorname{Exp}(\Omega(t))y_0.$$

On the second hand, we have

$$y'(t) = d\operatorname{Exp}_{\Omega(t)}(\Omega'(t))\operatorname{Exp}(\Omega(t))y_0.$$

□

[1] See also frozen-flow methods (Celledoni, Crouch, Grossman, Marthinsen, Owren,...).

Lie group methods - RKMK approach[1]

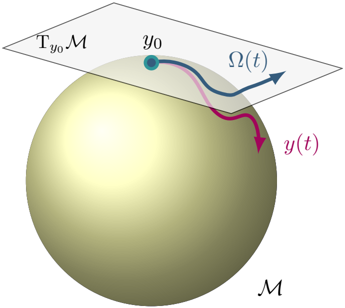
Proposition

Consider the ***ODE on a matrix Lie group***

$$y'(t) = A(y(t))y(t), \quad A\colon G \to \mathfrak{g}, \quad y_0 \in G.$$

Then, $y(t) = \mathrm{Exp}(\Omega(t))y_0$, *where the dynamics on* $\mathfrak{g}$ *is*

$$\Omega'(t) = d\,\mathrm{Exp}^{-1}_{\Omega(t)}(A(y(t))), \quad \Omega(0) = 0. \tag{*}$$



A RKMK method is a Runge-Kutta method applied to the lifted dynamics (*):

$$\Omega_n = \mathrm{RK}((*)), \quad y_{n+1} = \mathrm{Exp}(\Omega_n)y_n.$$

Remark

One can truncate $d\,\mathrm{Exp}^{-1}$ *or use retractions.*

[1] See also frozen-flow methods (Celledoni, Crouch, Grossman, Marthinsen, Owren,...).

Contents

Basics in differential geometry

Question: can we extend RKMK to any geometry?

Basics in differential geometry

Question: can we extend RKMK to any geometry?

What does it mean "any geometry"?

Basics in differential geometry

Question: can we extend RKMK to any geometry?

What does it mean "any geometry"?

We have

- A manifold $\mathcal{M}$
- Functions $\phi \in \mathcal{C}^\infty(\mathcal{M})$
- Smooth vector fields $f \in \mathfrak{X}(\mathcal{M})$, that is, $f(p) \in T_p\mathcal{M}$.

Basics in differential geometry

Question: can we extend RKMK to any geometry?

What does it mean "any geometry"?

We have

- A manifold $\mathcal{M}$
- Functions $\phi \in \mathcal{C}^\infty(\mathcal{M})$
- Smooth vector fields $f \in \mathfrak{X}(\mathcal{M})$, that is, $f(p) \in T_p\mathcal{M}$.

Lemma

Let the Jacobi bracket

$$[f,g]_J \rhd \phi := f \rhd (g \rhd \phi) - g \rhd (f \rhd \phi).$$

Then $(\mathfrak{X}(\mathcal{M}), [-,-]_J)$ *is a Lie algebra.*

A vector field $f \in \mathfrak{X}(\mathcal{M})$ defines the flow of the ODE $y' = f(y)$ on $\mathcal{M}$.

Connection

Definition

An **affine connection** is a bilinear mapping of vector fields

$$\rhd \colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}),$$

satisfying

$$(\phi f) \rhd g = \phi (f \rhd g), \quad f \rhd (\phi g) = f[\phi] g + \phi f \rhd g.$$

Connection

Definition

An **affine connection** is a bilinear mapping of vector fields

$$\rhd \colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}),$$

satisfying

$$(\phi f) \rhd g = \phi(f \rhd g), \quad f \rhd (\phi g) = f[\phi]g + \phi f \rhd g.$$

Example

In $\mathbb{R}^D$, the standard Euclidean connection is

$$f \rhd g = f^i \partial_i [g^j] \partial_j = g' f = \langle \nabla g, f \rangle.$$

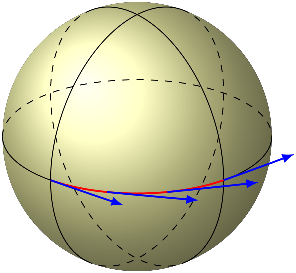
Given a frame basis E_i, $i = 1, \dots, D$, the ***Weitzenböck connection*** *is given by*

$$f \rhd g = f[g^i] E_i, \quad g = g^i E_i.$$

Parallel transport

Definition (Geodesic)

A parallel vector field $f \rhd f = 0$ defines a geodesic curve $y(t)$ as $y'(t) = f(y(t))$.
We write $\exp_p(tv)$.



Parallel transport

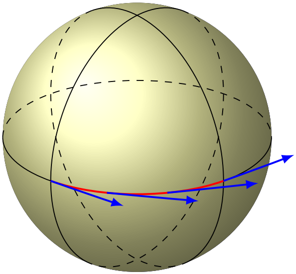
Definition (Geodesic)

A parallel vector field $f \triangleright f = 0$ defines a geodesic curve $y(t)$ as $y'(t) = f(y(t))$. We write $\exp_p(tv)$.

Definition (Parallel transport along geodesics)

The *parallel transport along geodesics*:

$$\Gamma_{tv}\colon T_p\mathcal{M} \to T_{\exp_p(tv)}\mathcal{M}, \quad p \in \mathcal{M}, \quad v \in T_p\mathcal{M}, \quad \Gamma_{tf}^{-1} = \exp(tf\triangleright).$$



Torsion and curvature

Definition

Torsion:

$$T(f,g) = f \rhd g - g \rhd f - [f,g]_J.$$

Curvature:

$$R(f,g)h = f \rhd (g \rhd h) - g \rhd (f \rhd h) - [f,g]_J \rhd h.$$

Torsion and curvature

Definition

Torsion:

$$T(f,g) = f \rhd g - g \rhd f - [f,g]_J.$$

Curvature:

$$R(f,g)h = f \rhd (g \rhd h) - g \rhd (f \rhd h) - [f,g]_J \rhd h.$$

Proposition (Bianchi identities)

The torsion and curvature of any affine connection on any manifold satisfy:

$$\sum_{\circlearrowleft X,Y,Z} T(T(X,Y),Z) + (X \rhd T)(Y,Z) - R(X,Y)Z = 0,$$

$$\sum_{\circlearrowleft X,Y,Z} (X \rhd R)(Y,Z) - R(X,T(Y,Z)) = 0.$$

Examples of geometries

Euclidean space $\mathbb{R}^d$: $T = 0$, $R = 0$.

Lie group: $\nabla T = 0$, $R = 0$.

Lemma

If $\nabla T = 0$, $R = 0$, $(\mathfrak{X}(\mathcal{M}), T)$ *is a Lie algebra.*

Examples of geometries

Euclidean space $\mathbb{R}^d$: $T = 0$, $R = 0$.

Lie group: $\nabla T = 0$, $R = 0$.

Lemma

If $\nabla T = 0$, $R = 0$, $(\mathfrak{X}(\mathcal{M}), T)$ is a Lie algebra.

Nomizu classification of invariant connections (1954):

Geometry	**Connection**	**Connection algebra**
Local Abelian Lie group	$T = 0$, $R = 0$	pre-Lie
Local Lie group	$\nabla T = 0$, $R = 0$	post-Lie
Local symmetric space	$T = 0$, $\nabla R = 0$	Lie admissible triple
Local reductive homogeneous space	$\nabla T = 0$, $\nabla R = 0$	post-Lie-Yamaguti

Examples of geometries

Euclidean space $\mathbb{R}^d$: $T = 0$, $R = 0$.

Lie group: $\nabla T = 0$, $R = 0$.

Lemma

If $\nabla T = 0$, $R = 0$, $(\mathfrak{X}(\mathcal{M}), T)$ *is a Lie algebra.*

Nomizu classification of invariant connections (1954):

Geometry	**Connection**	**Connection algebra**
Local Abelian Lie group	$T = 0$, $R = 0$	pre-Lie
Local Lie group	$\nabla T = 0$, $R = 0$	post-Lie
Local symmetric space	$T = 0$, $\nabla R = 0$	Lie admissible triple
Local reductive homogeneous space	$\nabla T = 0$, $\nabla R = 0$	post-Lie-Yamaguti

Main problem of current numerics on manifolds:
Riemannian: $T = 0$, Levi-Civita connection.
Almost all geometries in numerics: $R = 0$, Weitzenböck connection.

Contents

Jacobi fields

Definition

Given a geodesic $\gamma_t = \exp_p(tv)$ defined in a neighbourhood of p, a **Jacobi field** J is a vector field of the form

$$J(\gamma_t) = d_{tv} \exp_p(tw).$$

Equivalently, J satisfies

$$J(\gamma_t) = \frac{d}{ds}|_{s=0} \exp_p(tv + tsw).$$

Lemma (see Kobayashi, Nomizu)

The Jacobi field satisfies the equation

$$(\dot\gamma_t \cdot \dot\gamma_t) \rhd J(\gamma_t) = \dot\gamma_t \rhd (T_{\gamma_t}(\dot\gamma_t, J(\gamma_t))) + R_{\gamma_t}(\dot\gamma_t, J(\gamma_t))\dot\gamma_t.$$

The equivalent of $d\exp$

Define the trivialised differential of the exponential for $p \in \mathcal{M}$, $v, w \in T_p\mathcal{M}$,

$$\mathcal{E}_p \colon T_p\mathcal{M} \to \mathrm{End}_{\mathbb{R}}(T_p\mathcal{M}), \quad \mathcal{E}_p(v)w = \Gamma_v^{-1} d_v \exp_p(w).$$

Lemma

The operator $\mathcal{E}_p(tv)$ satisfies

$$t\frac{d^2}{dt^2}(t\mathcal{E}_p(tv)) - t\frac{d}{dt}(\mathcal{T}_p(tv)\mathcal{E}_p(tv)) = \mathcal{R}_p(tv)\mathcal{E}_p(tv),$$

where

$$\mathcal{T}_p(v)w = (\Gamma_{tv}^{-1} T_{\gamma_t})(v, w), \quad \mathcal{R}_p(v)w = (\Gamma_v^{-1} R_{\exp_p(v)})(v, w)v.$$

Lemma (Lie Polynomials)

Let $t_n(v) \colon w \to (v^{\cdot n} \rhd T_p)(v, w)$ and $r_n(v) \colon w \to (v^{\cdot n} \rhd R_p)(v, w)v$, then

$$\mathcal{T}_p(tv) = \sum_{n \geqslant 1} \frac{t^n}{(n-1)!} t_{n-1}(v), \quad \mathcal{R}_p(tv) = \sum_{n \geqslant 2} \frac{t^n}{(n-2)!} r_{n-2}(v).$$

Expansion of $\mathcal{E}_p(tv)$

Theorem (Generalisation of Gavrilov, 2012)

The Taylor expansion of $\mathcal{E}_p(v)w = \Gamma_v^{-1} d_v \exp_p(w)$ *satisfies*

$$\mathcal{E}_p(tv) = \sum_{n\geqslant 0} \frac{t^n}{(n+1)!} \sum_{P\in\mathrm{LiePol}, |P|=n} c_P P(v),$$

where

$$t_n(v)w = (v^{\cdot n} \rhd T_p)(v,w), \quad r_n(v)w = (v^{\cdot n} \rhd R_p)(v,w)v,$$

and

$$c_{t_p P} = \binom{p+1+|P|}{p} c_P, \quad c_{r_p P} = \binom{p+1+|P|}{p} c_P, \quad c_{\mathrm{id}} = 1.$$

First terms:

$$\mathcal{E}_p(tv) = \mathrm{id} + \frac{t}{2} t_0(v) + \frac{t^2}{3!}\left(2t_1 + r_0 + t_0^2\right)(v)$$
$$+ \frac{t^3}{4!}\left(3t_2 + 2r_1 + 3t_1 t_0 + 2t_0 t_1 + t_0 r_0 + r_0 t_0 + t_0^3\right)(v) + \cdots$$

The case of invariant connections

Corollary

Let an invariant affine connection $\rhd$, then we find

$$\begin{aligned}\mathcal{E}_p(tv) &= \sum_{n\geqslant 0} \frac{t^n}{(n+1)!} \sum_{P\in \mathrm{LiePol}_0, |P|=n} P(v)\\ &= \mathrm{id} + \frac{t}{2} t_0(v) + \frac{t^2}{3!}\Big(r_0 + t_0^2\Big)(v) + \frac{t^3}{4!}\Big(t_0 r_0 + r_0 t_0 + t_0^3\Big)(v)\\ &+ \frac{t^4}{5!}\Big(r_0^2 + t_0^2 r_0 + t_0 r_0 t_0 + r_0 t_0^2 + t_0^4\Big)(v)\\ &+ \frac{t^5}{6!}\Big(t_0 r_0^2 + r_0 t_0 r_0 + r_0^2 t_0 + t_0^3 r_0 + t_0^2 r_0 t_0 + t_0 r_0 t_0^2 + r_0 t_0^3 + t_0^5\Big)(v) + \cdots\end{aligned}$$

In particular for $\nabla T = 0$, $R = 0$, we recover the expansion of $d\exp$ in the matrix case

$$\mathcal{E}_p(tv) = \sum_{n\geqslant 0} \frac{t^n}{(n+1)!} T(v,-)^{\circ n}.$$

General BCH formula[2]

Corollary

Let an invariant affine connection $\rhd$, *then*

$$\mathcal{E}_p(tv)^{-1} = \mathrm{id} - \frac{t}{2} t_0(v) + t^2\Big(-\frac{1}{6} r_0 + \frac{1}{12} t_0^2\Big)(v) + t^3\Big(\frac{1}{24} t_0 r_0 + \frac{1}{24} r_0 t_0\Big)(v)$$
$$+ t^4\Big(\frac{7}{360} r_0^2 - \frac{1}{720} t_0^2 r_0 - \frac{1}{120} t_0 r_0 t_0 - \frac{1}{720} r_0 t_0^2 - \frac{1}{720} t_0^4\Big)(v) + \cdots$$

Theorem

The composition of geodesics satisfies

$$\exp_{\exp_p(v)}(\Gamma_v u) = \exp_p(BCH(u,v)),$$

where $BCH(u,v) = \omega(1)$ *and*

$$\omega'(t) = \mathcal{E}_p(\omega(t))^{-1} \Gamma_{\omega(t)}^{-1} \Gamma_{\Gamma_v tu} \Gamma_v u, \quad \omega(0) = v.$$

The case $\nabla T = 0$, $R = 0$ gives back the standard BCH formula.

[2] See also Gavrilov's double exponentials and Al-Kaabi, Ebrahimi-Fard, Manchon, Munthe-Kaas, 2025.

Lifting ODEs in general geometry

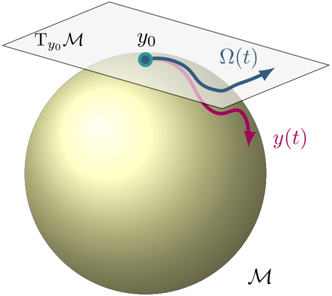
Theorem

Consider the ODE

$$y'(t) = f(y(t)), \quad y_0 = p, \quad f \in \mathfrak{X}(\mathcal{M}).$$

Then $y(t) = \exp_p(\Omega(t))$, *where*

$$\Omega'(t) = \mathcal{E}_p(\Omega(t))^{-1}\Gamma_{\Omega(t)}^{-1} f(y(t)), \quad \Omega(0) = 0.$$



The map $\Gamma_{\Omega}^{-1} f(y)$ is called the trivialisation of f in Lie-group methods.

If $f(y) = A(y)y$, $\Gamma_{\Omega}^{-1} f(y) = A(y)$

RKMK in any geometry

RKMK on Lie groups were introduced by H. Munthe-Kaas in the 90's, extended on symmetric spaces ($T = 0$, $\nabla R = 0$) in 2024.

Definition (General RKMK)

Let $\hat{f}(\exp_y(v)) = \Gamma_v^{-1} f(\exp_y(v))$. The new methods are

$$\theta^i = \sum_{j=1}^{s} a_{ij}\Omega^j,$$

$$\Omega^i = h\mathcal{E}_p(\theta^i)^{-1}\hat{f}(\exp_y(\theta^i)),$$

$$\psi_h(y) = \exp_y(\sum_{i=1}^{s} b_i\Omega^i).$$

Theorem

If the coefficients correspond to a ***Euclidean*** *order p RK method, the RKMK method is of order p.*

Conclusion

Summary:

- We provide a new fully general class of intrinsic methods for solving differential equations on manifolds.
- The approach is versatile and can be extended for solving deterministic and stochastic evolutionary problems.
- Ongoing numerical experiments.

Outlooks:

- Study of the algebra of Lie polynomials for constructing the simplest RKMK methods: Opening position in algebra/operads/numerics in 2026. Study of the algebraic structures appearing in general intrinsic integration.
- Creation of efficient high-order intrinsic sampling method on Riemannian manifolds (thesis of Sébastien Macé).
- Implementation of the new methods in Manifolds.jl (with P. Navaro and R. Bergmann).
- Multiscale dynamics on manifolds (for electromagnetics, ML, molecular dynamics).