Non-material Poincaré-Cosserat equations for geometrically exact sliding rods: Illustrative examples

Frédéric Boyer, Vincent Lebastard, Schucheng Zhang, Johann Herault

IMT-Atlantique – LS2N (France)

Journées du GDR Mecanique et Géométrie Différentielle

19-21 Novembre 2025

ONERA-Chatillon

Poincaré eq. = generalization of Lagrange eq. to systems whose config. space is a non-commutative Lie group ($\mathcal{C} = G$):

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \eta}\right) - ad_{\eta}^{T}\left(\frac{\partial l}{\partial \eta}\right) - X_g(l) = F_{\mathrm{ext}}^{\mathrm{nc}} \quad , \; \dot{g} = g\hat{\eta}$$

- Example 1 : $G = SO(3)$: Euler's eq. of the spinning top :

$$J\dot{\Omega} + \Omega \times J\Omega = C_{\mathrm{g}}(R) \;\; , \;\; \dot{R} = R\hat{\Omega}$$

- Example 2 : $G = SE(3)$: Newton-Euler's eq. of the rigid body :

$$\begin{pmatrix} J & m\hat{S} \\ m\hat{S}^T & m1 \end{pmatrix}\begin{pmatrix} \dot{\Omega} \\ \dot{V} \end{pmatrix} + \begin{pmatrix} mS \times (\Omega \times V) + \Omega \times J\Omega \\ (mS \times \Omega) \times \Omega + \Omega \times mV \end{pmatrix} = \begin{pmatrix} C_{\mathrm{g}} \\ N_{\mathrm{g}} \end{pmatrix} + \begin{pmatrix} C_{\mathrm{ext}}^{\mathrm{nc}} \\ N_{\mathrm{ext}}^{\mathrm{nc}} \end{pmatrix} \;\; , \;\; \dot{R} = R\hat{\Omega} \;\; , \;\; \dot{r} = RV$$

These eq. are deduced from least action principle on $\mathcal{C} = G$:

$$\delta \int_{t_a}^{t_b} l(g,\eta)\ dt = -\int_{t_a}^{t_b} \delta W_{\text{ext}}^{\text{nc}}\ dt,$$

Applying the same principle to a density of Lagrangian along one material dim. X:

$$\delta \int_{t_a}^{t_b} \int_0^1 \mathfrak{L}(g,\eta,\xi)\ dX\ dt = -\int_{t_a}^{t_b} \delta W_{\text{ext}}^{\text{nc}}\ dt,$$

➡ Poincaré-Cosserat PDEs along one material dimension:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\partial \eta}\right) - ad_\eta^T\left(\frac{\partial \mathfrak{L}}{\partial \eta}\right) + \frac{\partial}{\partial X}\left(\frac{\partial \mathfrak{L}}{\partial \xi}\right) - ad_\xi^T\left(\frac{\partial \mathfrak{L}}{\partial \xi}\right) - X_g(\mathfrak{L}) = \bar{F}_{\text{ext}}^{\text{nc}}$$

For $G = SE(3)$ ➡ Cosserat rods : Simo-Reissner, Kirchhoff…

This construction can be extended to non-material (pieces of) Kirchhoff rods:

$$\int_{t_a}^{t_b} \left(\int_{s_1}^{s_2} \delta \mathfrak{L} \, ds + [\mathfrak{L} \, \delta s]_{s_1}^{s_2} \right) dt - \int_{t_a}^{t_b} \left(\left[\Delta \zeta^T \left(\frac{\partial \mathfrak{L}}{\partial \eta} \right) \dot{s} \right]_{s_1}^{s_2} - \Delta W_{\mathrm{ext}}^{\mathrm{nc}} \right) dt = 0,$$

Applying this least action principle, gives the non-material Poincaré-Cosserat pdes:

$$\begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial \mathfrak{T}}{\partial \eta_D} \right) - ad_\eta^T \left(\frac{\partial \mathfrak{T}}{\partial \eta_D} \right) - \frac{\partial}{\partial s} \left(\frac{\partial \mathfrak{U}_{\mathrm{e}}}{\partial \xi} + \dot{s} \left(\frac{\partial \mathfrak{T}}{\partial \eta_D} \right) \right) + ad_\xi^T \left(\frac{\partial \mathfrak{U}_{\mathrm{e}}}{\partial \xi} + \dot{s} \left(\frac{\partial \mathfrak{T}}{\partial \eta_D} \right) \right) = \bar{F}_{\mathrm{g}} + \bar{F}_{\mathrm{ext}}^{\mathrm{nc}} \\ \left(\frac{\partial \mathfrak{U}_{\mathrm{e}}}{\partial \xi} \right)(s_1) = -F_- \ \text{, or } g(s_1) = g_- \quad , \quad \left(\frac{\partial \mathfrak{U}_{\mathrm{e}}}{\partial \xi} \right)(s_2) = F_+ \quad \text{: BCs} \\ \mathfrak{U}_{\mathrm{e}} = \frac{1}{2}(K - K_o)^T \mathcal{H}_a (K - K_o) + N^T (\Gamma - E_1) \quad \text{: Constitutive law of a Kirchhoff rod.} \end{cases}$$

Rk 1 : If $\dot{s}'(s) = 0$, i.e. the sliding velocity is uniform :

$$\frac{D}{Dt}\left(\frac{\partial \mathfrak{T}}{\partial \eta_D}\right) - ad^T_{\eta_D}\left(\frac{\partial \mathfrak{T}}{\partial \eta_D}\right) - \frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{U}_\mathrm{e}}{\partial \xi}\right) + ad^T_{\xi}\left(\frac{\partial \mathfrak{U}_\mathrm{e}}{\partial \xi}\right) = \bar{F}_\mathrm{g} + \bar{F}^\mathrm{nc}_\mathrm{ext}$$

F. Boyer, V. Lebastard, F. Candelier, F. Renda, « Extended Hamilton's principle applied to geometrically exact Kirchhoff sliding rods », *JSV*, 2022.

Rk 2 : Three cases depending of the status of the sliding variable:

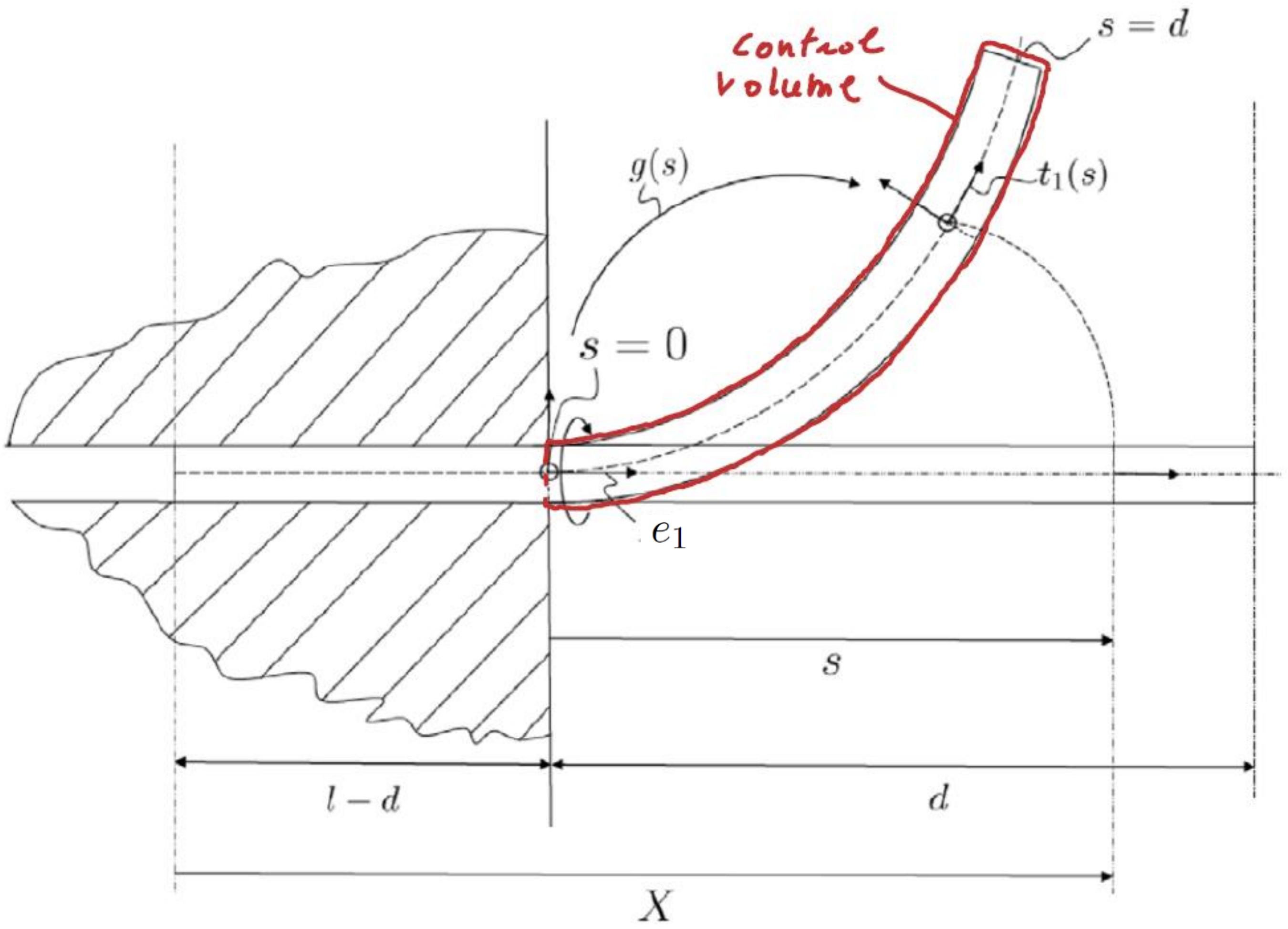
- $s(X) = s(X,t)$ imposed ➡ Non material Poincaré-Cosserat PDEs enough
- $s(X)$ free ➡ These pdes are supplemented with an ODE on s .
- $s = f(g)$ ➡ Flux of Lagrangian density appears in the BCs.

Rk 3 : These eq. naturally apply to stratified media sliding w.r.t. each others.

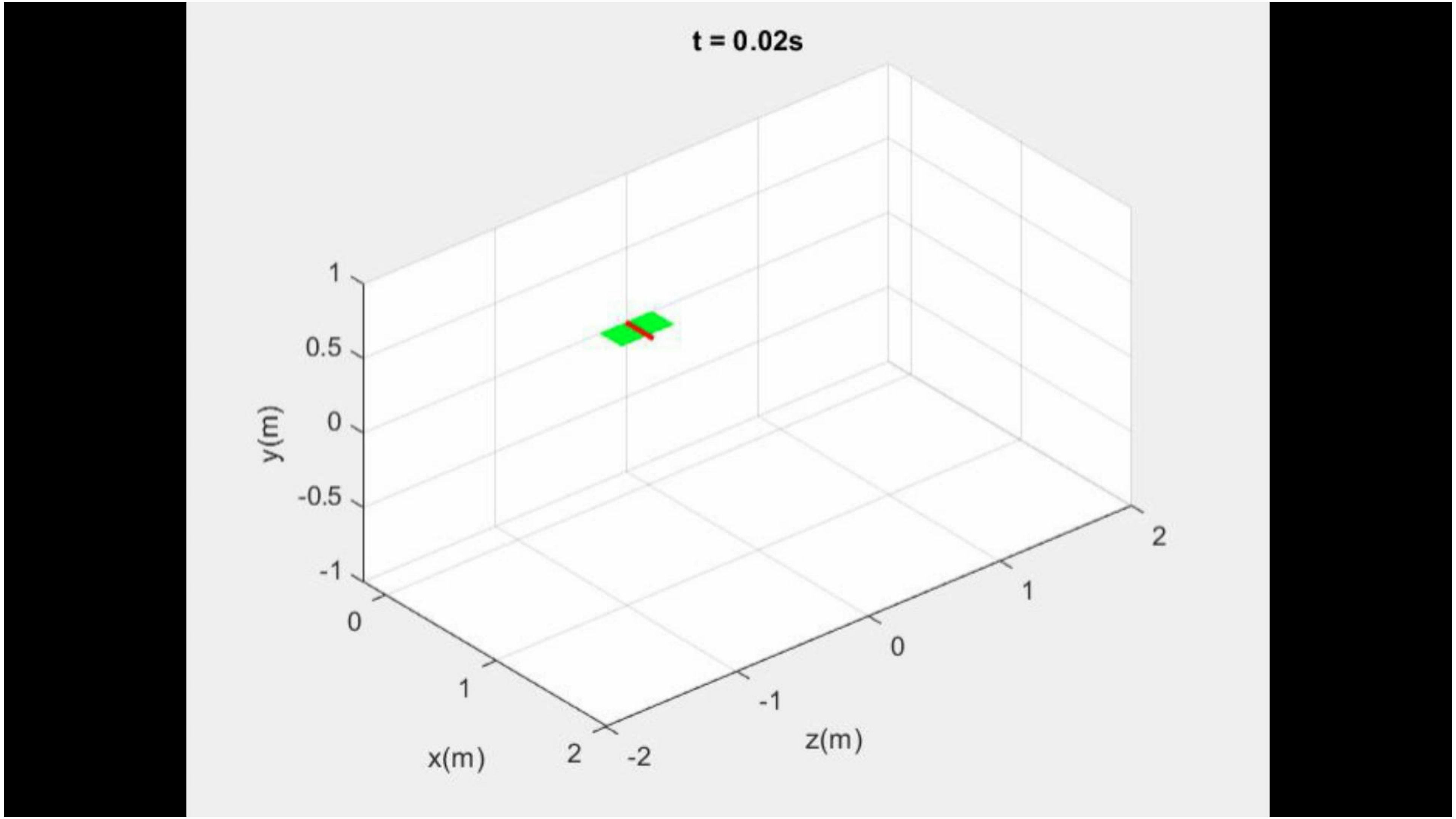
First example :

« A sliding rod »

Scheidl, J., Vetyukov, Y , “Review and perspectives in applied mechanics of axially moving flexible structures”, Acta Mechanica, 2023.



s = control tube cross-sections = non-material variable for rod cross-sections

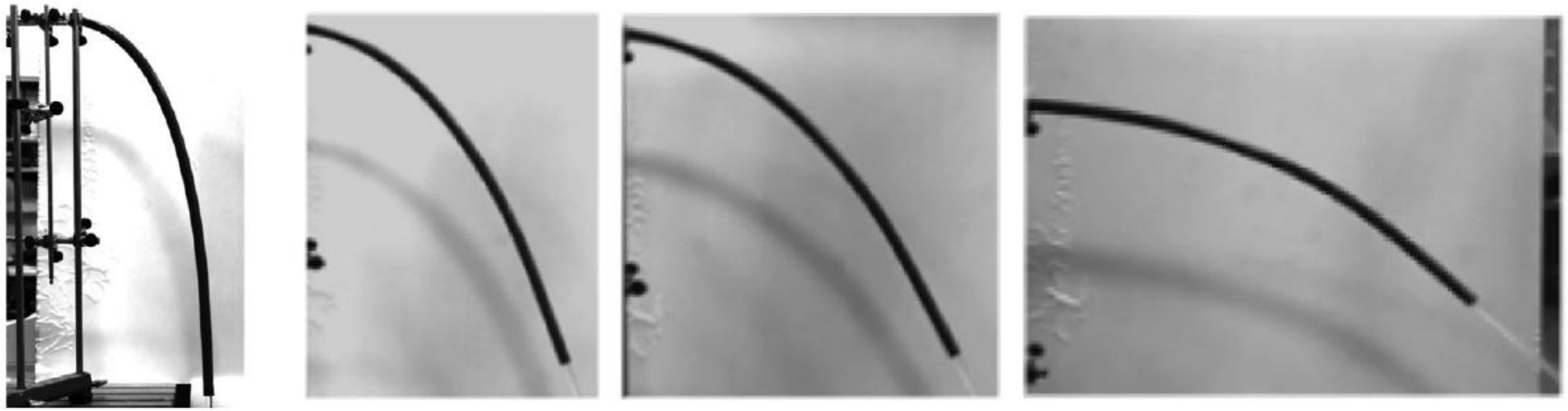


F. Boyer, V. Lebastard, F. Candelier, F. Renda, « Extended Hamilton’s principle applied to geometrically exact Kirchhoff sliding rods », *Journal of Sound and Vibration*, 2022.

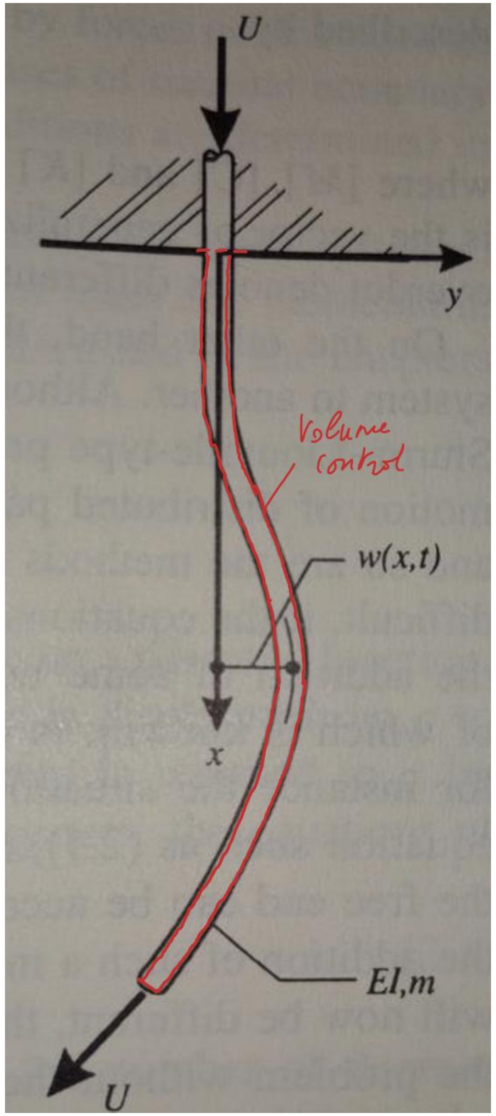
Second example :

« The garden pipe »

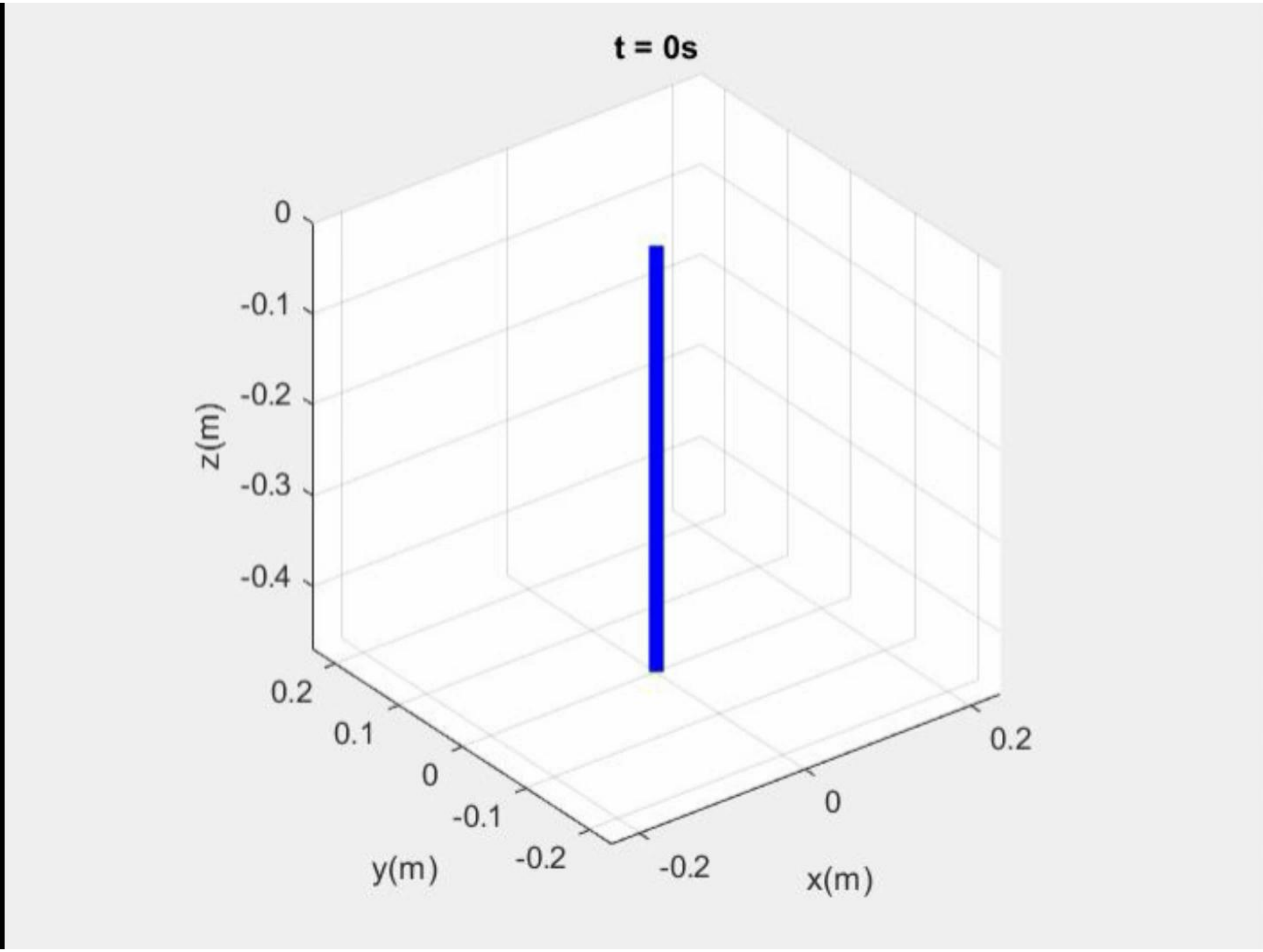
M.P. Païdoussis, N.T. Issid, « Dynamic stability of pipes conveying fluid », *Journal of Sound and Vibration*, Volume 33, Issue 3, 1974.



- Fluid is ideal (inviscid, incompressible)
- Fluid velocity uniform in pipe sections
- Pipe cross-section labels are non-material for fluid slices



Païdousis's book, 1988.


t = 0s
0
-0.1
-0.2
-0.3
-0.4
z(m)
0.2
0.1
0
-0.1
-0.2
y(m)
-0.2
0
0.2
x(m)

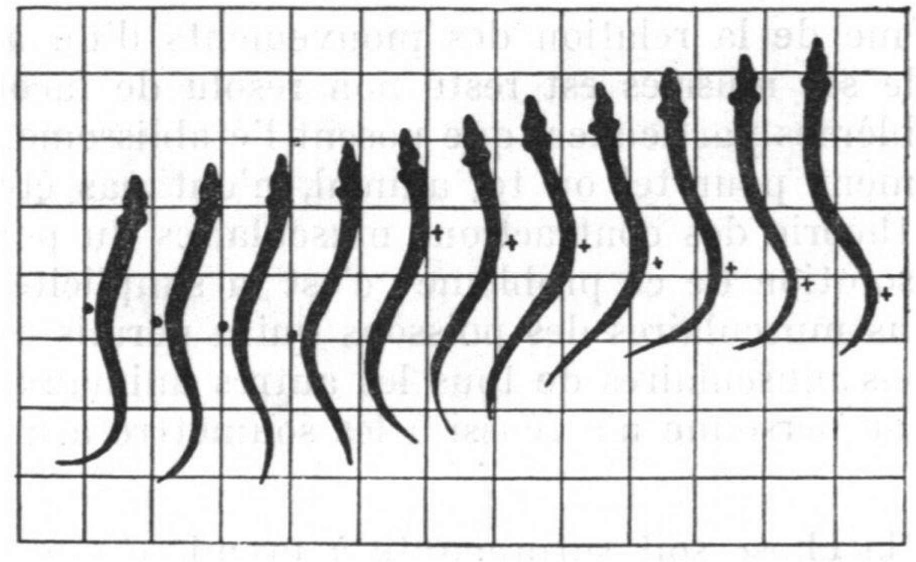
Third example :

« Large Amplitude Elongated Body swimming theory of J. Lighthill »

James Lighthill, “Large-amplitude elongated body theory of fish locomotion”. *Proc. R. Soc.* 179, 125–138 (1971).

At high Re, swimming gaits = wave of increasing curvature from head to tail…

J. Gray, J. Exp. Biol. 1933.

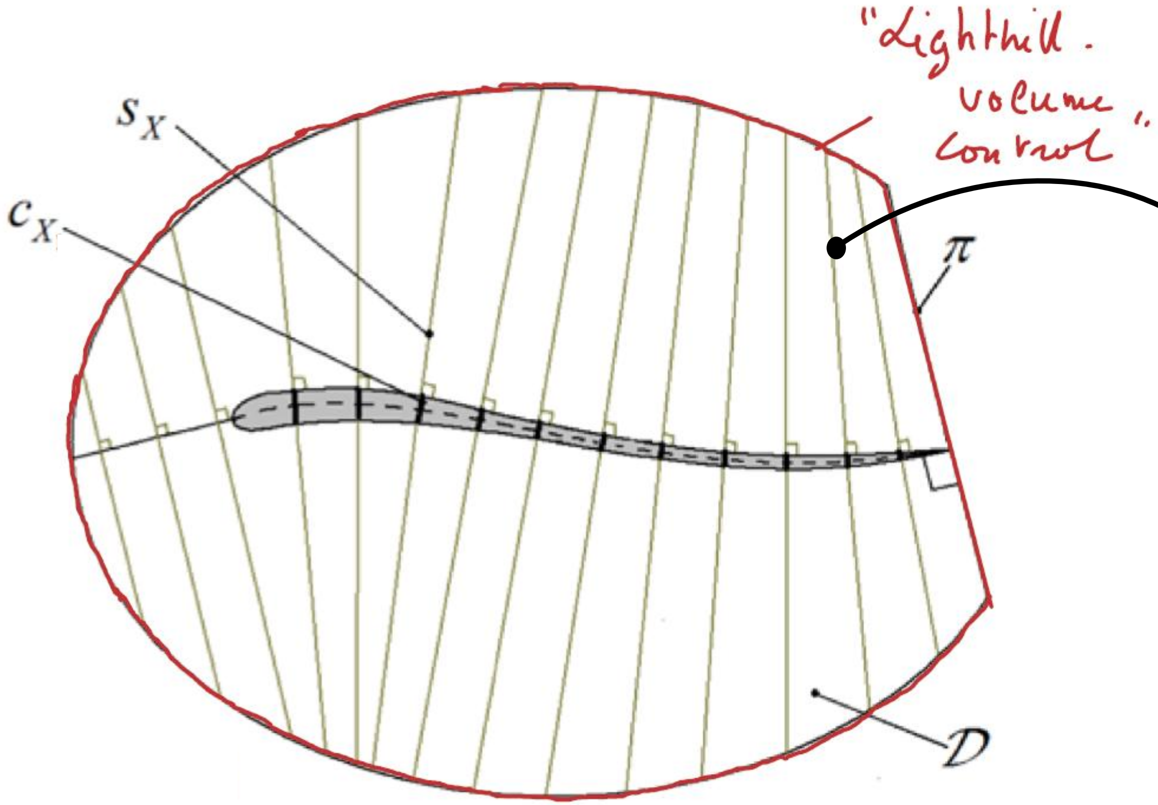


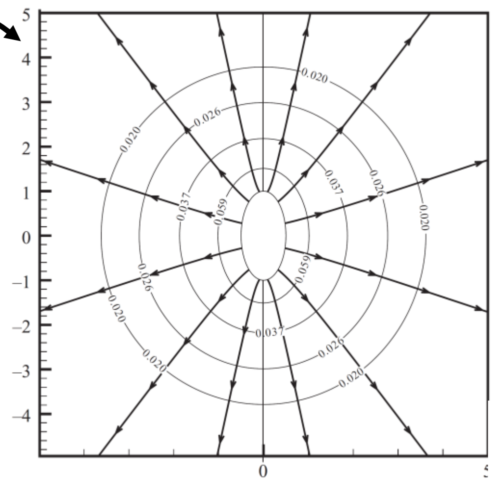
$$K(X,t) = (a_0 + a_1 X + a_2 X^2)\sin\left[2\pi\left(\frac{X}{\lambda} - \frac{t}{T}\right)\right],$$

Principle: *Each slice of the fluid is laterally accelerated by the body cross-sections as it sweeps past the body. Hence, the kinetic momentum of each fluid slice grows (from the head to the tail) before being shed into the wake, thus generating thrust to the fish by reaction.*

Ideal fluid and high aspect ratio : slender body theory of fluid mechanics.

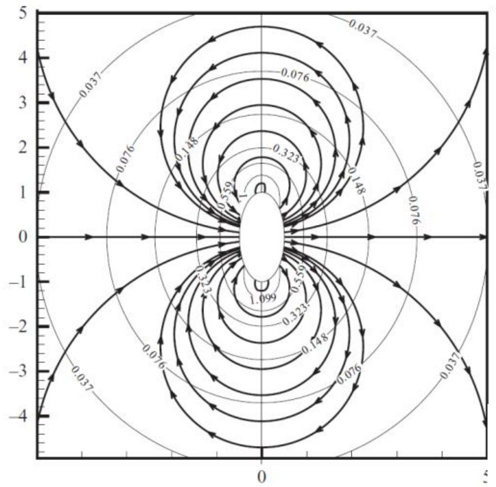
➡ The 3D flow inside $\mathcal{D}$ can be stratified into a stacking of planar flows (fluid Cosserat).





Planar flow of a source :
Models the axial penetration

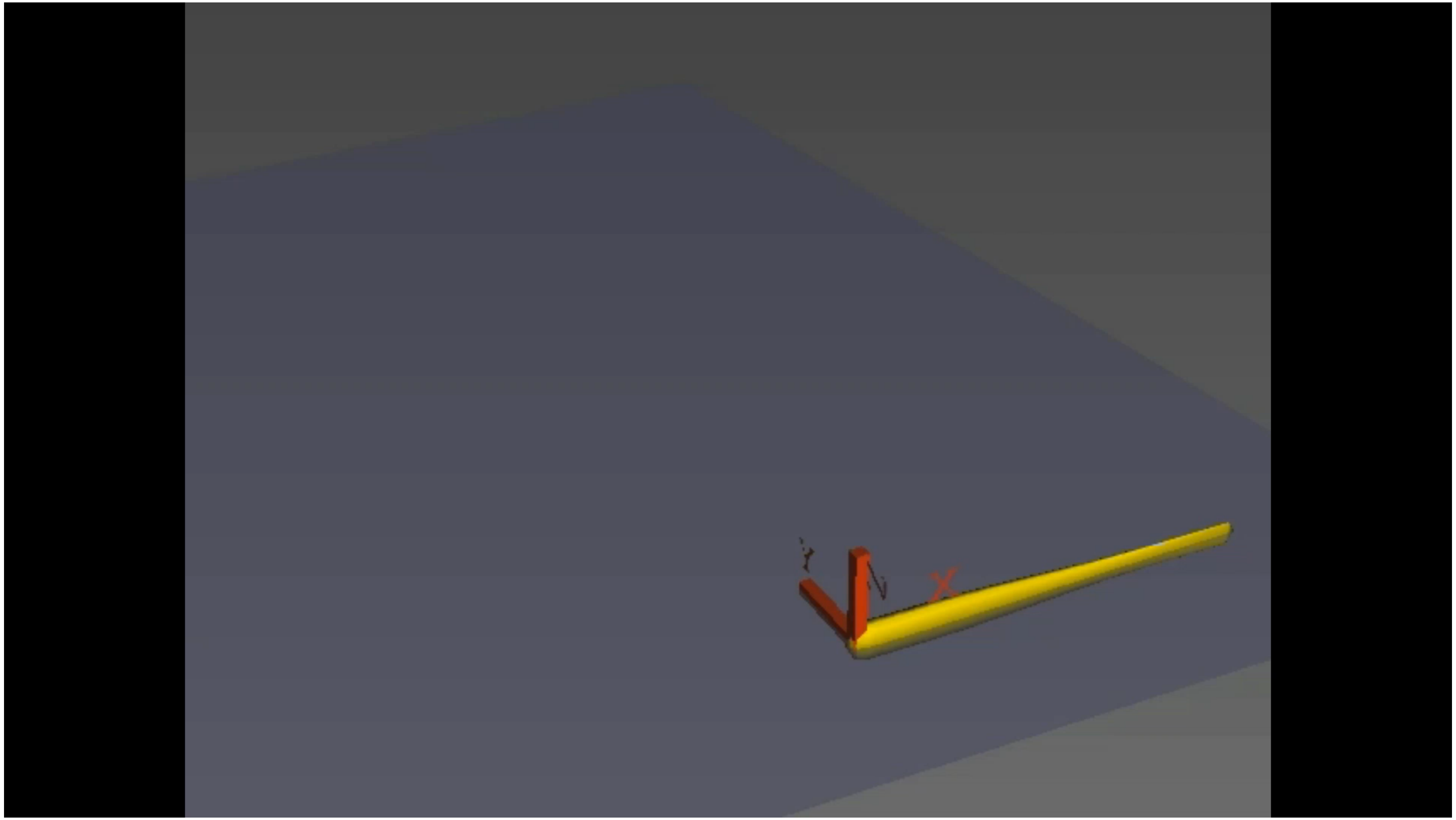
+



Planar flow of a doublet :
Models lateral oscillations

➡ Body cross-section labels are non-material for fluid slices ...

➡ $\mathfrak{T}_f(X)$ Density of fluid kinetic energy ➡ Non mat. Poincaré Cosserat eq. ➡ Extended Lighthill model of swimming



F. Boyer, M. Porez, A. Leroyer, « Poincaré–Cosserat Equations for the Lighthill Three-dimensional Large Amplitude Elongated Body Theory: Application to Robotics », *Journal of Nonlinear Science,* 20, 47–79 (2010).

- Fish= Cosserat beam internally governed by a constitutive (visco-elastic) law…

Video of a dead fish swimming in a Kvs…

$$\boldsymbol{v}_f = \left(v_\infty + \sum_i (-1)^i \frac{\Gamma}{2\pi r_i} \sin\theta_i \right) \boldsymbol{e}_1 + \sum_i (-1)^i \frac{\Gamma}{2\pi r_i} \cos\theta_i \boldsymbol{e}_2$$

F. Candelier, M. Porez, F. Boyer, « Note on the swimming of an elongated body in a non-uniform flow », *Journal of Fluid Mechanics*. 2013; 716:616-637.

F. Candelier, M. Porez, F. Boyer, « Note on the swimming of an elongated body in a non-uniform flow », *Journal of Fluid Mechanics*. 2013; 716:616-637.

Thank you for your attention !

Questions…?