Observer invariant port Hamiltonian systems: first steps

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23 juin 2026

Objective

Express a mechanical problem using observer-invariant port-Hamiltonian-like formulation, i.e. define

- a configuration state α
- a scalar energy functional $H(\alpha)$
- flows $\mathfrak{f}$ as a time-like variation of the state α
- efforts $\mathfrak{e} = \delta_\alpha H$ as the energy variation

so that the physical system is governed by as a closed-form system that relates flows and efforts by a skew-adjoint operator $\mathcal{S}$ as

$$\mathfrak{f} = \mathcal{S}\mathfrak{e}$$

and that is observer invariant.

Outline

1. Hypotheses and physical principle

2. Invariant to change of observer PHS / Interpretation

Hypotheses and principle

Hypothesis 1

The universe lies in a 4D Riemanian manifold with a metric of signature (+,-,-,-).

Hypothesis 2

The universe is occupied by physical [a] continuously distributed energy.

a. independent of the coordinate system.

Principle

The energy of any 4D-domain is left invariant by a vector field, which defines the velocity field.

4D manifold

Universe : $4D$ Riemannian manifold $\mathcal{M}$

- Metric tensor field $\boldsymbol{g}$
- Event $P \in \mathcal{M}$ identified by a set x of four coordinates $(x^\mu)_{\mu\in\{0,1,2,3\}}$
- Worldline $\gamma_P(s)$ curve in $\mathcal{M}$,

$$\gamma_P : \mathbb{R} \to \mathcal{M}, s \mapsto \gamma_P(s) = \{\phi_s(P)\}.$$

- 4-velocity noted $\boldsymbol{u}$:

$$u^\mu = \frac{dx^\mu}{ds} \quad \text{with} \quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu .$$

time-like future oriented unit vector :

$$u^\mu g_{\mu\nu} u^\nu = 1$$

4D World-tube

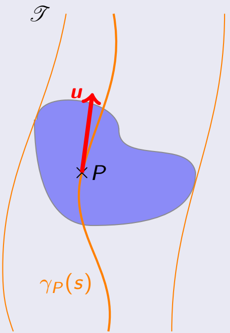

Figure – Representation of a world-tube $\mathscr{T}$ with world-lines.

Hypothesis 2 : Energy

Energy of a compact four-volume $\Omega \subset \mathcal{M}$

Given N $(1 \leq \mathcal{I} \leq \mathcal{N})$ function fields $\phi_{\mathcal{I}}(x)$ define the integral $\mathscr{E}_\Omega$:

$$\mathscr{E}_\Omega(A(\phi_{\mathcal{I}})) = \int_\Omega A(\phi_{\mathcal{I}}(x))\, vol_g$$

where $vol_g = \sqrt{-det(g)}\left(\mathrm{d}x^0(m) \wedge \mathrm{d}x^1(m) \wedge \mathrm{d}x^2(m) \wedge \mathrm{d}x^3(m)\right) = \sqrt{-det(g)}dx^4$
is the volume form associated with the metric $\boldsymbol{g}$.
Define the energy density $\mathcal{A} = A\sqrt{-det(g)}$.

Physical quantities are independent of the choice of a coordinate system

A is a scalar field

Description of the physics

Choice of the fields $\phi_{\mathcal{I}}$ and choice of the function $\mathcal{A}(\phi_{\mathcal{I}})$

The energy of any 4D-domain is left invariant by a vector field, which defines the velocity field.

The integral

$$\mathscr{E}_\Omega(A(\phi_{\mathcal{I}})) = \int_\Omega A(\phi_{\mathcal{I}})\ \sqrt{-det(g)}dx^4 = \int_\Omega \mathcal{A}(\phi_{\mathcal{I}})\ dx^4$$

is time-covariant, that is there exists a vector density field $\mathcal{B}$ such that

$$\sum_{\mathcal{I}} \frac{\partial \mathcal{A}}{\partial \phi_{\mathcal{I}}} \underset{u}{\mathscr{L}} \phi_{\mathcal{I}} = \nabla \cdot \mathcal{B}$$

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   - Case 1 : Mass
   - Case 2 : Mass - elasticity case

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Mass energy

$$H(f_\rho, g_{\mu\nu}) = \int_\Omega f_\rho c^2 vol_g = \int_\Omega f_\rho c^2 \sqrt{u^\mu g_{\mu\nu} u^\nu}\, vol_g = \int_\Omega f_\rho c^2 \sqrt{u^\mu g_{\mu\nu} u^\nu} \sqrt{-det(g)} \mathrm{d}x^4$$
$$= \int_\Omega \rho c^2 \sqrt{u^\mu g_{\mu\nu} u^\nu} \mathrm{d}x^4$$

where :

- $\sqrt{u^\mu g_{\mu\nu} u^\nu} = 1$ (definition of the quadri-velocity)
- f_ρ is a scalar function
- $\rho = f_\rho \sqrt{-det(g)}$ is the mass density (scalar density of weight 1).

State configuration and energy expression

$$\alpha = \begin{pmatrix} u^{\mu} \\ \mathcal{G}_{\mu\nu} \\ \varrho \end{pmatrix} \overset{physics}{=} \begin{pmatrix} u^{\mu} \\ g_{\mu\nu} \\ f_{\rho} c^2 \sqrt{-det(g)} \end{pmatrix}. \tag{1}$$

$$\begin{aligned} H(\alpha) &= \int_{\Omega} \varrho \sqrt{\mathcal{G}_{\mu\nu} u^{\mu} u^{\nu}} \mathrm{d}x^4 \\ &= \int_{\Omega} \mathcal{H}(\alpha) \mathrm{d}x^4 \end{aligned}$$

Flows

This allows us to define the flow of the PHS formulation as

$$\mathfrak{f} = \underset{u}{\mathscr{L}}\alpha = \begin{pmatrix} (\underset{u}{\mathscr{L}} u)^{\mu} \\ (\underset{u}{\mathscr{L}} \mathcal{G})_{\mu\nu} \\ \underset{u}{\mathscr{L}} \varrho \end{pmatrix}. \quad (2)$$

Physical meaning of the flows :

1. $\mathfrak{f}_u$ is the variation of u in the direction u, that is $\underset{u}{\mathscr{L}} u = 0$;
2. $\mathfrak{f}_g = 2d$ corresponds to (twice the) rate of deformation d ;
3. $\mathfrak{f}_\rho$ is the mass density variation w.r.t. u. We choose $\mathfrak{f}_\rho = 0$.

$$\mathfrak{f} = \begin{pmatrix} 0 \\ (\underset{u}{\mathscr{L}} g)_{\mu\nu} := 2d_{\mu\nu} \\ \underset{u}{\mathscr{L}} \left(f_\rho c^2 \sqrt{-det(g)} \right) = 0 \end{pmatrix} \quad (3)$$

Efforts

Remembering the energy

$$H(\alpha) = \int_\Omega \varrho \sqrt{\mathcal{G}_{\mu\nu} u^\mu u^\nu} \mathrm{d}x^4,$$

the efforts are defined by its variationnal derivatives

$$\mathfrak{e} = \begin{pmatrix} \delta_u H \\ \delta_{\mathcal{G}} H \\ \delta_\varrho H \end{pmatrix} = \begin{pmatrix} \varrho u_\mu \\ \frac{\varrho}{2} u^\mu u^\nu \\ \sqrt{\mathcal{G}_{\mu\nu} u^\mu u^\nu} \end{pmatrix} \tag{4}$$

Physical meaning of the efforts :

1. ρu^μ is the momentum,
2. $\varrho u^\mu u^\nu = T^{\mu\nu}$ is the momentum-energy tensor,
3. $\sqrt{\mathcal{G}_{\mu\nu} u^\mu u^\nu} = ||u||_g$.

$$\mathfrak{e} = \begin{pmatrix} \rho c^2 u_\mu \\ \frac{T^{\mu\nu}}{2} \\ \sqrt{\mathcal{G}_{\mu\nu} u^\mu u^\nu} \end{pmatrix} \tag{5}$$

Interconnection matrix

Flows and efforts are related by a skew-adjoint operator [a]

$$\begin{pmatrix} (\underset{u}{\mathscr{L}} u)^\mu \\ (\underset{u}{\mathscr{L}} \mathcal{G})_{\mu\nu} \\ \underset{u}{\mathscr{L}} \varrho \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\varrho}\nabla\cdot & 0 \\ (\nabla + \nabla^T)(\frac{\cdot}{\varrho}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_u H \\ \delta_{\mathcal{G}} H \\ \delta_\varrho H \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2d_{\mu\nu} \\ \underset{u}{\mathscr{L}} \varrho \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\varrho}\nabla\cdot & 0 \\ (\nabla + \nabla^T)(\frac{\cdot}{\varrho}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varrho u_\mu \\ \frac{\varrho}{2} u^\mu u^\nu \\ \sqrt{\mathcal{G}_{\mu\nu} u^\mu u^\nu} \end{pmatrix}$$

leading to the three equations

$$0 = \frac{2}{\varrho}\nabla \cdot \left(\frac{\varrho}{2} u^\mu u^\nu\right) = \frac{1}{\varrho}\nabla \cdot T$$

$$2d_{\mu\nu} = (\nabla + \nabla^T)(\frac{\varrho u_\mu}{\varrho}) = (\nabla + \nabla^T) u_\mu$$

$$\underset{u}{\mathscr{L}} \varrho = 0$$

a. Andrea Brugnoli, Denis Matignon et Joseph Morlier. “A linearly-implicit energy-momentum preserving scheme for geometrically nonlinear mechanics based on non-canonical Hamiltonian formulations”. In : *Nonlinear Dynamics* 113.20 (2025), p. 27539-27566. ISSN : 1573-269X. DOI : https://doi.org/10.1007/s11071-025-11601-6.

Interpretations

Stress-energy tensor conservation

At non relativistic limit, energy and momentum conservation (1st principle of thermodynamics and mechanical equilibrium) :

$$\nabla \cdot T = 0$$

Strain rate tensor

Strain rate definition in 4D :

$$2d_{\mu\nu} = \left(\nabla + \nabla^T\right) u_\mu$$

Mass conservation

Under the hypothesis of mass conservation : we choose

$$\underset{u}{\mathscr{L}} \varrho = 0$$

.

Power balance

$$\begin{pmatrix} 0 \\ 2d_{\mu\nu} \\ \underset{u}{\mathscr{L}}\varrho \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\varrho}\nabla\cdot & 0 \\ (\nabla+\nabla^T)(\frac{\cdot}{\varrho}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varrho u_\mu \\ \frac{\varrho}{2}u^\mu u^\nu \\ \sqrt{\mathcal{G}_{\mu\nu}u^\mu u^\nu} \end{pmatrix}$$

Power balance

$$\begin{aligned} \int_\Omega \mathfrak{e}^T J \mathfrak{e} \mathrm{d}x^4 &= \int_\Omega \left(\varrho u_\mu \frac{1}{\varrho} \nabla \cdot T^{\mu\nu} + \frac{1}{2} T^{\mu\nu}_\rho 2d_{\mu\nu} + \sqrt{\mathcal{G}_{\mu\nu}u^\mu u^\nu} . \underset{u}{\mathscr{L}}\varrho \right) \mathrm{d}x^4 \\ &= \int_\Omega \left(\varrho u_\mu \frac{1}{\varrho} .0 + \frac{1}{2} T^{\mu\nu}_\rho 2d_{\mu\nu} + 1.0 \right) \mathrm{d}x^4 \\ &= \int_\Omega T^{\mu\nu} d_{\mu\nu} \mathrm{d}x^4 = 0 \end{aligned}$$

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Mass and elastic energy

$$
\begin{aligned}
H(f_\rho, g_{\mu\nu}, \mathcal{C}^{\mu\nu\kappa\lambda}, e_{\mu\nu}) &= \int_\Omega \left(f_\rho c^2 + \frac{1}{2}\mathcal{C}^{\mu\nu\kappa\lambda} e_{\mu\nu} e_{\kappa\lambda} \right) vol_g \\
&= \int_\Omega \left(f_\rho c^2 \sqrt{u^\mu g_{\mu\nu} u^\nu} + \frac{1}{8}\mathcal{C}^{\mu\nu\kappa\lambda} (g_{\mu\nu} - b_{\mu\nu})(g_{\kappa\lambda} - b_{\kappa\lambda}) \right) \sqrt{-det(g}\\
&= \int_\Omega \left(f_\rho c^2 \sqrt{u^\mu g_{\mu\nu} u^\nu} + \frac{1}{2}\mathcal{C}^{\mu\nu\kappa\lambda} e_{\mu\nu} e_{\kappa\lambda} \right) \sqrt{-det(g)}\mathrm{d}x^4
\end{aligned}
$$

where $\mathcal{C}$ is the elasticity tensor and with the strain $e = \frac{1}{2}(g - b)$ defined by the metric g and the deformation tensor b.

State and energy functional

State and Hamiltonian expression

$$\alpha = \begin{pmatrix} u^{\mu} \\ \mathcal{G}_{\mu\nu} \\ \varrho \\ \mathcal{E}^{\mu\nu\kappa\lambda} \\ e_{\mu\nu} \end{pmatrix} \overset{physics}{=} \begin{pmatrix} u^{\mu} \\ g_{\mu\nu} \\ f_{\rho} c^2 \sqrt{-det(g)} = c^2 \rho \\ \sqrt{-det(g)}\mathcal{C}^{\mu\nu\kappa\lambda} \\ e_{\mu\nu} \end{pmatrix}, \tag{8}$$

$$H(\alpha) = \int_{\Omega} \left(\varrho \sqrt{\mathcal{G}_{\mu\nu} u^{\mu} u^{\nu}} + \frac{1}{2} \mathcal{E}^{\mu\nu\kappa\lambda} e_{\mu\nu} e_{\kappa\lambda} \right) \mathrm{d}x^4 = \int_{\Omega} \mathcal{H}(\alpha) \mathrm{d}x^4$$

PHS flows

This allows us to define the flow of the PHS formulation as

$$\mathfrak{f} = \underset{u}{\mathscr{L}} \alpha = \begin{pmatrix} \underset{u}{\mathscr{L}} u = 0 \\ \underset{u}{\mathscr{L}} \mathcal{G}_{\mu\nu} \\ \underset{u}{\mathscr{L}} \varrho \\ \underset{u}{\mathscr{L}} (\mathcal{E}^{\mu\nu\kappa\lambda}) \\ \underset{u}{\mathscr{L}} e_{\mu\nu} \end{pmatrix}. \tag{9}$$

Physical meaning :

1. $\underset{u}{\mathscr{L}} (\mathcal{E}^{\mu\nu\kappa\lambda}) = 0$ material properties are conserved (not related to strain)
2. $\underset{u}{\mathscr{L}} e_{\mu\nu} = \frac{1}{2} \underset{u}{\mathscr{L}} (g_{\mu\nu} - b_{\mu\nu}) = \frac{1}{2} \underset{u}{\mathscr{L}} g_{\mu\nu}$ since $\underset{u}{\mathscr{L}} b_{\mu\nu} = 0$.

$$\mathfrak{f} = \underset{u}{\mathscr{L}} \alpha = \begin{pmatrix} 0 \\ \underset{u}{\mathscr{L}} \mathcal{G}_{\mu\nu} := 2 d_{\mu\nu} \\ \underset{u}{\mathscr{L}} \varrho = 0 \\ \underset{u}{\mathscr{L}} (\mathcal{E}^{\mu\nu\kappa\lambda}) = 0 \\ \underset{u}{\mathscr{L}} e_{\mu\nu} := d_{\mu\nu} \end{pmatrix}. \tag{10}$$

Effort

$$H(\alpha) = \int_\Omega \left(\varrho \sqrt{u^\mu \mathcal{G}_{\mu\nu} u^\nu} + \frac{\mathcal{E}^{\mu\nu\kappa\lambda}}{2} e_{\mu\nu} e_{\kappa\lambda} \right) \mathrm{d}x^4$$

The efforts are defined by

$$\mathfrak{e} = \begin{pmatrix} \delta_u H \\ \delta_{\mathcal{G}} H \\ \delta_\varrho H \\ \delta_{\mathcal{E}} H \\ \delta_e H \end{pmatrix} = \begin{pmatrix} \varrho u_\mu \\ \varrho \frac{u^\mu u^\nu}{2} \\ \sqrt{u^\mu \mathcal{G}_{\mu\nu} u^\nu} \\ \frac{1}{2} e_{\mu\nu} e_{\kappa\lambda} \\ \mathcal{E}^{\mu\nu\kappa\lambda} e_{\kappa\lambda} \end{pmatrix} \tag{11}$$

Physical meaning :

1.
2.
3.
4.
5. $\mathcal{E}^{\mu\nu\kappa\lambda} e_{\mu\nu}$ the elastic part $T_{\mathcal{E}}$ of the stress-energy tensor T.

$$\mathfrak{e} = \begin{pmatrix} \rho c^2 u_\mu \\ \frac{1}{2} T_\rho^{\mu\nu} \\ \sqrt{\mathcal{G}_{\mu\nu} u^\mu u^\nu} \\ \frac{1}{2} e_{\mu\nu} e_{\kappa\lambda} \\ T_{\mathcal{E}}^{\mu\nu} \end{pmatrix} \tag{12}$$

Interconnection matrix

Flows and efforts are related by a skew-adjoint operator [a]

$$\begin{pmatrix} (\underset{u}{\mathscr{L}} u)^{\mu} \\ (\underset{u}{\mathscr{L}} \mathcal{G})_{\mu\nu} \\ \underset{u}{\mathscr{L}} \varrho \\ \underset{u}{\mathscr{L}} (\mathcal{E}^{\mu\nu\kappa\lambda}) \\ \underset{u}{\mathscr{L}} e_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\varrho}\nabla\cdot & 0 & 0 & \frac{1}{\varrho}\nabla\cdot \\ (\nabla+\nabla^T)(\frac{\cdot}{\varrho}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (\nabla+\nabla^T)(\frac{\cdot}{2\varrho}) & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_u H(X) \\ \delta_{\mathcal{G}} H(X) \\ \delta_{\varrho} H(X) \\ \delta_{\mathcal{E}} H \\ \delta_e H \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2d_{\mu\nu} \\ \underset{u}{\mathscr{L}} \varrho \\ \underset{u}{\mathscr{L}} (\mathcal{E}^{\mu\nu\kappa\lambda}) \\ d_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\varrho}\nabla\cdot & 0 & 0 & \frac{1}{\varrho}\nabla\cdot \\ (\nabla+\nabla^T)(\frac{\cdot}{\varrho}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (\nabla+\nabla^T)(\frac{\cdot}{2\varrho}) & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varrho u_{\mu} \\ \frac{1}{2} T_{\rho}^{\mu\nu} \\ \sqrt{\mathcal{G}_{\mu\nu} u^{\mu} u^{\nu}} \\ \frac{1}{2} e_{\mu\nu} e_{\kappa\lambda} \\ T_{\mathcal{E}}^{\mu\nu} \end{pmatrix}$$

a. Brugnoli, Matignon et Morlier, "A linearly-implicit energy-momentum preserving scheme for geometrically nonlinear mechanics based on non-canonical Hamiltonian formulations".

Interpretations

Stress-energy tensor

$$T^{\mu\nu} = T_{\rho}^{\mu\nu} + T_{\mathcal{E}}^{\mu\nu} = \varrho u^{\mu} u^{\nu} + \mathcal{E}^{\mu\nu\kappa\lambda} e_{\kappa\lambda}$$

Five equations

$$0 = \frac{2}{\varrho} \nabla \cdot \left(\frac{\varrho}{2} u^{\mu} u^{\nu} \right) + \frac{1}{\varrho} \nabla \cdot \left(\mathcal{E}^{\mu\nu\kappa\lambda} e_{\kappa\lambda} \right) = \frac{1}{\varrho} \nabla \cdot \left(T_{\rho}^{\mu\nu} + T_{\mathcal{E}}^{\mu\nu} \right)$$

$$2 d_{\mu\nu} = (\nabla + \nabla^{T})(\frac{\varrho u_{\mu}}{\varrho}) = (\nabla + \nabla^{T}) u_{\mu}$$

$$\underset{u}{\mathscr{L}} \varrho = 0$$

$$\underset{u}{\mathscr{L}} (\mathcal{E}^{\mu\nu\kappa\lambda}) = 0$$

$$d_{\mu\nu} = (\nabla + \nabla^{T})(\frac{\varrho u_{\mu}}{2\varrho}) = \frac{1}{2} (\nabla + \nabla^{T}) u_{\mu}$$

Power balance

$$\begin{pmatrix} 0 \\ 2d_{\mu\nu} \\ \underset{u}{\mathscr{L}}\varrho \\ \underset{u}{\mathscr{L}}(\mathcal{E}^{\mu\nu\kappa\lambda}) \\ d_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\varrho}\nabla\cdot & 0 & 0 & \frac{1}{\varrho}\nabla\cdot \\ (\nabla+\nabla^T)(\frac{\cdot}{\varrho}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (\nabla+\nabla^T)(\frac{\cdot}{2\varrho}) & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varrho u_\mu \\ \frac{1}{2}T_\rho^{\mu\nu} \\ 1 \\ \frac{1}{2}e_{\mu\nu}e_{\kappa\lambda} \\ T_\mathcal{E}^{\mu\nu} \end{pmatrix}$$

Power balance

$$\begin{aligned}\int_\Omega \mathfrak{e}^T J\mathfrak{e}\,\mathrm{d}x^4 &= \int_\Omega \left(\varrho u_\mu \frac{1}{\varrho}\nabla\cdot T^{\mu\nu} + \frac{1}{2}T_\rho^{\mu\nu}2d_{\mu\nu} + 1.\underset{u}{\mathscr{L}}\varrho + \frac{1}{2}e_{\mu\nu}e_{\kappa\lambda}.\underset{u}{\mathscr{L}}\mathcal{E}^{\mu\nu\kappa\lambda} + T_\mathcal{E}^{\mu\nu}d_{\mu\nu}\right)\mathrm{d}x^4 \\ &= \int_\Omega \left(\varrho u_\mu \frac{1}{\varrho}.0 + \frac{1}{2}T_\rho^{\mu\nu}2d_{\mu\nu} + 1.0 + \frac{1}{2}e_{\mu\nu}e_{\kappa\lambda}.0 + T_\mathcal{E}^{\mu\nu}d_{\mu\nu}\right)\mathrm{d}x^4 \\ &= \int_\Omega T^{\mu\nu}d_{\mu\nu}\mathrm{d}x^4 = 0\end{aligned}$$

Conclusion

Results

- Observer-invariant port-Hamiltonian formulation for the case
  - of the mass
  - of the mass with elasticity

Outlook

- Modelling and numerical simulation for large transformations models.